Presentations, Invariance, and Definability

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Gedanken ohne Inhalt sind leer,
Anschauungen ohne Begriffe
sind blind.

Immanuel Kant

The medium is the message.

Marshall McLuhan

Motivation

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- But this identification is hardly universal throughout mathematics and computer science.
- For example, a graph is often presented as a drawing of nodes in the plane together with arcs joining them to indicate the edges, or as an adjacency matrix whose axes are labelled by the vertices in some arbitrary order and whose entries indicate the presence or absence of an edge.

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- From our point of view, the significant feature of presentations is not concreteness or perceptual accessibility, but rather the additional information they carry about an abstract structure and the availability of that information to the conceptual apparatus of a logical language.
- We introduce the notion of presentation-invariant definability to compare the information revealed by various modes of presentation.

Structures and Presentations

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- In this talk, almost all the presentations we discuss will be given by orderings of the nodes.
- But orderings by no means exhaust the class of presentations that are of interest from the point of view of computation and cognition.

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- A presentation scheme $\mathbb{P}$ for $\mathcal{G}$ is *elementary* (alternatively, an *elementary presentation* of $\mathcal{G}$) if and only if for some first order sentence θ and for all $G = \langle V, E\rangle \in \mathcal{G}$ and all binary relations P on V, $\langle V, E, P\rangle \in \mathbb{P}$ if and only if $\langle V, E, P\rangle \models \theta$.

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- Evidently, if $\mathbb{P}$ is elementary, then $\mathbb{P}$ is logical.

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- $\mathbb{L}$ is the class of all finite graphs equipped with arbitrary linear orders of their nodes.
- Let $G = \langle V, E\rangle$ be a finite connected graph. P is a *traversal* of G if and only if P is a linear order of V and for every a other than the P-least element of V, there is a b P-less-than a such that $\langle b, a\rangle \in E$. A traversal of a not necessarily connected graph G is an ordered sum of traversals of its connected components.

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- $\mathbb{T}$ is the class of all finite graphs equipped with arbitrary traversals of their nodes.
- Note that $\mathbb{U}$, $\mathbb{L}$ and $\mathbb{T}$ are elementary presentations of the class of finite graphs.

Structures and Presentations

$\mathbb{T}$ is elementary

$\langle V, E, P\rangle$ is a traversal of $G = \langle V, E\rangle$ just in case it satisfies the conjunction of the following elementary conditions:

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- P is a strict linear ordering of V (for which we write $<$).
- For all $a, b, c \in V$, if $a < b < c$ and Eac and $\neg Eab$, then $(\exists d < b)Edb$.

Presentation Invariance

Definition

Let $\mathcal{G}$ be a class of graphs, let $\mathbb{P}$ be a presentation scheme for $\mathcal{G}$, and let $\mathfrak{Q}$ be a boolean query (in the signature of $\mathbb{P}$).

- We say that $\mathfrak{Q}$ is $\mathbb{P}$-*invariant* if and only if for all $\mathrm{G} \in \mathcal{G}$ and all presentations H and H' of G in $\mathbb{P}$, $\mathrm{H} \in \mathfrak{Q}$ if and only if $\mathrm{H}' \in \mathfrak{Q}$. We say that a sentence θ of a logical language is $\mathbb{P}$-*invariant* if and only if it defines a $\mathbb{P}$-invariant boolean query.

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The Data Independence Principle

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- Yet there may be computing devices, natural or artificial, with pure forms of inner intuition other than our own.
- And query processing may exploit information beyond mere order when it is available.
- All of which provides potential scientific and technological motivation for the study of invariance with respect to presentation schemes other than $\mathbb{L}$.

Invariant Definability

Definition

Let $\mathcal{G}$ be a class of graphs, let $\mathcal{K}$ be a subclass of $\mathcal{G}$, and let $\mathbb{P}$ and $\mathbb{Q}$ be presentation schemes for $\mathcal{G}$.

- We say that $\mathcal{K}$ is $\mathbb{P}$-*invariantly elementary over* $\mathcal{G}$ if and only if there is a $\mathbb{P}$-invariant first order sentence φ such that for all $\mathrm{G} \in \mathcal{G}$, $\mathrm{G} \in \mathcal{K}$ if and only if for some presentation $\mathrm{H} \in \mathbb{P}$ of G, $\mathrm{H} \models \varphi$.

