4 Lecture 01.27

On 01.27, we listed the numbers which are powers of truth-functional schemata over $X = \{p, q, r\}$.

- First note that for every $S, S' \in \mathbb{S}(X)$ the power of S = the power of S' if and only if $|\mathbb{P}_X(S)| = |\mathbb{P}_X(S')|$, where we use $|U|$ to denote the number of members of the finite set U.
- In particular, if $\mathfrak{P} = \mathbb{P}_X(S)$, then the power of $S = 2^{(8-|\mathfrak{P}|)}$.
- It follows at once that for each $S \in \mathbb{S}(X)$, the power of $S = 2^i$, for some $0 \leq i \leq 8$.

We then considered the question, "what is the length of the longest succinct list of schemata with power 32?" If we apply what we've just learned, we see that a schema has power 32 if and only if exactly three truth assignments satisfy it. Hence the length of the longest such succinct list is exactly the number of subsets of size three contained in a set of size eight.

This led to an interlude on permutations and combinations: how many ways can we select k members of a set of size n? There is an ambiguity here: are we counting modes of selection, which involve the order of choices, or collections of members selected, where the order of selection is irrelevant. Once we recognize the ambiguity, we can proceed to count both. We introduced notation for each: $(n)_k$ for the number of ordered sequences of k distinct elements that can be drawn from a set of size n and $\binom{n}{k}$ for the number of subsets of size k that are included in set of size n. To evaluate $(n)_k$ we argued as follows. Suppose we think of counting the ways n students could fill a row of length k in a lecture hall. Let's suppose the seats are labelled $1, 2, \ldots, k$. There are n choices for the student to fill seat 1; once that seat is filled, there are $n-1$ choices for the student to fill seat 2; and so on until there are $(n-k)+1$ choices for the student to fill seat k. Hence, by the product rule, there are $n \cdot (n-1) \cdots ((n-k)+1)$ ways of filling all k seats, that is, $(n)_k = n \cdot (n-1) \cdots ((n-k)+1)$. Now that we have counted the number of ordered sequences, we can see how to count the number of subsets. By the same reasoning, each subset of size k appears as the content of $k \cdot (k-1) \cdots 2 \cdot 1$ ordered sequences of length k; this number is called k *factorial* and is often abbreviated as $k!$. Hence,

$$\binom{n}{k} = \frac{(n)_k}{k!}.$$

Observe that

$$(n)_k = \frac{n!}{(n-k)!}$$

from which it follows that

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

This last formulation makes transparent a symmetry in the values of $\binom{n}{k}$, namely, for every k between 0 and n, $\binom{n}{k} = \binom{n}{n-k}$. This accords nicely with the observation that complementation induces a one-one correspondence between the subsets of size k and the subsets of size $(n-k)$ that can be selected from a set of size n. Note also that it determines in a non-arbitrary way that the value of 0! is 1.

Let's not forget how this all began. The length of the longest succinct list of schemata with power 32 is $\binom{8}{3} = 56$.