

## 7 Lecture 02.08

The central occupation of this week's classes will be an approach to establishing the decidability of satisfiability of pure monadic schemata complementary to that developed in sections 25 and 26 of *Deductive Logic*. Our approach introduces notions that we will elaborate further, when we turn to study polyadic quantificational logic.

As a warm-up to the main event, we noted that we now have three (equivalent) ways of viewing structures, each of which may contribute a useful perspective, depending on the problem to hand. These are

- the Canonical View, which consists of specifying the universe of discourse and extensions for each of the (finitely many) predicate letters in play,
- the Types View, which consists of specifying a universe of discourse and sorting it into types, that is, maximally specific descriptions that can be framed in terms of the predicate letters in play, and
- the Venn View, which pictures the extensions of all the predicate letters in play as intersecting regions contained in a rectangle that represents the universe of discourse.

We will prove the following *Small Model Theorem* for monadic logic; the decidability of satisfiability of pure monadic schemata is a corollary to this result.

**Theorem 1** *Let  $S$  be a pure monadic schema containing occurrences of at most  $n$  distinct monadic predicate letters. If  $S$  is satisfiable then there is a structure  $A$  of size at most  $2^n$  such that  $A \models S$ .*

The proof of Theorem 1 rests on the following lemma. In order to state the lemma, we need to introduce some new concepts. Suppose without loss of generality that we restrict our attention to monadic schemata in which only the predicate letters  $F$  and  $G$  occur. We say that two structures  $A$  and  $B$  are *monadically similar* if and only if they satisfy exactly the same pure monadic schemata. We explore a sufficient condition for the monadic similarity of structures.

A function  $h$  is a mapping from one set, called the *domain* of  $h$  to another set (it may be the same set), called the *range* of  $h$ . For every element  $a$  of the domain of  $h$  we write " $h(a)$ " to denote the element of the range of  $h$  to which it is mapped. We sometimes call  $h(a)$  the  *$h$  image* of  $a$  or the *image* of  $a$  under  $h$ . We sometimes use the notation

$$h : X \longrightarrow Y$$

to indicate that  $h$  is a function with domain  $X$  and range  $Y$ . If  $h : X \longrightarrow Y$  we say that  $h$  is *onto* if and only if for every  $b \in Y$  there is an  $a \in X$  such that  $h(a) = b$ . In this case, we will also say that  $h$  is *surjective*.

Let  $A$  and  $B$  be structures. We call  $h$  a *homomorphism from  $A$  onto  $B$*  just in case  $h$  is an onto function with domain  $U^A$  and range  $U^B$  satisfying the following condition: for every monadic predicate letter  $P$  and every  $m \in U^A$ ,

$$m \in P^A \quad \text{if and only if} \quad h(m) \in P^B.$$

If there is a homomorphism from  $A$  onto  $B$ , we say that  $B$  is a *surjective homomorphic image* of  $A$ . The next lemma provides a useful sufficient condition for monadic similarity.

**Lemma 1** *Let  $A$  and  $B$  be structures. If there is a homomorphism from  $A$  onto  $B$ , then  $A$  is monadically similar to  $B$ .*

*Proof:* Let  $A$  and  $B$  be structures and suppose that  $h$  is a homomorphism of  $A$  onto  $B$ . It suffices to show that for every simple monadic schema  $S$ ,

$$A \models S \quad \text{if and only if} \quad B \models S,$$

since every pure monadic schema is a truth-functional compound of simple monadic schemata.

We begin by observing that for every  $c \in U^A$  and every one variable open schema  $S$ ,  $A$  makes  $S$  true with respect to the assignment of  $c$  to “ $x$ ,” if and only if  $B$  makes  $S$  true with respect to the assignment of  $h(c)$  to “ $x$ .” This follows immediately from the fact that  $h$  is a homomorphism.

Consider the simple schema  $S$  and suppose that  $S$  is the existential quantification of the one variable open schema  $T$  (the case of universal quantification is treated similarly). Suppose  $A \models S$ . Then, for some  $c \in U^A$ ,  $A$  makes  $T$  true with respect to the assignment of  $c$  to “ $x$ .” It follows that  $B$  makes  $T$  true with respect to the assignment of  $h(c)$  to “ $x$ .” Hence,  $B \models S$ .

Conversely, suppose  $B \models S$ . Then, for some  $c \in U^B$ ,  $B$  makes  $T$  true with respect to the assignment of  $c$  to “ $x$ .” Since  $h$  is surjective, there is a  $d \in U^A$  with  $h(d) = c$ . It follows at once that  $A$  makes  $T$  true with respect to the assignment of  $d$  to “ $x$ .” Hence,  $A \models S$ . ■

We recall our discussion of element types:

- $T_1(x) : Fx \wedge Gx$
- $T_2(x) : Fx \wedge \neg Gx$
- $T_3(x) : \neg Fx \wedge Gx$
- $T_4(x) : \neg Fx \wedge \neg Gx$

We say that a structure *realizes* a given type  $T_i$  just in case it makes the existential simple schema  $(\exists x)T_i$  true.

**Example 1** *The following structure realizes all four of the types listed above.*

$$A : U^A = \{1, 2, 3, 4\}, F^A = \{1, 3\}, G^A = \{1, 2\}$$

*Moreover, the 14 proper substructures of  $A$  realize exactly the fourteen proper nonempty subsets of the types listed above.*

Lemma 1 yields a useful necessary and sufficient condition for monadic similarity.

**Lemma 2** *A and B realize the same types if and only if they are monadically similar.*

*Proof:* If  $A$  and  $B$  realize the same types, then there is a single structure  $C$  which is a surjective homomorphic image of both  $A$  and  $B$ . Therefore, by our earlier result,  $A$  is monadically similar to  $C$  and  $B$  is monadically similar to  $C$ . It follows at once that  $A$  is monadically similar to  $B$ . The reverse implication follows immediately from the fact realization of a type is expressed by a pure monadic schema. ■

Theorem 1 is an immediate corollary to Lemma 2.

*Proof*(of Theorem 1): It follows at once from Lemma 2 and Example 1, that there is a collection  $X$  of 15 structures each of size  $\leq 4$  such that for any pure monadic schema  $S$  involving only the predicate letters “ $F$ ” and “ $G$ ,” if  $S$  is satisfiable, then there is a structure  $A \in X$  such that  $A \models S$ . More generally, there is a collection  $X$  of  $2^{(2^n)} - 1$  structures each of size  $\leq 2^n$  such that for any pure monadic schema  $S$  involving only the predicate letters “ $F_1$ ,”  $\dots$  “ $F_n$ ,” if  $S$  is satisfiable, then there is a structure  $A \in X$  such that  $A \models S$ . ■

**Corollary 1** *There is a decision procedure to determine whether a pure monadic schema is satisfiable.*