1 Lecture 01.13

We began with the question: Is there a numerically diverse group of Philadelphians? (We call a group of people numerically diverse if no two people in the group have the same number of friends in the group - we assume groups are of size at least two and that friendship is always mutual.) We demonstrated that the answer is no by an application of

1 *The Pigeonhole Principle: If you distribute m pigeons into n pigeonholes and $m \geq n+1$, then some hole contains at least two pigeons.*

We argued as follows. Suppose we have a group $G = \{1, \ldots, n\}$ of n people (we use numerals to name the people for privacy concerns). For brevity, let's write p_{ij} to signify that i is a friend of j. We assume friendship is symmetric, that is, if p_{ij}, then p_{ji}, for all $i, j \in G$, and irreflexive, that is, it is not the case that p_{ii}, for all $i \in G$. Let's write $f(i)$ for the number of friends of i, that is, the number of j such that p_{ji}. Since friendship is irreflexive, the possible values of f are the n numbers $0, 1, \ldots, n-1$. We are thinking of these values as the pigeonholes for application of the principle 1 and the members of G as being placed in these holes by f. We want to argue that the value of f must agree on at least two members of G. But so far, since we have n members of G and n pigeonholes into which they are sorted by f, we may not yet draw that conclusion via principle 1. But now we consider the question, "can f really take all the values from 0 to $n-1$?" In particular, can it take on both the value 0 and the value $n-1$. We argue that the answer is no. Suppose that there is some i with $f(i) = 0$, that is, for every j, it is not the case that p_{ji}. Then, by symmetry, for every j, it is not the case that p_{ij}. So, if i has no friends, then the maximum number of friends of any j is $n-2$, that is, f cannot take on the value $n-1$. Thus, the possible values of f are the $n-1$ numbers $0, \ldots, n-2$. But now, by principle 1, we can conclude that f takes on the same value for at least two members of G. This concludes our argument that there cannot be a numerically diverse group of Philadelphians. We proceeded to show that this argument presupposes that there are finitely many Philadelphians, by presenting an example of a numerically diverse infinite group.

Say a group of people has three-mutuality if it contains either a group of three mutual friends or a group of three mutual strangers. How large a group of people can lack three-mutuality? We showed that the largest such group has five members, that is, there is a pattern of friendship among a group of five people (a "friendship pentagon") that lacks three-mutuality, whereas every pattern of friendship among six or more people has three-mutuality. For the latter, we argued as follows, by application of a more sophisticated version of principle 1

2 *The Mean Pigeonhole Principle: If you distribute m pigeons into n pigeonholes and $m \geq k \cdot n + 1$, then some hole contains at least $k+1$ pigeons.*

(Note that 1 is just the special case of 2 for $k = 1$.) Let $G = \{1, \ldots, 6\}$ and sort the five people $2, \ldots, 6$ into two pigeonholes according to the truth value, true ($\top$) or false ($\perp$) of $p_{12}, \ldots, p_{16}$, again using p_{ij} to signify i is a friend of j. By principle 2, one of these holes, suppose it's the $\top$ one, contains at least three members of G. Now, either two of these are friends, in which case they, together with 1 form a collection of three mutual friends, or none of them of friends, in which case they themselves form a collection of three mutual strangers. The argument is analogous, in the case that three members of G were sorted into the $\perp$ pigeonhole. We discussed the problem of establishing that for every natural number n there is a k such that every group of size at least k has n-mutuality.

We mentioned that the course will explore relationships and that love differs from friendship in that there are narcissists (so we can't assume the relation is irreflexive) and is not always requited (so we can't assume the relationship is symmetric). How many different patterns of love might obtain among a group of four people, call them $1, 2, 3, 4$. Now, we decided to recycle the sentence letters and use p_{ij} to signify the statement that i loves j; we noted that 16 sentence letters would be required to record all the relevant statements. Since each pattern of love among $1, 2, 3, 4$ is determined by assigning one of the truth values $\top$ or $\perp$ to each of these 16 sentence letters, we concluded that the number of such patterns is 2^{16}. Why? Because there are two assignments to p_{11} and for each of these, there are two assignments to p_{12}, and thus $2 \cdot 2 = 2^2$ assignments to them jointly (this observation is given the exalted title, "The Product Rule"). Thus, by iterating application of the product rule another fourteen times, we arrive at the conclusion that there are 2^{16} possible truth assignments to the 16 sentence letters. We marveled at the fact that there are as many as 65,536 different potential love-scenarios at a table for four.