6 Lecture 02.03

On 02.03, we continued our study of monadic quantification theory. Consider again the example given at the end of the last class (at least as reported in the memoir of that class).

- x is an even number $\wedge$ x is a prime number
- $(\exists x)(x$ is an even number $\wedge$ x is a prime number$)$

As noted, the first of these sentences is not simply true or false, it is true or false with respect to an assignment to the variable "x"; we say in this instance that the occurrences of the variable "x" are *free* in this sentence. On the other hand, the occurrences of the variable x are *bound* by the existential quantifier in the second sentence; this sentence is true or false independent of any assignment to the variable x. Note that a variable may have both free and bound occurrences within a single sentence:

- $(\exists x)(x$ is an even number$)$ $\wedge$ $(x$ is a prime number$)$;

and may have occurrences bound by distinct quantifiers:

- $(\exists x)(x$ is an even number$)$ $\wedge$ $(\exists x)(x$ is a prime number$)$.

Next we consider the use of the universal quantifier. We can render the statement

- all numbers are even or odd

as

- $(\forall x)$ $[(x$ is an even number$)$ or $(x$ is an odd number$)]$.

The last statement is true, just in case whatever integer is assigned to the variable x satisfies the open statement within the square brackets. Here we see the contextual determination of a *universe of discourse* – when we say "all numbers" in this context, we intend that the variable of quantification range over all integers and not, for example, all complex numbers.

As we did in the case of truth-functional logic, we will introduce a schematic language for monadic quantificational logic. We specify the following categories of monadic schemata.

- A *one variable open schema* is a truth functional compound of expressions such as
  $Fx, Gx, Hx, \ldots$.
- A *simple monadic schema* is the existential or universal quantification of a one variable open schema with variable of quantification x.
- A *pure monadic schema* is a truth functional compound of simple monadic schemata.

We introduce *structures* as interpretations of monadic schemata. These play the role that truth-assignments played in the context of truth-functional logic. In order to specify a structure A for a schema S we need to

- specify a nonempty set U^A, the universe of A;
- specify sets $F^A, G^A, \ldots$ each of which is a subset of U^A as the extensions of the monadic predicate letters which occur in S;
- specify an element $a \in U^A$ to assign to the variable x, if x occurs free in S.

When the variable x has no free occurrences in the schema S, we write $A \models S$ as shorthand for "the schema S is true in the structure A," alternatively "the structure A satisfies the schema S." Otherwise, we write $A \models S[a]$ as shorthand for "the structure A satisfies the schema S relative to the assignment of a to the variable x."

We extend the notions of validity, satisfiability, implication, and equivalence to monadic quantificational schemata.

- A monadic schema S is *valid* if and only if for every structure $A, A \models S$.
- A monadic schema S is *satisiable* if and only if for some structure $A, A \models S$.
- A monadic schema S *implies* a monadic schema T if and only if for every structure A, if $A \models S$, then $A \models T$.
- Monadic schemata S and T are equivalent if and only if S implies T, and T implies S.

We discussed how to count the number of structures with a fixed universe of discourse that satisfy a given schema. We asked, how many structures with universe of discourse $U = \{1, 2, 3, 4, 5, 6\}$ interpreting the monadic predicate letters F and G satisfy the schema

$$S : \quad (\forall x)(Fx \supset Gx).$$

We observed that a structure A satisfies S if and only if $F^A \subseteq G^A$. So we need to determine the number, call it n, of pairs of subsets Y, Z of U with $Y \subseteq Z$. By using what we learned earlier about binomial coefficients, we see that

$$n = \sum_{i=0}^{i=6} \binom{6}{i} 2^i = \sum_{i=0}^{i=6} \binom{6}{i} 2^i \cdot 1^{6-i} = (2+1)^6 = 3^6.$$

The next to last equality is justified by the celebrated *Binomial Theorem.* For those of us with no taste for binomial coefficients, we will discuss a much simpler and direct combinatorial argument for the conclusion that $n = 3^6$.

Consider the following four one variable open schemata; we will call them (element) types.

- $T_1(x) : Fx \wedge Gx$
- $T_2(x) : Fx \wedge \neg Gx$
- $T_3(x) : \neg Fx \wedge Gx$
- $T_4(x) : \neg Fx \wedge \neg Gx$

Note that a structure A satisfies the schema S if and only if it contains no element satisfying the type T_2. Since a structure is determined by the type of each of its elements, there are as many structures with universe U satisfying S as there are ways of sorting the members of U into the three remaining types. For each of the six members of U, there are three types into which it could be sorted, so by the product rule, the number of structures satisfying S is 3^6.

If R and R^* are monadic schemata we say that a structure A is a *counterexample* to the claim that R implies R^* if and only if $A \models R$ and $A \not\models R^*$. We continued with the preceding example and counted the number of counterexamples to the claim that the schema S implies the schema

$$T : \qquad (\forall x)(Gx \supset Fx).$$

Again, we suppose that our structures have universe of discourse U and interpret exactly the monadic predicate letters F and G. If a structure A satisfies both S and T, then $F^A = G^A$. Hence, of the 3^6 structures satisfying S, the number that also satisfy T is 2^6, that is, the number of subsets of U, assigned within a single structure to both F and G. So the number of counterexamples to the claim that S implies T is $3^6 - 2^6$.