9 Lecture 02.17

On 02.17, we commenced our study of polyadic quantification theory. This topic will remain our focus through the end of the Term. As opposed to truth-functional and monadic logic which, as we've seen, are of limited expressive power, polyadic quantification theory allows for faithful schematization of vast tracts of scientific discourse. But we begin, not with science, but with literature.

Consider the sentences

- Romeo loves Juliet.
- Someone loves Juliet.
- Romeo loves someone.

The first sentence implies the second and the third sentence. We can schematize the second, by making use of the monadic predicate "◯ loves Juliet" thus

$$(\exists x)(x \text{ loves Juliet}).$$

And we can schematize the third, by making use of the monadic predicate "Romeo loves ◯" thus

$$(\exists x)(\text{Romeo loves } x).$$

But if we wish to schematize the sentence "someone loves someone," which is also implied by the first sentence above, we need to expand our resources to include *dyadic predicates*.

- $\boxed{1}$ loves $\boxed{2}$
- $\langle \text{Romeo}, \text{Juliet} \rangle$ is in the extension of "$\boxed{1}$ loves $\boxed{2}$."
- $(\exists x)(\exists y)(x \text{ loves } y)$

The extension of a dyadic predicate is a set of *ordered* pairs.

- $\langle 45, 47 \rangle$ is in the extension of "$\boxed{1} \leq \boxed{2}$."
- $\langle 45, 47 \rangle$ is not in the extension of "$\boxed{2} \leq \boxed{1}$."
- $\langle 47, 45 \rangle$ is in the extension of "$\boxed{2} \leq \boxed{1}$."

Similarly, the extension of a triadic predicate, such as "$\boxed{1}$ is further from $\boxed{2}$ than it is from $\boxed{3}$," is a set of ordered triples.

Consider the following statements involving alternation of quantifiers.

- Everyone loves someone (or other).

$$S_1 : \quad (\forall x)(\exists y)(x \text{ loves } y).$$

- There is someone whom everyone loves.

$$S_2 : \quad (\exists y)(\forall x)(x \text{ loves } y).$$

- Everyone is loved by someone.

$$S_3 : \quad (\forall y)(\exists x)(x \text{ loves } y).$$

- Someone loves everyone.

$$S_4 : \quad (\exists x)(\forall y)(x \text{ loves } y).$$

The second statement implies the first, and the fourth implies the third. We gave counterexamples to show that no other implications obtain. Consider the following three structures A, B, C.

Structure	Universe	Extension of L
A	$\{a, b\}$	$\{\langle a, a\rangle, \langle b, b\rangle\}$
B	$\{a, b\}$	$\{\langle b, b\rangle, \langle a, b\rangle\}$
C	$\{a, b\}$	$\{\langle b, b\rangle, \langle b, a\rangle\}$

Note that $A \models S_1$ and $A \models S_3$, while $A \not\models S_2$ and $A \not\models S_4$, from which it follows, by definition, that S_1 does not imply S_2, nor does S_3 imply S_4. Moreover $B \models S_2$, but $B \not\models S_3$, and $C \models S_4$, but $C \not\models S_1$; thus S_2 does not imply S_3, and S_4 does not imply S_1. Failure of the remaining (non-trivial) implications now follows. For example, S_1 does not imply S_4, for otherwise, since S_2 implies S_1, and S_4 implies S_3, it would follow that S_2 implies S_3, to which B is a counterexample. We summarize the results of this discussion in the following matrix $\langle a_{ij} \mid 1 \leq i, j \leq 4\rangle$, where $a_{ij} = 1$ if and only if the schema in the i-th row implies the schema in the j-th column.

S_i implies S_j	S_1	S_2	S_3	S_4
S_1	1	0	0	0
S_2	1	1	0	0
S_3	0	0	1	0
S_4	0	0	1	1

We proceeded to explore "scope ambiguities." Consider the statement, "everybody loves a lover." We observed that "x is a lover" can be schematized as $(\exists y)Lxy$, and corresponding to the two readings, "everybody loves someone who is a lover", and "if someone is a lover, then everybody loves her" we have the respective schematizations:

- $$(\forall z)(\exists x)((\exists y)Lxy \wedge Lzx), \text{ versus}$$

- $$(\forall x)((\exists y)Lxy \supset (\forall z)Lzx).$$

We observed that a structure A satisfies the second schema if and only if either L^A is empty or $L^A = U^A \times U^A$, the cartesian product of the universe of A with itself. On the other hand, if a structure B satisfies the first schema, then L^B is non-empty; moreover, if B consists of a pair of requiting lovers at least one of whom is not a narcissist, B satisfies the first, but not the second, schema. Thus, neither disambiguation of the original sentence implies the other.

We went on to discuss several important properties of relations.

- L^A is *reflexive* if and only if
$$A \models (\forall x)Lxx.$$

- L^A is *irreflexive* if and only if
$$A \models (\forall x)\neg Lxx.$$

- L^A is *symmetric* if and only if
$$A \models (\forall x)(\forall y)(Lxy \supset Lyx).$$

- L^A is *asymmetric* if and only if
$$A \models (\forall x)(\forall y)(Lxy \supset \neg Lyx).$$

- L^A is *transitive* if and only if
$$A \models (\forall x)(\forall y)(\forall z)(Lxy \supset (Lyz \supset Lxz)).$$

- A is a *simple graph* if and only if L^A is irreflexive and symmetric.