19 Lecture 04.06

On 04.06, we proved

Theorem 1 *If $V \subseteq U^B$ is definable over B, then V is finite or V is co-finite.*

Proof: Suppose to the contrary, that there is a set V, definable over B, which is neither finite nor co-finite, and suppose that the schema $S(x)$ defines V over B. We derive a contradiction from this hypothesis. Let $\Lambda = \{S \mid B \models S\}$; Λ is the set of all schemata true in the structure B and is often called the *complete theory* of B. Let y and z be fresh variables which occur nowhere in Λ, $S(x)$, or any of the schemata $S^n(x)$ for $n \geq 0$ defined above. Define the set of schemata Γ as follows.

$$\Gamma = \Lambda \cup \{S(y), \neg S(z)\} \cup \{\neg S^n(y), \neg S^n(z) \mid n \geq 0\}.$$

Let Δ be a finite subset of Γ. It follows from the fact that both $S[B]$ and $\neg S[B]$ are infinite, that Δ is satisfied by B with suitable assignments from U^B to the variables y and z. Hence, by the Compactness Theorem, Γ itself is satisfiable. Of course, if the structure C satisfies Γ, then C is not isomorphic to B since the the elements of U^C assigned to y and z in C (call them a and b respectively) are not reachable in C from the unique element of C with no predecessor. We will show that there is an automorphism h of C with $h(a) = b$. This will yield the desired contradiction, since $C \models S(y|a)$ and $C \models \neg S(z|b)$. Note that B, and hence C, satisfy the following infinite list of schemata, let's call it Ω.

- $(\exists x)(\forall y)((\forall z)\neg Lzy \equiv x = y)$
- $(\forall x)(\exists y)(\forall z)(Lxz \equiv z = y)$
- $(\forall x)(\forall y)(\forall z)((Lxz \land Lyz) \supset x = y)$
- $(\forall x)\neg Lxx$

  $\vdots$

  $(\forall x)(\forall y_1)\ldots(\forall y_n)\neg(Lxy_1 \land Ly_1y_2 \ldots \land Ly_nx)$

  $\vdots$

The first three schemata of Ω guarantee that L^C is an injective functional relation which is "almost" surjective – there is a unique element of U^C which lacks a pre-image under the function whose graph is L^C. Note that this guarantees that U^C is infinite. The final infinite list of schemata of Ω guarantee that the the function whose graph is L^C contains no finite cycles. Since C is not isomorphic to B all this implies that C consists of an L^C chain that is isomorphic to B and a non-empty set of L^C chains each of which is isomorphic to $\mathbb{Z}$ (the set of all integers) equipped with its usual successor relation. But, since a and b must lie on one or two of these "$\mathbb{Z}$-chains," there is an automorphism h of C with $h(a) = b$. ∎

19.1 Addendum

The following material, though not covered in class, provides another interesting application of the Compactness Theorem that can hardly be passed over in the context of our discussion of the infinite set of sentences Ω. We will return to study Theorem 2 before the Term ends.

Recall that a set of schemata Λ implies a schema S if and only if for every structure A, if $A \models \Lambda$, then $A \models S$. We write $\Lambda \models S$ as shorthand for Λ implies S. This "overloading" of the symbol "$\models$" is harmless since no ambiguity can arise; in the case of "implies" the lefthand argument to "$\models$" is a set of sentences, in the case of "satisfies" the lefthand argument to "$\models$" is a structure. We write $\mathsf{Cn}(\Lambda)$ for $\{S \mid \Lambda \models S\}$, the set of schemata implied by Λ. Now to the question: Is there a schema S such that $\mathsf{Cn}(\{S\}) = \mathsf{Cn}(\Omega)$? Or, as some might put it, Is $\mathsf{Cn}(\Omega)$ *finitely axiomatizable*? In order to answer this question, let's recast the Compactness Theorem as a result about implication.

Theorem 2 (Compactness for Implication) *Suppose Λ is a set of schemata and S is a schema. If $\Lambda \models S$, then there is a finite $\Delta \subseteq \Lambda$ such that $\Delta \models S$.*

Proof: Suppose that $\Lambda \models S$. It follows that $\Lambda \cup \{\neg S\}$ is not satisfiable. Hence, by the Compactness Theorem, there is a finite $\Delta \subseteq \Lambda$ such that $\Delta \cup \{\neg S\}$ is not satisfiable. Therefore, there is a finite $\Delta \subseteq \Lambda$ such that $\Delta \models S$. ■

Corollary 1 *There is no schema S such that $\mathsf{Cn}(\{S\}) = \mathsf{Cn}(\Omega)$.*

Proof: Suppose that $\mathsf{Cn}(\{S\}) = \mathsf{Cn}(\Omega)$, for some schema S. It follows that $\Omega \models S$, and hence, by Compactness for Implication, for some finite $\Delta \subseteq \Omega$, $\Delta \models S$. But for large enough n, $\Delta \not\models (\forall x)(\forall y_1)\ldots(\forall y_n)\neg(Lxy_1 \wedge Ly_1y_2 \ldots \wedge Ly_nx)$.
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