2 Lecture 01.20

Today, we began our systematic treatment of truth-functional logic. You should have read sections 1-9 of *Deductive Logic* to prepare for class.

Per our conversation last time, we will use sentence letters to schematize statements, that is, sentences (of natural language) which are true or false. We will study ways of forming compound statements from simpler statements; insofar as we will restrict our study to the formation of compound statements whose truth value depends only on the truth values of the simpler statements out of which they are composed, we will be able to interpret these schemata via truth assignments to sentence letters, and retain full access to their logical powers thereby. Thus the term, *truth-functional* logic.

Consider again schematizing the statements "i loves j", $1 \leq i, j, \leq 4$, using the sentence letters p_{ij}; for example, the sentence letter p_{11} schematizes the statement "1 loves 1", or briefly, "1 is a narcissist." Suppose we wish to write down truth-functional schemata using these sentence letters, thus interpreted, that are true just in case

1. all of 1, 2, 3, and 4 are narcissists;
2. none of 1, 2, 3, and 4 are narcissists;
3. at least one of 1, 2, 3, and 4 is a narcissist;
4. an odd number of 1, 2, 3, and 4 are narcissists.

In order to do so, we introduce the following truth-functional connectives. For each connective, we display its truth-functional interpretation via a table indicating the truth value of the compound schema as a function of the truth values of its components.

- Conjunction:

p	q	$p \wedge q$
$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\bot$
$\bot$	$\bot$	$\bot$

- Negation:

p	$\neg p$
$\top$	$\bot$
$\bot$	$\top$

- Inclusive Disjuntion

p	q	$p \vee q$
$\top$	$\top$	$\top$
$\top$	$\bot$	$\top$
$\bot$	$\top$	$\top$
$\bot$	$\bot$	$\bot$

- Exclusive Disjunction

p	q	$p \oplus q$
$\top$	$\top$	$\bot$
$\top$	$\bot$	$\top$
$\bot$	$\top$	$\top$
$\bot$	$\bot$	$\bot$

We can now schematize conditions 1 – 4 above as follows.

S1: $((p_{11} \wedge p_{22}) \wedge p_{33}) \wedge p_{44}$

S2: $((\neg p_{11} \wedge \neg p_{22}) \wedge \neg p_{33}) \wedge \neg p_{44}$

S3: $((p_{11} \vee p_{22}) \vee p_{33}) \vee p_{44}$

S4: $((p_{11} \oplus p_{22}) \oplus p_{33}) \oplus p_{44}$

The first three are quite straightforward to verify; the fourth requires some explanation. It will be useful to introduce some terminology to facilitate the explanation, which is given by Proposition 1 below.

Let X be a set of sentence letters. A truth assignment A for X is a mapping which associates with each sentence letter $q \in X$ one of the two truth values $\top$ or $\bot$; we write $A(q)$ for the value that A associates to q. Suppose S is a truth-functional schema such that every sentence letter with an occurrence in S is a member of X. We say a truth assignment A for X satisfies such a schema S ($A \models S$) if and only if S receives the value $\top$ relative to the truth assignment A.

Proposition 1 *For every $n \geq 2$ and every set $X = \{q_1, \ldots, q_n\}$ of n distinct sentence letters, a truth assignment A for X satisfies the schema*

$$S_n : (\ldots(q_1 \oplus q_2)\ldots \oplus q_n)$$

if and only if A assigns an odd number of the sentence letters in X the value $\top$.

Proof: We proved the proposition by induction on n.

- Basis: Examination of the truth table for $\oplus$ suffices to establish the proposition for the case $n = 2$.

