17 Lecture 03.30

On 03.30, we began to look at definability in infinite structures. We first analyzed definability in the infinite graph A described as follows:

- $U^A = \mathbb{Z}$, the set of all integers, $\{\ldots -2, -1, 0, 1, 2, \ldots\}$;
- $L^A = \{\langle i, j\rangle \mid j \text{ is the absolute value of } i\}$. (Recall that the absolute value of an integer i is i, if $i \geq 0$, and is $-i$, if $i < 0$.)

We observed that every permutation g of $\mathbb{Z}^+$ can be extended to an automorphism h of A by setting $h(i) = g(i)$, for $i \in \mathbb{Z}^+$; $h(0) = 0$; and $h(i) = -g(-i)$, for $i < 0$. Let's write $\mathbb{Z}^-$ for the set of negative integers. Thus, $\mathsf{Orbs}(A, \mathsf{Aut}(A)) = \{\mathbb{Z}^+, \{0\}, \mathbb{Z}^-\}$. Each orbit of $\mathsf{Aut}(A)$ acting on U^A is definable:

- $S_1[A] = \mathbb{Z}^+$, where $S_1(x)$ is $(\exists y)(y \neq x \wedge Lyx)$;
- $S_2[A] = \mathbb{Z}^-$, where $S_2(x)$ is $(\forall y)\neg Lyx$;
- $S_3[A] = \{0\}$, where $S_3(x)$ is $\neg S_1(x) \wedge \neg S_2(x)$.

Hence, there are exactly eight sets definable in A:

1. $\emptyset$,
2. $\{0\}$,
3. $\mathbb{Z}^+$,
4. $\mathbb{Z}^-$,
5. $\mathbb{Z}^+ \cup \mathbb{Z}^-$,
6. $\mathbb{Z}^+ \cup \{0\}$,
7. $\mathbb{Z}^- \cup \{0\}$,
8. $\mathbb{Z}$.

At this point, one of you raised a very interesting question: to what extent can the structure A itself be specified by a schema. As we'd already discussed:

Theorem 1 *If D is a finite graph, then there is a schema S such that for every graph D',*

$$D' \models S \text{ if and only if } D' \cong D.$$

The following result, in sharp contrast, shows that *no* infinite structure can be perfectly described by schemata. In order to state the result, we need to define $\mathsf{Th}(D)$, the *complete theory of* D:

$$\mathsf{Th}(D) = \{S \mid S \text{ is a schema and } D \models S\}.$$

Theorem 2 *For every infinite graph D, there is a graph D', $D' \models \mathsf{Th}(D)$ and $D' \ncong D$.*

Theorem 2 is a corollary to the Compactness Theorem for polyadic quantification theory, a fundamental result we will study next week.

We next looked at another infinite structure B where definability behaves very differently. B is described as follows:

- $U^B = \mathbb{Z}^+ \cup \{0\}$;
- $L^B = \{\langle i, j\rangle \mid j = i + 1\}$.

We first observed that $\mathsf{Aut}(B) = \{e\}$, that is, B is a rigid structure. This can be established by mathematical induction. Suppose h is an automorphism of B. Since 0 is the only node of B with in-degree 0, we must have $h(0) = 0$. Now suppose, as induction hypothesis, that $h(n) = n$. Since $n + 1$ is the only member of U^B to which n is related, it follows from the hypothesis that h is an automorphism that $h(n + 1) = n + 1$. This completes the induction; thus, for all $k \in U^B$, $h(k) = k$. Hence, $\mathsf{Aut}(B) = \{e\}$.