8 Lecture 02.10

We concluded the proof of the Small Model Theorem. Again, we focused on the case of schemata involving only the monadic predicate letters F and G. We drew pictures, in "Types View", of 15 structures $A_1, \ldots, A_{15}$ each with universe of discourse included in $\{1, 2, 3, 4\}$ and no two with the same universe of discourse. Recall the element types:

- $T_1(x) : Fx \wedge Gx$
- $T_2(x) : Fx \wedge \neg Gx$
- $T_3(x) : \neg Fx \wedge Gx$
- $T_4(x) : \neg Fx \wedge \neg Gx$

We constructed the structures A_i by letting j realize the type $T_j(x)$ for each $j \in U^{A_i}$. Let A be an arbitrary structure. It is clear from our construction that there is an i such that A_i realizes exactly the same types as A. Moreover, since A_i has exactly one element realizing any type that it realizes, A_i is a surjective homomorphic image of A. It follows at once from the result of our last class that A is monadically similar to A_i, that is, they satisfy exactly the same set of pure monadic schemata. Having thus concluded the proof, we turned to deriving a list of useful corollaries.

Corollary 1 *1. For every schema S, if S is satisfiable, then there is an $1 \leq i \leq 15$ such that $A_i \models S$.*

2. There is an algorithmic decision procedure to determine whether a schema S is satisfiable.

3. Schema S implies schema T if and only if

$$\{i \mid A_i \models S \text{ and } 1 \leq i \leq 15\} \subseteq \{i \mid A_i \models T \text{ and } 1 \leq i \leq 15\}.$$

4. Schemata S and T are equivalent if and only if

$$\{i \mid A_i \models S \text{ and } 1 \leq i \leq 15\} = \{i \mid A_i \models T \text{ and } 1 \leq i \leq 15\}.$$

With these results in hand, we proceeded to analyze the expressive power of monadic schemata. Recall the notions deployed in Problem Set 2, but now upgraded to apply to monadic schemata.

- A list of pure monadic schemata is *succinct* if and only if no two schemata on the list are equivalent.
- A pure monadic schema *implies a list of schemata* if and only if it implies every schema on the list.
- The *power* of a pure monadic schema is the length of a longest succinct list of pure monadic schemata it implies.

We continued to focus on the vocabulary consisting of the monadic predicate letters F and G and answered the following questions.

Question 1 *For which numbers n is there a schema S whose power is n?*

Answer: It follows from Corollary 1, parts (3) and (4), that the power of a schema S is determined by the size j of $\{i \mid A_i \models S \text{ and } 1 \leq i \leq 15\}$, in particular, the power of S is 2^{15-j}; for pure schemata S, j may be any number between 0 and 15. This answers Question 1.

Question 2 *What is the length of a longest succinct list of pure schemata satisfied by exactly 4 structures with universe of discourse $\{1, 2, 3, 4\}$?*

Answer: Let $\mathbb{V} = \{A \mid U^A = \{1, 2, 3, 4\}\}$. Recall that $A \approx_M B$ if and only if for all pure monadic schemata S, $A \models S$ if and only $B \models S$. For $A \in \mathbb{V}$, let $\hat{A} = \{B \in \mathbb{V} \mid B \approx_M A\}$. In order to answer the question, it suffices to determine the size of $\hat{A}$ for each $A \in \mathbb{V}$. First, note that the size of $\hat{A}$ is determined by the number of types realized by A. We computed these sizes:

- If A realizes exactly 1 type, then the size of $\hat{A}$ is 1. There are $\binom{4}{1}$ structures in $\mathbb{V}$ satisfying exactly 1 type.
- If A realizes exactly 2 types, then the size of $\hat{A}$ is $2^4 - 2$. There are $\binom{4}{2}$ structures in $\mathbb{V}$ satisfying exactly 2 types.
- If A realizes exactly 3 types, then the size of $\hat{A}$ is $\binom{4}{2} \cdot 3!$. There are $\binom{4}{3}$ structures in $\mathbb{V}$ satisfying exactly 3 types.
- If A realizes exactly 4 types, then the size of $\hat{A}$ is $4!$. There are $\binom{4}{4}$ structures in $\mathbb{V}$ satisfying exactly 4 types.

It is now easy to see that the answer to Question 2 is 1; in particular, one such list consists of the single schema

$$(\forall x)(Fx \land Gx) \lor (\forall x)(Fx \land \neg Gx) \lor (\forall x)(\neg Fx \land Gx) \lor (\forall x)(\neg Fx \land \neg Gx).$$