Logic and Probability, EPFL, Fall 2011
Lecture Notes 2
Val Tannen

6 Warm-Up on 0-1 Properties

Consider graphs constructed randomly on the set of vertices $\{1, \ldots, n\}$. At first, we assume that each (unordered) pair of vertices is connected by an edge with probability 1/2. It follows that the underlying (finite) probability space consists of $2^{\binom{n}{2}}$ equiprobable graphs. We denote by G_n a random graph on n vertices.

A *triangle* in such a graph is a clique of size three, i.e., three distinct nodes each two of which are connected by an edge. What is the probability that a random graph on n vertices contains at least one triangle? Because the possible worlds are equiprobable $\Pr(G_n \text{ has triangles})$ can be computed by counting all graphs that have at least one triangle and dividing by $2^{\binom{n}{2}}$. This counting may be a hard combinatorial task but it turns out that we can easily obtain some interesting information about these probabilities:

Proposition 6.1 $\lim_{n\to\infty} \Pr(G_n \text{ has triangles}) = 1$

Proof For each $1 \le i \ne j \ne k \ne i \le n$ let T_{ijk} denote the event that in the random graph the vertices i, j, k form a triangle, obviously an event of probability 1/8. Using the monotonicity of probability:

$$\Pr(G_n \text{ has a triangle}) = \Pr(\bigcup_{i,j,k} T_{ijk}) \ge \Pr(T_{123} \cup T_{456} \cup \cdots)$$

The events $T_{123}, T_{456}, \ldots$ are not disjoint but they are independent. Hence

$$\Pr(T_{123} \cup T_{456} \cup \cdots) = \left(1 - \left(1 - \frac{1}{8}\right)^{\lfloor n/3 \rfloor}\right)$$

The proposition follows by taking limits. $\square$

The terminology for this case, when the limit is 1, is that "almost all graphs have triangles" or "there is almost surely a triangle". Let us now show that "almost no graphs have isolated nodes" or "there is almost never an isolated node".

Proposition 6.2 $\lim_{n\to\infty} \Pr(G_n \text{ has isolated nodes}) = 0$

Proof For each $1 \le i \le n$ denote by I_i the event that node i is isolated. There are $2^{\binom{n-1}{2}}$ graphs in which i is isolated hence

$$\Pr(I_i) = \frac{2^{\binom{n-1}{2}}}{2^{\binom{n}{2}}}$$

Now

$$\Pr(G_n \text{ has isolated nodes}) = \Pr(I_1 \cup \cdots \cup I_n) \le \Pr(I_1) + \cdots + \Pr(I_n) = \frac{n 2^{\binom{n-1}{2}}}{2^{\binom{n}{2}}} = \frac{n}{2^{n-1}}$$

Again the proposition follows by taking limits. $\square$

Now, for any graph property P let us introduce the notation

$$\mu(P) \;=\; \lim_{n\to\infty} \Pr(G_n \text{ has } P)$$

provided, of course, that the limit exists. When the property P is expressible in FOL this limit behaves quite remarkably as the following interesting theorem due to Glebskii, Kogan, Liogonki, and Talanov (1969), and independently to Fagin (1976) states.

Theorem 6.3 (0-1 Law for FOL) *For any FO sentence φ, $\mu(\varphi)$ exists and is either 0 or 1.*

Although we provided separate proofs, the results about "has triangle" and "has isolated node" fall under the purview of this more general theorem because they can be expressed in FOL. Indeed, we can use for example a binary predicate $E(i, j)$ that holds exactly when the undirected edge between i and j is in the graph (hence $E(j, i)$ holds also). Then

$$\text{has triangles}: \quad \exists x, y, z\ x \neq y \neq z \neq x \wedge E(x, y) \wedge E(y, z) \wedge E(z, x)$$

$$\text{has isolated nodes}: \quad \exists x.\forall y\ \neg E(x, y)$$

On the other hand, it is easy to see that for the "parity" property $\mu(\text{has even number of edges}) = 1/2$ so Theorem 6.3 can be used to conclude that parity cannot be expressed in FOL or in any logic that admits a 0-1 law.

Note also that the limit $\mu(\text{has even number of vertices})$ does not even exist.

Finally, we shall see that $\mu(\text{connectivity}) = 1$ and $\mu(\text{acyclicity}) = 0$ even though these properties are not in FOL. Indeed it turns out out that 0-1 laws still hold for logics that are stronger than FOL (in particular logics with "fixpoints").

Another issue is that Theorem 6.3 tells us that $\mu(\text{ has triangles})$ exists and is either 0 or 1 but it does not tell us *which*! Same for $\mu(\text{ has isolated nodes})$. Can it be determined from the FO sentence that expresses the property whether the limit is 0 or 1?

7 Generalized Random Graphs

It turns out also that it is worthwhile considering random graphs where the probability that each edge occurs is not necessarily 1/2, but any constant probability in $[0, 1]$ and it may even be a function of the number n of nodes of the graph, for example, $p(n) = \frac{1}{n}$, $p(n) = \frac{1}{n^2}$, $p(n) = \frac{\ln n}{n}$.