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- We write $\mathbb{P}(\mathcal{G})$ for the collection of $\mathcal{K} \subseteq \mathcal{G}$ which are $\mathbb{P}$-invariantly elementary over $\mathcal{G}$. We say presentation scheme $\mathbb{P}$ *is at least as revealing as* $\mathbb{Q}$ on $\mathcal{G}$ if and only if $\mathbb{Q}(\mathcal{G}) \subseteq \mathbb{P}(\mathcal{G})$, and we say that $\mathbb{P}$ and $\mathbb{Q}$ are $\mathcal{G}$-*equivalent* if and only if $\mathbb{P}(\mathcal{G}) = \mathbb{Q}(\mathcal{G})$.

Comparing Presentation Schemes over all Finite Graphs

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- G contains a cycle if and only for each traversal of G there is a vertex with two neighbors prior to it in the traversal.
- G is connected if and only if for each traversal of G, every vertex other than the first has a neighbor prior to it in the traversal.

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- Bipartiteness is $\mathbb{T}^2$-invariantly elementary over the class of all finite graphs.

Comparing Presentation Schemes over all Finite Graphs

Bipartiteness is not $\mathbb{T}$-invariantly elementary

- Let $<$ be the usual linear order on the natural numbers and define $H_n = \langle \{1, \ldots, n\}, E_n \rangle$ where $\langle i, j \rangle \in E_n$ if and only if $i = j + 1$ or $j = i + 1$ or ($i = 1$ and $j = n$) or ($j = 1$ and $i = n$).

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- Note that
  - $\langle H_n, < \rangle$ is a traversal and
  - H_n is bipartite if and only if n is even.
- It follows at once that bipartiteness is not $\mathbb{T}$-invariantly elementary, for otherwise, the set of even length linear orders would be elementary over the class of finite linear orders.

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- Recall that a graph is not bipartite if and only if it has an odd cycle.
- And recall that a connected graph has an odd cycle if and only if its square is connected.
- Therefore G is bipartite if and only if every component of G gives rise to more than one component in G^2. But this is an elementary property of a traversal of G^2.

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- Thus, $\mathbb{T}^2 = \mathbb{T}^{\mathrm{I}}$, where for all $\mathrm{G} \in \mathcal{G}$, $\mathrm{I}(\mathrm{G}) = \mathrm{G}^2$.
- In the context of database management systems, an interpretation amounts to a "view." Thus, "presentation via interpretation" encapsulates the notion of presenting a view of the data.

Presentation Invariance via Intepretations

Definition

Let $\mathcal{C}$ be a class of structures and let $\mathbb{P}^{\mathrm{I}}$ be a presentation scheme for $\mathcal{C}$ via I.

- We say that a boolean query $\mathfrak{Q} \subseteq \mathrm{I}[\mathcal{C}]$ is $\mathbb{P}^{\mathrm{I}}$-*invariant* if and only if for all $A \in \mathcal{C}$ and all $\mathbb{P}^{\mathrm{I}}$-presentations B and B' of A, $\mathrm{B} \in \mathfrak{Q}$ if and only if $\mathrm{B}' \in \mathfrak{Q}$.

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- We say that a sentence θ of a logical language is $\mathbb{P}^{\mathrm{I}}$-*invariant* if and only if it defines a $\mathbb{P}^{\mathrm{I}}$-invariant boolean query.

Elementary Invariant Definability via Interpretations

Definition

Let $\mathcal{C}$ be a class of finite structures and let $\mathfrak{Q} \subseteq \mathcal{C}$ be a boolean query.

- We say that $\mathfrak{Q}$ is $\mathbb{P}^{\mathsf{E}}$-*invariantly elementary over* $\mathcal{C}$ if and only if there is a first-order definable interpretation I and $\mathbb{P}^{\mathrm{I}}$-invariant first order sentence φ such that for all $A \in \mathcal{C}$, $A \in \mathfrak{Q}$ if and only if for some $\mathbb{P}^{\mathrm{I}}$-presentation B of A, $\mathrm{B} \models \varphi$.

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- We write $\mathbb{P}^{\mathsf{E}}(\mathcal{C})$ for the collection of boolean queries $\mathfrak{Q} \subseteq \mathcal{C}$ which are $\mathbb{P}^{\mathsf{E}}$-invariantly elementary over $\mathcal{C}$ and $\mathbb{P}^{\mathsf{E}}$ for the the collection of boolean queries which are $\mathbb{P}^{\mathsf{E}}$-invariantly elementary over the class of all finite structures (of a fixed relational signature).

A Characterization of Logspace

We say a finite structure $\mathcal{A}$ is *situated* if and only if its universe is an initial segment of the natural numbers and its signature includes the distinguished relation symbol BIT with $\langle i, j\rangle \in \mathsf{BIT}^{\mathcal{A}}$ just in case the i^{th} bit of j is 1. We write L for the collection of boolean queries which are computable in logarithmic space.

