16 Lecture 03.23

On 03.23, we continued to look at the use of automorphisms, now as a tool for analyzing which sets are definable in a structure.

Up to this point we have neglected schemata containing free variables. Today we will correct this oversight. Consider the schema

$$S(x): \quad (\exists y)(\forall z)(Lxz \equiv z = y).$$

Let A be a graph. We define $S[A] = \{a \in U^A \mid A \models S[x|a]\}$, that is, $S[A]$ is the set of nodes of A that satisfy the schema $S(x)$ in A when assigned to the variable x. We call $S[A]$ the *set defined by* $S(x)$ in A. In the case to hand, if A is a simple graph, then $S[A]$ is the set of nodes of A of degree 1.

Given a graph A, we will consider which subsets of U^A are *definable subsets of* A, that is for which $V \subseteq U^A$ is there a schema $S[x]$ such that $S[A] = V$. In the case of finite graphs, we will be able to give an entirely satisfactory analysis in terms of the symmetries of A, that is, the collection of automorphisms of A. Recall that h is an *automorphism* of A if and only if h is a bijection of U^A onto U^A and for all $a, b \in U^A$,

$$\langle a, b\rangle \in L^A \text{ if and only if } \langle h(a), h(b)\rangle \in L^A.$$

In other words, h is an automorphism of A if and only if h is an isomorphism of A onto itself. We define $\mathsf{Aut}(A) = \{h \mid h \text{ is an automorphism of } A\}$. The following theorem is fundamental.

Theorem 1 *Let A be a graph and $h \in \mathsf{Aut}(A)$. For every $a \in U^A$ and every schema $S(x)$,*

$$A \models S[x|a] \textit{ if and only if } A \models S[x|h(a)].$$

If f is a function with domain U and $V \subseteq U$, we define $f[V] = \{f(a) \mid a \in V\}$ (the f *image* of V). With this notation in hand, we can now state a corollary to Theorem 1 which bears on definability.

Corollary 1 *Let A be a graph and $h \in \mathsf{Aut}(A)$. If V is a definable subset of A, then $h[V] = V$.*

Thus, in order to show that V is *not* a definable subset of A it suffices to exhibit an $h \in \mathsf{Aut}(A)$ and $a \in V$ such that $h(a) \notin V$. Moreover, in the case of finite structures, the converse of Corollary 1 is true.

Theorem 2 *Let A be a finite graph and $V \subseteq U^A$. V is a definable subset of A, if for every $h \in \mathsf{Aut}(A)$, $h[V] = V$.*

In order to prove Theorem 2, and to apply it to questions of counting definable sets, it will be useful to introduce the notion of the *orbit of a node* $a \in U^A$ *under the action of* $\mathsf{Aut}(A)$:

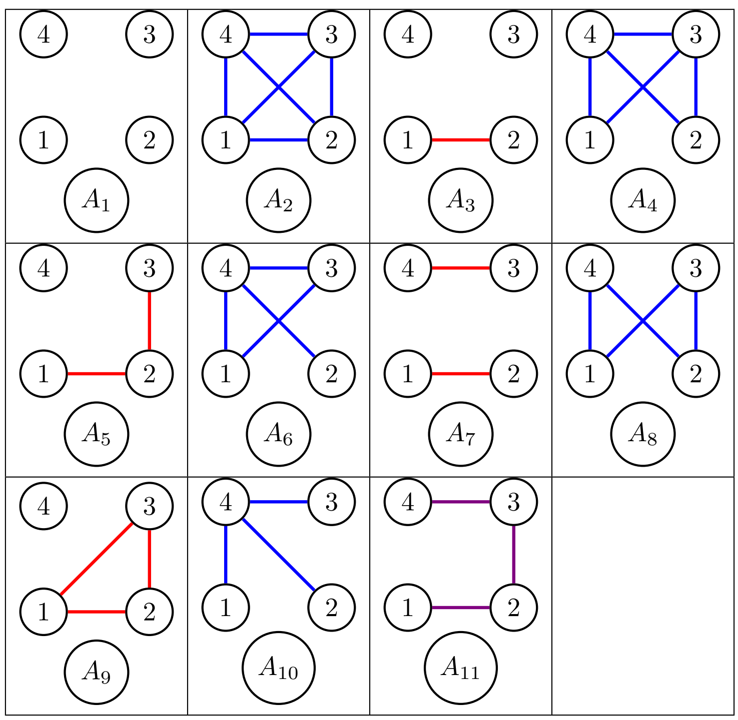
$$\mathsf{orb}(a, \mathsf{Aut}(A)) = \{h(a) \mid h \in \mathsf{Aut}(A)\}.$$

