13 Lecture 03.02

On 03.02, we practiced counting the number of labelled simple graphs that satisfy various conditions that can be expressed by quantificational schemata. You asked wonderful questions that drove our investigation forward and highlighted many points that would otherwise have remained in the shadows. I may not be able to do justice to the full breadth of our conversation, but I'll summarize at least some of the main topics.

Since the set $\{1, \ldots, n\}$ occurs so often in our conversation, we decided to use the common abbreviation $[n]$ to denote it. We recalled that for a schema S we'd defined $\mathsf{mod}(S, n) = \{A \mid A \models S \text{ and } U^A = [n]\}$. Recall that a simple graph is 2-regualr if and only if it satisfies the schema:

- $\mathsf{2reg}$: $(\forall x)(\exists^{=2} y)Lxy$

Let S be the conjunction of $\mathsf{2reg}$ and SG. We calculated $|\mathsf{mod}(S, 6)|$. We began by reminding ourselves that if A is finite and $A \models S$ then A is a disjoint union of cycles. This led immediately to the observation that if $A \in \mathsf{mod}(S, 6)$ then A must consist of two disjoint triangles, or a single hexagon. So in order to complete our calculation, we just need to determine how many distinct ways we can label a structure of one or the other of these shapes. Suppose the unlabeled structure $\mathbb{T}$ consists of two triangles, call them the top triangle and the bottom triangle. We can label the top triangle with any set $X \subseteq [6]$ of size three, leaving $[6] - X$ to label the bottom triangle. At first blush, this suggests that there are $\binom{6}{3}$ distinct labelings of $\mathbb{T}$. But notice that we get the same labeled structure, if we use $[6] - X$ to label the top triangle, and X to label the bottom triangle, so there are $\binom{6}{3}/2 = 10$ distinct labelings of $\mathbb{T}$. Next, suppose the unlabeled structure $\mathbb{H}$ consists of a single hexagon. We used our prior calculation that there are 6! strict linear orders of [n] to calculate the number of distinct labelings of $\mathbb{H}$. For each such linear order, we can "wrap it around" the hexagon starting from a fixed position to arrive at a labeling. It is clear that the reverse of any order gives the same labeling as the order itself, as do each of the orders that arise by starting at the i-th position of the given order, for $i > 1$, and continuing on beyond the sixth position with the first $i - 1$ elements of the given order. Thus, the total number of labelings of $\mathbb{H}$ is $6!/(6 \cdot 2) = 60$. It follows that $|\mathsf{mod}(S, 6)| = 10 + 60 = 70$.

We next turned our attention to Problem 1 in Problem Set 5, since that offers considerable opportunity to practice counting. Here are the relevant definitions and part of the problem itself.

- Irr to abbreviate the schema $(\forall x)\neg Lxx$,
- Sym to abbreviate the schema $(\forall x)(\forall y)(Lxy \supset Lyx)$, and
- SG to abbreviate the conjunction of Irr and Sym. Structures that satisfy SG are called *simple graphs*.

- The *order* of a graph A (written $\mathsf{o}rd(A)$) is $|U^A|$. The *size* of a simple graph A (written $\mathsf{s}ize(A)$) is $|L^A|/2$. This corresponds to the number of "undirected edges" of A.
- If S is a schema, we write $\mathsf{mod}(S, n)$ for the set of structures A such that $A \models S$ and $U^A = \{1, \ldots, n\}$.
- Let K be a set of simple graphs. We call A a *size maximal* member of K if and only if $A \in K$ and for every $B \in K$, $\mathsf{s}ize(A) \geq \mathsf{s}ize(B)$.
- For $n \geq 2$, we let $\Delta_n(x_1, \ldots, x_n)$ abbreviate the schema:

$$x_1 \neq x_2 \wedge x_1 \neq x_3 \ldots \wedge x_{n-1} \neq x_n.$$

- For $n \geq 3$, we let C_n abbreviate the schema:

$$(\exists x_1) \ldots (\exists x_n)(\Delta_n(x_1, \ldots, x_n) \wedge Lx_1x_2 \wedge Lx_2x_3 \wedge \ldots L_{n-1}x_n \wedge Lx_nx_1).$$

Let S_1 be $\mathsf{SG} \wedge \neg C_3 \wedge \neg C_4 \wedge \neg C_5 \wedge \neg C_6$.

- How many structures are size maximal members of $\mathsf{mod}(S_1, 6)$?