Theorem

Let $\mathfrak{Q}$ be a boolean query on the class of situated finite structures. The following are equivalent.

- $\mathfrak{Q} \in \mathsf{L}$.

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- $\mathfrak{Q} \in \mathbb{T}^{\mathsf{E}}$.

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Proof Sketch I

- We first show that for every query $\mathfrak{Q}$ on situated finite graphs, if $\mathfrak{Q} \in \mathbb{T}^{\mathsf{E}}$, then $\mathfrak{Q} \in \mathsf{L}$.

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- It is well known that $\mathsf{FO} \subseteq \mathsf{L}$.

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- It is well known that $\mathsf{FO} \subseteq \mathsf{L}$.
- Hence, it suffices to show that for every simple graph G a traversal of G may be computed in logarithmic space in the size of G.

A Characterization of Logspace

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- Reingold established that there is a "universal traversal sequence" (UTC) which for every connected simple graph G, yields a walk covering all the nodes of G, computable in logarithmic space in the size of G.
- Given a simple graph G (not necessarily connected), we show how to use such a universal traversal sequence to construct in logarithmic space, a traversal of G.

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- As a result, we may compute, in logspace, a preorder of the connected components of G by the $<$ order of their $<$-least members.
- Within each connected component C, we order nodes by their first occurrence in a UTC of C beginning at the $<$-least member of C.
- It is easy to see that the resulting total order is a traversal of G and that the logspace computability of the UTC implies that this total order is logspace computable.

A Characterization of Logspace

Proof Sketch III

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- We use the result of Lewis and Papadimitriou that $\mathsf{L} \subseteq \mathsf{SL}$, and show that $\mathsf{SL} \subseteq \mathbb{T}^{\mathsf{E}}$. ($\mathsf{SL}$ is symmetric logspace.)

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- We use the result of Lewis and Papadimitriou that $\mathsf{L} \subseteq \mathsf{SL}$, and show that $\mathsf{SL} \subseteq \mathbb{T}^{\mathsf{E}}$. (SL is symmetric logspace.)
- Let $\mathfrak{Q}$ be a boolean query on situated structures of signature σ and suppose M is a Turing machine witnessing the membership of $\mathfrak{Q}$ in SL.

A Characterization of Logspace

Proof Sketch IV

- With the help of BIT, we construct a formula $\tau_M(\overline{x}, \overline{y})$ so that for finite structures A of signature σ, $\tau_M[A]$ is the transition graph of M with input A, which is, by hypothesis, a simple graph.

A Characterization of Logspace

Proof Sketch IV

- With the help of BIT, we construct a formula $\tau_M(\overline{x}, \overline{y})$ so that for finite structures A of signature σ, $\tau_M[A]$ is the transition graph of M with input A, which is, by hypothesis, a simple graph.
- Now, M accepts A if and only if there is a path in $\tau_M[A]$ from the start state of M to a final state.

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- With the help of BIT, we construct a formula $\tau_M(\overline{x}, \overline{y})$ so that for finite structures A of signature σ, $\tau_M[A]$ is the transition graph of M with input A, which is, by hypothesis, a simple graph.
- Now, M accepts A if and only if there is a path in $\tau_M[A]$ from the start state of M to a final state.
- But reachability for undirected graphs is traversal-invariant first-order definable. Therefore, $\mathfrak{Q} \in \mathbb{T}^{\mathsf{E}}$. $\square$

Partial-order presentations

- We introduce a new combinatorial parameter which naturally generalizes the notion of vertex separation number from linear layouts of graphs to layouts which are tree-like partial orders.

Partial-order presentations

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- We begin by defining a modest generalization of the notion of a normal spanning tree.

Normal presentations

Definition

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- Let $\mathbb{N}$ be the presentation scheme consisting of normal presentations of finite simple graphs.
- Observe that $\mathbb{N}$ is an elementary presentation scheme. We next exploit normal presentations to introduce the concept of node separation number.

Partial-order presentations

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- $\nu(G, H)$ (the node separation number of G with respect to H) is $\max(\{|B_a| \mid a \in V\}) - 1$.

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- $\nu(G, H)$ (the node separation number of G with respect to H) is $\max(\{|B_a| \mid a \in V\}) - 1$.
- $\nu(G)$ (the node separation number of G) is the minimum over all normal presentations H of G of $\nu(G, H)$.

