Friendly Logics, Fall 2015, Homework 2

Problem 1 We discussed in class that both Gödel's sentence σ_G and its negation are unprovable and one of them must be true so there must be some true but unprovable sentence. In fact, we *can* say more. σ_G asserts its own unprovability and so ... it is the one that should be true! Show that this is indeed so.

Problem 2 Prove Löb's Theorem: *for any sentence* τ *if* $PA \vdash \Box\tau \Rightarrow \tau$ *then* $PA \vdash \tau$. **Hints:** Use the fixed point lemma (lemma 4.1) in lecture notes 4 to construct σ_L such that $PA \vdash \sigma_L \Leftrightarrow \Box\sigma_L \Rightarrow \tau$. After re-reading the proof of Gödel's Second, use the Hilbert-Bernays derivability conditions (lemma 5.1) to show that $PA \vdash \Box\sigma_L \Rightarrow \Box\tau$. Then do more work :)

Problem 3 Show that the parity query *is* FO definable over finite models if the vocabulary contains at least a *binary relation symbol.*

Problem 4 Use a compactness argument to show that acyclicity is not FO definable over *all* undirected simple graphs.

Problem 5 Assume a vocabulary with just one binary relation symbol E. Prove that for any finite model $\mathcal{A}$ there exists a sentence $\sigma_{\mathcal{A}}$ such that for any model $\mathcal{B}$ we have $\mathcal{B} \models \sigma_{\mathcal{A}}$ iff $\mathcal{B} \simeq \mathcal{A}$.

Problem 6 Let k, m, n be positive integers. Prove that the following are equivalent:

(i) $L_m \sim_k L_n$

(ii) $m = n$ or, both $m, n \geq 2^k - 1$

Hints: by induction on $\min(m, n)$, showing, along the way, that for every $s \geq 1$, $L_m \sim_{s+1} L_n$ iff

- $\forall a \in L_m \ \exists b \in L_n \ L_m^{>a} \sim_s L_n^{>b} \quad \& \quad L_m^{<a} \sim_s L_n^{<b}$, and
- $\forall b \in L_n \ \exists a \in L_m \ L_m^{>a} \sim_s L_n^{>b} \quad \& \quad L_m^{<a} \sim_s L_n^{<b}$

Problems 7-10 Using reductions from the non-definability of parity over linear orders show that connectivity (P7), acyclicity (P8), "treeness" (P9), and "bipartiteness" (P10) are not definable over finite undirected simple graphs.