3 Lecture 01.25

We explored the expressive power of truth-functional logic. (In lecture, we illustrated the proof of Theorem 1 below with an example built over the set of sentence letters $X = \{p, q, r\}$; here, we will give a general treatment, for comparison, and for completeness.) At the end of the last lecture, we suggested using the notion of the proposition expressed by a schema as an intuitive vehicle for pursuing this investigation. Since the semantical correlate of a truth-functional schema is a set of truth assignments to some finite set of sentence letters, we can frame the question of the *expressive completeness of truth-functional logic* in terms of propositions. Let X be a non-empty finite set of sentence letters. We deploy the notation: $\mathbb{A}(X)$ for the set of truth assignments to the sentence letters X, and and $\mathbb{S}(X)$ for the set of truth-functional schemata compounded from sentence letters all of which are members of X. If $\mathfrak{P} \subseteq \mathbb{A}(X)$, we call $\mathfrak{P}$ a *proposition over* X. We will establish

Theorem 1 (Expressive Completeness of Truth-functional Logic) *Let X be a non-empty finite set of sentence letters and let $\mathfrak{P}$ be a proposition over X. There is a schema $S \in \mathbb{S}(X)$ such that $\mathbb{P}_X(S) = \mathfrak{P}$.*

For the proof of Theorem 1, the following terminology and lemma will be useful.

Definition 1 *Let X be a non-empty finite set of sentence letters and let $S \in \mathbb{S}_X$.*

- *S is a* literal *over X just in case $S = p$ or $S = \neg p$, for some $p \in X$.*
- *S is a* term *over X just in case S is a conjunction of literals over X (we allow conjunctions of length 1).*
- *S is in disjunctive normal form over X if and only if S is a disjunction of terms over X (we allow disjunctions of length 1).*

If Λ is a set of literals over X we write $\bigwedge \Lambda$ to abbreviate a term which is formed as a conjunction of the literals in Λ. Similarly, if Γ is a set of terms over X we write $\bigvee \Gamma$ to abbreviate a schema in disjunctive normal form which is formed as a disjunction of the terms in Γ.

Lemma 1 *Let X be a non-empty finite set of sentence letters. For every $A \in \mathbb{A}(X)$ there is a schema T_A which is a term over X such that for every $A' \in \mathbb{A}(X)$*

$$A' \models T_A \quad \textit{if and only if} \quad A' = A.$$

Proof: Let X be a finite set of sentence letters and suppose $A \in \mathbb{A}(X)$. For each $p \in X$, let $l_p = p$, if $A \models p$, and let $l_p = \neg p$, if $A \not\models p$. Let $\Lambda = \{l_p \mid p \in X\}$ and let $T_A = \bigwedge \Lambda$. It is easy to verify that for every $A' \in \mathbb{A}(X)$, $A' \models T_A$ if and only if $A' = A$. ■

Proof of Theorem 1: Fix a finite non-empty set of sentence letters X and suppose $\mathfrak{P}$ is a proposition over X. If $\mathfrak{P} = \emptyset$, then pick $p \in X$ and note that

$\mathbb{P}_X(p \wedge \neg p) = \mathfrak{P}$. Otherwise, for each $A \in \mathfrak{P}$, choose a term T_A, as guaranteed to exist by Lemma 1, such that for every $A' \in \mathbb{A}(X)$, $A' \models T_A$ if and only if $A' = A$. Let $\Gamma = \{T_A \mid A \in \mathfrak{P}\}$ and let $S = \bigvee \Gamma$. It is easy to verify that $\mathbb{P}_X(S) = \mathfrak{P}$. ∎

Corollary 1 *Every truth-functional schema is equivalent to a schema in disjunctive normal form.*

Problem Set 2 introduces the following useful terminology. All schemata are drawn from $\mathbb{S}(X)$ for a fixed non-empty finite set of sentence letters X.

- A list of truth-functional schemata is *succinct* if and only if no two schemata on the list are equivalent.
- A truth-functional schema *implies a list of schemata* if and only if it implies every schema on the list.
- The *power* of a truth-functional schema is the length of a longest succinct list of schemata it implies.

For concreteness, we considered $X = \{p, q, r\}$. What is the length of a longest succinct list of truth-functional schemata over X? We arrived at the answer by proving an *upper bound* and a *lower bound* on this length.

- Upper bound: It is easy to verify that schemata S and S' are equivalent if and only if $\mathbb{P}(S) = \mathbb{P}(S')$. Hence, the length of a succinct list of schemata cannot exceed the number of propositions over X, that is, the number of subsets of the set $\mathbb{A}(X)$. The size of X is 3, so the size of $\mathbb{A}(X)$ is 2^3, since determining a truth assignment to X involves three binary choices. By the same reasoning, the number of propositions over X is 2^{2^3}, since determining a proposition involves deciding, for each of the 2^3 truth assignments, whether to include or omit it. Hence, the length of the longest succinct list is no more than 256.
- Lower bound: By Theorem 1, for every proposition over X, there is a schema expressing it. Since schemata expressing distinct propositions are not equivalent, it follows at once that there is a succinct list of schemata of length 256.

We proceeded to compute the power, as defined above, of an exemplary schema; let's do $p \wedge (q \vee r)$ here. Note that a schema S implies a schema S' if and only if $\mathbb{P}(S) \subseteq \mathbb{P}(S')$. Thus, the power of S is the number of sets Z satisfying the condition:

$$\mathbb{P}(S) \subseteq Z \subseteq \mathbb{A}(X). \tag{1}$$

The size of $\mathfrak{P} = \mathbb{P}(p \wedge (q \vee r))$ is 3, so the size of $\mathbb{A}(X) - \mathfrak{P} = 5$. It follows at once that $2^5 = 32$ sets Z satisfy condition (1); hence, the power of $p \wedge (q \vee r)$ is 32. We ended by posing two questions. "What is the power of $p \wedge \neg p$?" "What is the power of $p \vee \neg p$?"