Partial-order presentations

Node separation number and tree-width

- It is easy to verify that if $H = \langle G, \leq \rangle$ is a normal presentation of G and T is a tree on node set V with $\leq_T = \leq$, then $\langle T, \{B_a \mid a \in V\} \rangle$ is a tree decomposition of G.

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- It follows at once that $\nu(G) \geq \text{tw}(G)$ (the tree-width of G).
- On the other hand, a straightforward generalization of an argument due to Nancy Kinnersley, which establishes the identity of vertex separation number and path-width, yields the reverse inequality $\nu(G) \leq \text{tw}(G)$.

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- The proof yields a particularly simple normal form for minimum width tree decompositions as follows.

Node separation number and tree-width

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Theorem

Suppose $G = \langle V, E\rangle$ *has tree-width* k*. Then there is tree* T *with node set* V *and root* $r \in V$ *such that* $H = \langle G, \leq_T^r\rangle$ *is a normal presentation of* G*, and* $\langle T, \{B_a \mid a \in V\}\rangle$ *is a tree decomposition of* G *of width* $k + 1$*.*

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For every finite simple graph G, $\nu(G) = \mathrm{tw}(G)$.

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Definition

Let $\mathcal{S}_k$ be the set of graphs G with $\nu(G) \leq k$ and let
$\mathbb{S}_k = \{H \in \mathbb{N} \mid \nu(G, H) \leq k\}$.
Observe that for every k, $\mathbb{S}_k$ is an elementary presentation scheme.

Partial-order presentations

Computing normal presentations

Theorem

For every for every k *there is a linear time algorithm* α_k *and a logspace algorithm* β_k *such that for every* $G \in \mathcal{S}_k$*,* $\alpha_k(G)$ *and* $\beta_k(G)$ *are normal presentations of* G*, and* $\alpha_k(G), \beta_k(G) \in \mathbb{S}_k$*.*

Node separation number and tree-width: Proof sketches

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Claim

It is easy to verify that if $H = \langle G, \leq \rangle$ *is a normal presentation of* $G = \langle V, E \rangle$ *and* T *is a tree on node set* V *with* $\leq_T = \leq$*, then* $\langle T, \{B_a \mid a \in V\} \rangle$ *is a tree decomposition of* G*.*

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- It follows immediately from the definition of B_a, that $a \in B_a$ for every $a \in V$;
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- So each node and each edge of G is covered by a bag.
- It only remains to show that for every $a \in V$, the set $C_a = \{b \in V \mid a \in B_b\}$ is connected in T.
- But note that a is the $\leq_T$ minimal element of C_a and for every $b \in C_a$ and $a \leq_T c \leq_T b$, $c \in C_a$. $\square$

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- Let $\langle T^*, \{B^*_t \mid t \in T^*\}\rangle$ be a tree decomposition of a simple graph $G = \langle V, E\rangle$.

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- Root the tree T^* at an arbitrarily chosen node r with $B^*_r \neq \emptyset$; this induces a tree-like partial-order $\leq^*$ of T^*.
- For each $a \in V$, let $m(a)$ be the $\leq^*$-least $t \in T^*$ with $a \in B^*_t$. Note that $m(a)$ is well defined since the set $\{t \in T^* \mid a \in B^*_t\}$ is connected in T^*.

Node separation number and tree-width: Proof sketches

- Let $\leq$ be a tree-like partial-order of V satisfying the condition that for all $a, b \in V$, if $m(a) \neq m(b)$, then $a \leq b$ if and only if $m(a) \leq^* m(b)$, and for all $a \in V$, the restriction of $\leq$ to $\{b \mid m(b) = m(a)\}$ is a linear ordering. We show:

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(1) $H = \langle G, \leq \rangle$ is a normal presentation and

(2) the tree decomposition $\langle T, \{B_a \mid a \in V\} \rangle$ induced by H has width no greater than that of $\langle T^*, \{B^*_t \mid t \in T^*\} \rangle$; in particular, for all $a \in V$, $B_a \subseteq B^*_{m(a)}$.

Node separation number and tree-width: Proof sketches

(1): We show that $\leq$ is a normal partial order of G. Let $a, b \in V$ with Eab. If $m(a) = m(b)$, then a and b are $\leq$-comparable, by definition.

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Since, $\leq^*$ is tree-like, this implies that a and b are comparable with respect to $\leq^*$ and hence, by definition, with respect to $\leq$ as well. $\square$

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Let t be $\leq^*$-least with $b, c \in B^*_t$. It follows that $m(b) \leq^* m(a) \leq^* t$.

Hence, $b \in B^*_{m(a)}$, since the set $\{t \in T^* \mid b \in B^*_t\}$ is connected in T^*. $\square$