12 Lecture 02.29

On 02.29, we counted the number of finite structures with universe of discourse $\{1, \ldots, n\}$ that satisfy various conditions. We'd already noted that there are 2^{n^2} graphs and $2^{\binom{n}{2}}$ simple graphs with universe of discourse $\{1, \ldots, n\}$. We began by showing that

- $|\mathsf{mod}(\mathsf{Fun}, n)| = n^n$;
- $|\mathsf{mod}((\mathsf{Fun} \wedge \mathsf{Inj}), n)| = n!$;
- $|\mathsf{mod}(\mathsf{Asy}, n)| = 3^{\binom{n}{2}}$;
- $|\mathsf{mod}(\mathsf{Tour}, n)| = 2^{\binom{n}{2}}$;
- $|\mathsf{mod}(\mathsf{SLO}, n)| = n!$;
- $|\mathsf{mod}(\mathsf{Bfun}, n)| = n^{n^2}$.

In each case, clear thinking and the product rule sufficed for the calculation.

Next, we discussed a more substantial counting problem that arises in applications of simple graphs to model resource allocations and various other phenomena. We call a simple graph A *bipartite* if and only if there is a partition X, Y of U^A such that for every $a, b \in U^A$, if $\langle a, b\rangle \in L^A$, then either $a \in X$ and $b \in Y$, or $a \in Y$ and $b \in X$. We say a simple graph B is a *subgraph* of A if and only if $U^B \subseteq U^A$ and for every $a, b \in U^B$, if $\langle a, b\rangle \in L^B$, then $\langle a, b\rangle \in L^A$. We write $B \subseteq A$ to abbreviate B is a subgraph of A. A 1-regular simple graph is called a *matching*. If $B \subseteq A$, $U^B = U^A$, and B is a matching, we say that B is a *perfect matching* of A. The $n-$*clique* K_n is the graph with universe of discourse $\{1, \ldots, n\}$ satisfying the schema $(\forall x)(\forall y)(x \neq y \supset Lxy)$. In an earlier lecture, we suggested the entertainment, "Count the number of perfect matchings contained in K_n, for every n." It is easy to see that the answer is 0, if n is odd, and $\neq 0$ for even n. We left this as a continuing source of amusement, and turned instead to counting the number of perfect matchings contained in a finite, 2-regular, bipartite simple graph A. We drew such a graph on the board with six nodes in each partition class and, for a reality check, confirmed that in every k-regular bipartite graph the number of nodes in the two partition classes is the same. Good, at least this necessary condition for the existence of a perfect matching obtains. We next recalled that every finite 2-regular graph is a union of disjoint cycles. We also noted that every cycle in a bipartite graph has an even number of edges. Then we observed that every even length cycle admits exactly two perfect matchings. Now we recognized we could answer the counting problem. Let $\gamma(A)$ be the number of cycles in a finite, 2-regular, bipartite simple graph A. Then the number of perfect matchings contained in A is $2^{\gamma(A)}$.