Definition 7.1 *Denote by $G[n, p(n)]$ the random graph with nodes $\{1, \ldots, n\}$ where for any two distinct nodes there is an edge between them with probability $p(n)$, independently of any other pairs of nodes. This yields a discrete probability space whose possible worlds are graphs on the same set of nodes and where the probability of each possible world is $p(n)^m(1 - p(n))^{\binom{n}{2}-m}$ where m is its number of edges. (Thus, the possible worlds are not equiprobable anymore.)*

For a given property P of graphs we are interested in the behavior of $\Pr(G[n, p(n)] \text{ has } P)$ for large n that is, in

$$\mu_{p(n)}(P) = \lim_{n\to\infty} \Pr(G[n, p(n)] \text{ has } P)$$

when this limit exists. We will often write $\mu(P)$ when the probability space is clear from the context. And we will be especially interested in those properties P which can be expressed by FO sentences. As we explained in the equiprobable case, when $\mu(P) = 1$ we say that P holds (or is true) *almost always (or almost surely)* and when $\mu(P) = 0$ we say that P holds (or is true) *almost never.*

Lemma 7.1 *Let P and Q be any properties of graphs (not necessarily just FO sentences). We have*

(a) $\mu(P) = 1$ iff $\mu(\neg P) = 0$. (P holds almost surely iff its negation holds almost never.)

(b) If $\mu(P) = \mu(Q) = 1$, then $\mu(P \wedge Q) = 1$. (If P and Q both hold almost always then so does their conjunction.)

(c) If $\mu(P) = \mu(Q) = 0$, then $\mu(P \vee Q) = 0$. (If P and Q both hold almost never then so does their disjunction.)

(d) Suppose P implies Q, then $\mu(P) = 1 \Rightarrow \mu(Q) = 1$ (if P holds almost surely then so does Q) and $\mu(Q) = 0 \Rightarrow \mu(P) = 0$ (if Q holds almost never then so does P).

The proofs use just the basic laws of probability and will be assigned as **homework exercises**. We will frequently use these facts.

8 The Extension Axioms

Definition 8.1 (Extension Axioms). *Let r and s be natural numbers. Then the extension axiom $EA_{r,s}$ says that for any distinct $x_1, \ldots, x_r, y_1, \ldots, y_s$, there exists z distinct from x's and y's such that for all i $(i = 1, \ldots, r)$, there is an edge between z and x_i, and for all j $(j = 1, \ldots, s)$, there is no edge between z and y_j. In FOL $EA_{r,s}$ is given by the following sentence:*

$$\forall x_1, \ldots, x_r \forall y_1, \ldots, y_s \; Distinct(x_1, \ldots, x_r, y_1, \ldots, y_s)$$
$$\rightarrow \; \exists z \; Distinct(z, x_1, \ldots, x_r, y_1, \ldots, y_s) \; \wedge \; \bigwedge_{i=1}^{r} E(z, x_i) \; \wedge \; \bigwedge_{j=1}^{s} \neg E(z, y_j)$$

where

$$Distinct(u, v, w, \ldots) \text{ stands for } u \neq v \wedge v \neq w \wedge u \neq w \wedge \cdots$$

Lemma 8.1 *Suppose $p(n) = p \neq 0, 1$ is constant (does not depend on n). (call this the random graph of constant probability). Then, for any fixed r, s we have $\mu(EA_{r,s}) = 1$, i.e., $EA_{r,s}$ holds almost surely.*

Proof For fixed $x_1, \ldots, x_r, y_1, \ldots, y_s$ we can compute the probability that there is no z such that

$$Distinct(z, x_1, \ldots, x_r, y_1, \ldots, y_s) \wedge \mathop{E}_{i=1}^{r}(z, x_i) \wedge \neg \mathop{E}_{j=1}^{s}(z, y_j)$$

Indeed

$$\Pr(\text{no } z) = (1 - p^r(1 - p)^s)^{n-r-s}$$

Now let $W_1, \ldots, W_m$ where $m = \binom{n}{r}\binom{n-r}{s}$ be all the events corresponding to there is no z as above for all the possible choices of r distinct x's and s's distinct y's, also distinct from the x's. These events may not be disjoint but we can still apply a standard inequality, and moreover all these events have the same probability, computed above. Therefore

$$\Pr(G[n,p] \models \neg EA_{r,s}) = \Pr(W_1 \cup \cdots \cup W_m) \leq \Pr(W_1) + \cdots \Pr(W_m) = \binom{n}{r}\binom{n-r}{s}(1 - p^r(1-p)^s)^{n-r-s}$$

Taking limits we get $\mu(EA_{r,s}) = 0$ and by Lemma 7.1(a) $\mu(EA_{r,s}) = 1$. $\square$

9 Connectivity, acyclicity, "treeness"

It is known that connectivity and acyclicity are not expressible in FOL. Nonetheless, we shall see that we can use the 0-1 status of some extension axioms to derive a 0-1 status for them!

Example 9.1 *In the random graph of constant probability the property "is connected" holds almost surely.*

Proof Consider the extension axiom $EA_{2,0}$, which states that $\forall x \neq y, \exists z$ different from x and y, such that $E(x,z) \wedge E(y,z)$. Therefore, any graph with at least three nodes that satisfies $EA_{2,0}$ is such any two of its nodes are connected by a path of length 2, hence the graph is connected. Since the extension axioms hold almost surely by Lemma 8.1 Lemma 7.1(d) that "is connected" is also almost surely true.

Remark Our argument seems to require that the graphs have at least three nodes (actually two, can you see this?)... but there are two ways out of this "problem". One is to observe that one can strengthen Lemma 7.1(d) by asking that there be some N such that "P implies Q" holds just for all graphs with more than N nodes, rather than all graphs. Another way out is to take the conjunction between $EA_{2,0}$ and the property that the graph has at least three nodes. The latter is obviously almost surely true and therefore their conjunction is too, by Lemma 7.1 (b).□

Example 9.2 *The properties "is acyclic" and "is a tree" hold almost never.*

Proof Notice that $EA_{1,0} \wedge EA_{2,0}$ implies that there exists a triangle and hence a cycle. As before, it follows that "has a cycle" holds almost surely and by Lemma 7.1(a) "is acyclic" holds almost never. Since "is a tree" implies "is acyclic" it follows by Lemma 7.1(d) that "is a tree" holds almost never. □.