First we noted that the schemata $\neg C_3, \neg C_4, \neg C_5, \neg C_6$ are satisfied by a structure if and only if it is triangle-free, square-free, pentagon-free, and hexagon-free, respectively, so their conjunction, $\neg C_3 \wedge \neg C_4 \wedge \neg C_5 \wedge \neg C_6$, is satisfied by a simple graph of order ≤ 6 if and only if it is acyclic, that is, contains no cycles. With this in mind, we listed all the unlabeled acyclic simple graphs of orders 2, 3, and 4, identified those which are size maximal, and counted their labelings with [2], [3], and [4], respectively. For order 2, this was easy: there is the empty graph and the single edge. Obviously, the single edge is size maximal, and equally obviously, it has exactly one labeling. Almost as easy is the case of order three. Here the empty graph, a single edge together with an isolated node, and a simple path consisting of two nodes of degree one and one of degree two, exhaust the unlabeled acyclic graphs of order three. The path is clearly the unique edge maximal one among them, since the edge counts are 0, 1, and 2. We counted the labelings of the path by noting that each such labeling is determined by a choice of label for the node of degree 2, and since there are three such choices (namely 1, 2, and 3) there are three labelings. In handling the order 4 case, we noted that a size maximal acyclic simple graph of order 4 has a size maximal acyclic simple graph of order 3 as a subgraph. So, in order to construct such graphs, we just need to add a node and some edges to the the path of order three, and determine which such additions are acyclic. After some thought, we arrived at the conclusion that there are two size maximal unlabeled acyclic simple graphs of order 4, each having exactly 3 edges, namely the path with four nodes and three edges (let's call this $\mathbb{P}$) and the three-pointed star, that is, the graph with one node of degree 3, and three nodes of degree 1 (and let's call this one $\mathbb{A}$). Reasoning as we did in the case of the path with three nodes and two edges, we arrived at the conclusion that there are four labelings of the star, one for each choice of label for the node of degree 3. When it came to counting labelings of the path, an interesting discussion of symmetries arose. We introduced the notions of isomorphism and automorphism to crystallize the relevant notion of symmetry.

- Let A and B be graphs and let f be a function with domain U^A and range U^B. f is an *isomorphism from A onto B* if and only if f is a bijection and for all $i, j \in U^A$, $\langle i, j\rangle \in L^A$ if and only if $\langle f(i), f(j)\rangle \in L^B$.

- Let A be a graph and let f be a function with domain U^A and range U^A. f is an *automorphism of A* if and only if f is an isomorphism of A onto A.

- Let A be a graph with $U^A = [n]$ and let f be a permutation of $[n]$, that is, a bijection of $[n]$ onto $[n]$. We define $f[A]$ to be the graph with universe $[n]$ such that for all $i, j \in [n]$, $\langle f(i), f(j)\rangle \in L^{f([A])}$ if and only if $\langle i, j\rangle \in L^A$.

It follows from the definition of $f[A]$ that f is an isomorphism of A onto $f[A]$, in particular, f is an automorphism of A if and only if $f[A] = A$.

Now, back to counting the labelings of $\mathbb{P}$. One direct way of doing this is to note that there are 4! labeled *directed paths* with three edges, and that each labeling of a simple path corresponds to two distinct labeled directed paths, depending on which end of the simple path we start from. Therefore, there are $4!/2 = 12$ distinct labelings of $\mathbb{P}$. Another way to count uses the symmetries of $\mathbb{P}$. Fix one labeling A of $\mathbb{P}$, say the one with an edge between 1 and 2, between 2 and 3, and between 3 and 4. For each of the 4! permutations f from [4] onto [4] (call the set of these permutations $\mathbb{S}_4$) note that exactly two of them are automorphisms of A: the identity permutation id which maps each i to itself, and the permutation g defined by

i	$g(i)$
1	4
2	3
3	2
4	1

which rotates the path about its center. As f ranges over $\mathbb{S}_4$, $f[A]$ ranges over all the labelings of $\mathbb{P}$. Since for each f, both the composition of f with id and the composition of f with g yield the same labeling, the 4! members of $\mathbb{S}_4$ yield only $4!/2$ distinct labelings. Whew! Our second counting method yields the same result as our first. Just to be sure, we checked that for a fixed labeling B of $\mathbb{A}$, six of the permutations in $\mathbb{S}_4$ are automorphisms of B (the ones that leave the node of degree three fixed and permute the remaining three nodes arbitrarily) and thus there are $4!/6$ distinct labeling of $\mathbb{A}$, again the same result as we achieved by our direct method of counting. Thus, we conclude in the end that there are 16 size maximal members of $\mathsf{mod}(S_1, 4)$.