- Induction Step: Suppose the proposition holds for a number $k \geq 2$, that is, for every truth assignment A for $\{q_1, \ldots, q_k\}$, $A \models S_k$ if and only if A assigns an odd number of the sentence letters in $\{q_1, \ldots, q_k\}$ the value $\top$; this is our induction hypothesis. We proceed to show that the proposition also holds for $k + 1$. Let A' be an assignment to the sentence letters $\{q_1, \ldots, q_{k+1}\}$ and let A be its restriction to $\{q_1, \ldots, q_k\}$. We consider two cases. First, suppose that $A'(q_{k+1}) = \top$. In this case, $A' \models S_{k+1}$ if and only if $A \not\models S_k$ if and only if (by our induction hypothesis) A assigns an even number of the sentence letters $\{q_1, \ldots, q_k\}$ the value $\top$. Hence, if $A'(q_{k+1}) = \top$, then $A' \models S_{k+1}$ if and only if A' assigns an odd number of the sentence letters in $\{q_1, \ldots, q_{k+1}\}$ the value $\top$. On the other hand, suppose that $A'(q_{k+1}) = \bot$. In this case, $A' \models S_{k+1}$ if and only if $A \models S_k$ if and only if (by our induction hypothesis) A assigns an odd number of the sentence letters $\{q_1, \ldots, q_k\}$ the value $\top$. Hence, if $A'(q_{k+1}) = \bot$, then $A' \models S_{k+1}$ if and only if A' assigns an odd number of the sentence letters in $\{q_1, \ldots, q_{k+1}\}$ the value $\top$. This concludes the proof, since either $A'(q_{k+1}) = \top$ or $A'(q_{k+1}) = \bot$. ■

We returned to our potential lovers and restricted attention to just two of them, 1 and 2. We asked how we could express the statement that all love is requited among these two. The natural mode of expression is: if 1 loves 2, then 2 loves 1, and if 2 loves 1, then 1 loves 2. In order to render this directly, we introduced the

- Material Conditional

p	q	$p \supset q$
$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\top$
$\bot$	$\bot$	$\top$

Now, using the sentence letter $p_{11}, p_{12}, p_{21}, p_{22}$ as earlier interpreted, we can express the happy state that all love among 1 and 2 is requited by the schema

$$R : (p_{12} \supset p_{21}) \wedge (p_{21} \supset p_{12}).$$

We asked in how many of the possible love scenarios among 1 and 2 is all love requited, and we computed that the answer is eight out of a total of sixteen such scenarios, by determining how many truth assignments to the sentence letters $p_{11}, p_{12}, p_{21}, p_{22}$ satisfy the schema R.

We emphasized that the satisfaction relation is the fundamental semantic relation, it is where language and the world meet; in the case to hand, language consists of truth-functional schemata and the possible worlds they describe are truth assignments to sentence letters. As the course progresses, we will encounter more textured representations of the world (relational structures) and richer languages to describe them (monadic and polyadic quantification theory). We now define some of the central notions of truth-functional logic in terms of satisfaction. These definitions will generalize directly to the more textured structures and richer languages we encounter later. For the following definitions, we suppose that S and T are truth-functional schemata and that A ranges over truth assignments to sets of sentence letters which include all those that occur in either S or T.

- S implies T if and only if for every truth assignment A, if $A \models S$, then $A \models T$.

- S is equivalent to T if and only if S implies T and T implies S.

- S is satisfiable if and only if for some A, $A \models S$.
- S is valid if and only if every truth assignment satisfies S.

We noted various equivalences, for example,

- $p \oplus q$ is equivalent to $q \oplus p$ (commutativity of exclusive disjunction)
- $(p \oplus q) \oplus r$ is equivalent to $p \oplus (q \oplus r)$ (associativity of exclusive disjunction).

We noted that both conjunction and inclusive disjunction are also commutative and associative, whereas the material conditional is neither. We encouraged the audience to think of examples of (binary) truth-functional connectives which are commutative but not associative, and associative but not commutative.

We introduced one further connective $\equiv$, the material biconditional. We specified its truth-functional interpretation by indicating that $p \equiv q$ is truth-functionally equivalent to both $(p \supset q) \wedge (q \supset p)$ and $\neg(p \oplus q)$.

Finally, we suggested that, as a heuristic, it is sometimes useful to think of a schema S as expressing a proposition, to whit, the set of truth assignments A that satisfy S; of course, this needs to be relativized to a collection of sentence letters X which includes all those occurring in S. We suggested the notation: $\mathbb{P}_X(S) = \{A \mid A \text{ is a truth assignment for } X \text{ and } A \models S\}$. When we use this notation without the subscript X, we assume A is a truth assignment for exactly the set of sentence letters with occurrences in S.