5 Lecture 02.01

On 02.01, we concluded our study of truth-functional logic. We observed that the finitary character of the semantics for truth-functional logic immediately yields an algorithm to decide the satisfiability of schemata of truth-functional logic. In particular, suppose $S \in \mathbb{S}(X)$ for some finite set of sentence letters X. Note first that for each truth-assignment $A \in \mathbb{A}(X)$ there is a simple and efficient algorithm, call it M, to determine whether $A \models S$. Thus, in order to test the satisfiability of S, we need only list $\mathbb{A}(X)$ in some canonical order $A_1, \ldots, A_{2^{|X|}}$ and use M to test whether the successive A_i satisfy S. Of course, this algorithm is not efficient, in the sense that it's running time is potentially exponential in the length of its input. The question whether there is an efficient algorithm to decide the satisfiability of truth-functional schemata is generally regarded as one of the most significant open mathematical problems of our time – for further information visit http://www.claymath.org/millennium-problems/p-vs-np-problem.

We then initiated our study of monadic quantification theory. Statements have significant logical form beyond the structure that can be exhibited in terms of truth-functional compounding. For example, the conjunction of the first two statements below implies, but does not truth-functionally imply, the third.

- All collies are mortal.
- Lassie is a collie.
- Lassie is mortal.

In order to analyze this example, we considered the following statements.

- Lassie is a collie.
- Scout is a collie.
- Rin-Tin-Tin is a collie.

These statements share the *monadic predicate* "◯ is a collie." Monadic predicates, unlike statements, are not true or false; rather, they are *true of* some objects and *false of* other objects. For example, "◯ is a prime number" is true of 2,3,5 and 7, and false of all even numbers greater than 2.

The *extension* of a monadic predicate is the collection of objects of which it is true. The extension of the monadic predicate "◯ is an even number" is the set $\{2, 4, 6, \ldots\}$. The extension of the monadic predicate "◯ is an even prime number" is the set $\{2\}$. The extension of the monadic predicate "◯ is an even prime number greater than 2" is the empty set. Distinct monadic predicates may have the same extension. For example, the extension of the predicate "◯ is a warm–blooded reptile" is also the empty set as is the extension of the predicate "◯ is a collie weighing more than 300 kilograms." We say that monadic predicates with the same extension are *coextensive*. We will focus on statements whose truth depends only on the extensions of the monadic predicates which occur in them. We call such sentential contexts in which interchange of coextensive predicates preserves truth–value *extensional*. Our focus on extensional contexts is the natural continuation of our earlier focus on truth–functional contexts.

Consider again the argument above. Intuitively, the validity of this argument does not depend on the particular name "Lassie" being used; it would be equally valid with any name in place of "Lassie." This generality may be brought out by the use of variables in place of particular names. We will form new expressions called *open sentences* by putting variables "$x, y, z, \dots$" for the placeholders in monadic predicates. Open sentences are not statements. They are true or false with respect to assignments of values to the variables they contain. For example, the open sentence "x is an even number" is true with respect to the assignment of 16 to "x" and false with respect to the assignment of 17 to "x" and false with respect to the assignment of Lassie to "x."

We may form compounds of open sentences using truth–functional connectives. For example, the following open sentences are truth–functionally complex.

- If x is divisible by six, then x is divisible by three.
- x is a collie and it is not the case that x weighs more than 300 kg.

We may use our prior understanding of the truth–functional connectives to determine the truth–values of such open sentences with respect to particular assignments of values to their variables.

We proceeded to introduce the existential quantifier. Consider the statement, "there is an even prime number." We render this statement as the application of the existential quantifier to the open sentence,

- "x is an even number $\wedge$ x is a prime number," thus
- $(\exists x)(x$ is an even number $\wedge$ x is a prime number$)$.

This last sentence is true just in case there is an assignment of some object to the variable x with respect to which the preceding open sentence is true.