1 Lecture 01.11

We began with the question: Is there a numerically diverse group of Philadelphians? (We call a group of people numerically diverse if no two people in the group have the same number of friends in the group - we assume groups are of size at least two and that friendship is always mutual.) We demonstrated that the answer is no by an application of

Example 1 *The Pigeonhole Principle: If you distribute m pigeons into n pigeonholes and $m \geq n+1$, then some hole contains at least two pigeons.*

We argued as follows. Suppose we have a group $G = \{1, \ldots, n\}$ of n people (we use numerals to name the people for privacy concerns). For brevity, let's write p_{ij} to signify that i is a friend of j. We assume friendship is symmetric, that is, if p_{ij}, then p_{ji}, for all $i, j \in G$, and irreflexive, that is, it is not the case that p_{ii}, for all $i \in G$. Let's write $f(i)$ for the number of friends of i, that is, the number of j such that p_{ji}. Since friendship is irreflexive, the possible values of f are the n numbers $0, 1, \ldots, n-1$. We are thinking of these values as the pigeonholes for application of the principle 1 and the members of G as being placed in these holes by f. We want to argue that the value of f must agree on at least two members of G. But so far, since we have n members of G and n pigeonholes into which they are sorted by f, we may not yet draw that conclusion via principle 1. But now we consider the question, "can f really take all the values from 0 to $n-1$?" In particular, can it take on both the value 0 and the value $n-1$. We argue that the answer is no. Suppose that there is some i with $f(i) = 0$, that is, for every j, it is not the case that p_{ji}. Then, by symmetry, for every j, it is not the case that p_{ij}. So, if i has no friends, then the maximum number of friends of any j is $n-2$, that is, f cannot take on the value $n-1$. Thus, the possible values of f are the $n-1$ numbers $0, \ldots, n-2$. But now, by principle 1, we can conclude that f takes on the same value for at least two members of G. This concludes our argument that there cannot be a numerically diverse group of Philadelphians.

We mentioned that the course will explore relationships and that love differs from friendship in that there are narcissists (so we can't assume the relation is irreflexive) and is not always requited (so we can't assume the relationship is symmetric). We observed that this difference between friendship and love allows the existence of numerically diverse groups of lovers, that is, groups where each person in the group loves a different number of people in the group. Consider, for example, a group of four people, call them $1, 2, 3, 4$, and suppose that 1 doesn't love anyone, 2 loves 1, 3 loves both 1 and 2, and 4 loves all of $1, 2$, and 3, and that this exhausts all the love among our group of four. We achieve numerical diversity at the sacrifice of requital.

How many different patterns of love might obtain among a group of four people, again call them $1, 2, 3, 4$. Now, we decided to recycle the sentence letters and use p_{ij} to signify the statement that i loves j; we noted that 16 sentence letters would be required to record all the relevant statements. Since each pattern of love among $1, 2, 3, 4$ is determined by assigning one of the truth values

$\top$ or $\perp$ to each of these 16 sentence letters, we concluded that the number of such patterns is 2^{16}. Why? Because there are two assignments to p_{11} and for each of these, there are two assignments to p_{12}, and thus $2 \cdot 2 = 2^2$ assignments to them jointly (this observation is given the exalted title, "The Product Rule"). Thus, by iterating application of the product rule another fourteen times, we arrive at the conclusion that there are 2^{16} possible truth assignments to the 16 sentence letters. We marveled at the fact that there are as many as 65,536 different potential love-scenarios at a table for four.

On the other hand, we considered how tame friendship is as compared with love, in terms of the number of possible friendship-scenarios. In virtue of the fact that friendship is symmetric and irreflexive, a friendship-scenario is determined by assigning one of the truth values $\top$ or $\perp$ to each of the 6 sentence letters p_{ij}, for $1 \leq i < j \leq 4$. Hence, there are only $2^6 = 64$ possible patterns of friendship among the group of four, less than 1/1000 of the number of potential love-scenarios.