18 Lecture 04.04

We continued our analysis of definability in the structure B that we began on 03.30. The argument we presented at the end of class on 03.30 to show that B is rigid suggests that for every $k \in U^B$, $\{k\}$ is definable over B. Let's show this, again by induction. First, the schema $S^0(x) : (\forall y)\neg Lyx$ defines $\{0\}$ over B. Next, as induction hypothesis, suppose that $S^n(x)$ defines $\{n\}$ over B. Let z be a variable which does not occur anywhere in $S^n(x)$ and let $S^n(z)$ be the result of replacing x with z at all its occurrences in $S^n(x)$. then the schema $(\exists z)(S^n(z) \wedge Lzx)$ defines $\{n + 1\}$ over B. this completes the induction and establishes that for every $k \in U^B$, $\{k\}$ is definable over B. It follows at once that every finite subset of U^B and every co-finite subset of U^B is definable over B.

What other subsets of U^B are definable over B? Note that since B is rigid, there is no possibility of exhibiting an automorphism h of B with $h[X] \neq X$, that is, the "automorphism method" is powerless to establish the undefinability of any subset of U^B in B. Could it be that every subset of U^B is definable over B? We show at once that for every infinite structure C there is a subset $X \subseteq U^C$ which is *not* definable over C. This result is a corollary to the celebrated Cantor Diagonal Theorem.

Theorem 1 (Cantor) *Let U be a set and let $\{V_a \mid a \in U\}$ be a collection of subsets of U indexed by U, that is, for each $a \in U$, V_a is a subset of U. Then there is subset W of U such that for all $a \in U$, $W \neq V_a$.*

Proof: Let $W = \{a \mid a \notin V_a\}$. Thus, for every $a \in U$, $a \in W$ if and only if $a \notin V_a$. It follows that for all $a \in U$, $W \neq V_a$. ■

In order to apply Theorem 1 to questions about definable sets we require the following result.

Theorem 2 *For every structure C, there is a sequence $V_1, V_2, \ldots$ of subsets of U^C such that for every set X definable over C, there is an i such that $X = V_i$.*

Proof: Every schema is a finite sequence of symbols drawn from a finite alphabet. Thus, we may arrange all schemata $S(x)$ in a list $S_1(x), S_2(x), \ldots$, first ordered by length, and then within length, alphabetically. We obtain a list $V_1, V_2, \ldots$ of all the sets definable over C by setting $V_i = S_i[C]$ for all i. ■

The following result is an immediate consequence of Theorems 1 and 2.

Corollary 1 *For every infinite structure C there is a subset $X \subseteq U^C$ which is not definable over C.*

Of course, this gives us no idea which particular sets are not definable over a given infinite structure. In the case of the graph B introduced above, we will show that if a set is neither finite nor co-finite, it is *not* definable over B. In order to establish this, we will deploy one of the fundamental properties of polyadic quantification theory: *compactness*. First, some definitions requisite to state the Compactness Theorem for polyadic quantification theory.

- A schema S is *satisfiable* if and only if for some structure A, $A \models S$.
- A set of schemata Γ is *satisfiable* if and only if there is structure A such that for every schema $S \in \Gamma$, $A \models S$.
- A set of schemata Γ is *finitely satisfiable* if and only if for every finite set $\Delta \subseteq \Gamma$, Δ is satisfiable.

Theorem 3 (Compactness Theorem) *For every set Γ of schemata of polyadic quantification theory, if Γ is finitely satisfiable, then Γ is satisfiable.*

Though the Compactness Theorem makes no mention of the notion of a derivation, one of its well-known proofs proceeds via the elaboration of a sound and complete formal system for logical deduction. This development will occupy our attention for much of the remainder of the Term. But for the moment, let's see how we can apply the Compactness Theorem to complete the analysis of the definable subsets of the structure B specified above.

Theorem 4 *If $V \subseteq U^B$ is definable over B, then V is finite or V is co-finite.*