We define $\mathsf{Orbs}(A, \mathsf{Aut}(A)) = \{\mathsf{orb}(a, \mathsf{Aut}(A)) \mid a \in U^A\}$. As a corollary to Corollary 1 and Theorem 2 we have:

Corollary 2 *Let A be a finite graph and $V \subseteq U^A$. V is a definable subset of A if and only if either $V = \emptyset$ or there is a sequence of sets $O_1, \ldots, O_k$, where each $O_i \in \mathsf{Orbs}(A)$, and $V = O_1 \cup \ldots \cup O_k$.*

It follows at once from Corollary 2, that if A is a finite graph, then the number of definable subsets of A is $2^{|\mathsf{Orbs}(A, \mathsf{Aut}(A))|}$.

We proceeded to give a complete analysis of the definable subsets of simple graphs with four nodes. First, we classified all the members of $\mathsf{mod}(\mathsf{SG}, 4)$ up to isomorphism. We discovered that any maximal collection of pairwise non-isomorphic graphs in $\mathsf{mod}(\mathsf{SG}, 4)$ has exactly 11 members. We listed such a collection $A_1, \ldots, A_{11}$ and calculated $|\mathsf{orb}(A_i, \mathbb{S}_4)|$ and $|\mathsf{Aut}(A_i)|$ for each $1 \leq i \leq 11$. See the tables below. The *complement* A^c of a simple graph A is defined as follows: $U^{A^c} = U^A$; for $a \neq b$, $\langle a, b\rangle \in L^{A^c}$ if and only if $\langle a, b\rangle \notin L^A$. In the table of graphs below, each A_i with i odd, is drawn in red, and $A_{i+1} = A_i^c$ is drawn in blue. The exceptional graph A_{11} is drawn in purple since it is isomorphic to its own complement.



Note that $\mathsf{Aut}(A) = \mathsf{Aut}(A^c)$, for every simple graph A. This made it quick work to complete the following table.

A_i	$\vert\mathsf{orb}(A_i, \mathbb{S}_4)\vert$	$\vert\mathsf{Aut}(A_i)\vert$
A_1	1	24
A_2	1	24
A_3	6	4
A_4	6	4
A_5	12	2
A_6	12	2
A_7	3	8
A_8	3	8
A_9	4	6
A_{10}	4	6
A_{11}	12	2

Note the "verification" of the result predicted by the Orbit-Stabilizer Theorem: $|\mathsf{orb}(A_i, \mathbb{S}_4)| \cdot |\mathsf{Aut}(A_i)| = |\mathbb{S}_4| (= 24)$.

We introduced the notion of rigidity: a graph A is *rigid* if and only if $\mathsf{Aut}(A) = \{e\}$, that is , A has no non-trivial automorphisms. We noted that no member of $\mathsf{mod}(\mathsf{SG}, 4)$, is rigid, and mused about the question: "what is the least n such that $\mathsf{mod}(\mathsf{SG}, n)$ contains a rigid graph?"

16.1 Addendum

We began, but did not complete, a systematic account of which sets are definable in the structures $A_1, \ldots, A_{11}$. The following table, together with Corollary 2, suffices. We write $\mathsf{Orbs}(A, \mathsf{Aut}(A))$ to denote the collection of orbits of $\mathsf{Aut}(A)$ acting on U^A. We list only the odd numbered structures, since, as already observed, $\mathsf{Aut}(A) = \mathsf{Aut}(A^c)$.

A_i	$\mathsf{Orbs}(A_i, \mathsf{Aut}(A_i))$
A_1	$\{[4]\}$
A_3	$\{\{1, 2\}, \{3, 4\}\}$
A_5	$\{\{2\}, \{4\}, \{1, 3\}\}$
A_7	$\{[4]\}$
A_9	$\{\{1, 2, 3\}, \{4\}\}$
A_{11}	$\{\{1, 4\}, \{2, 3\}\}$