Friendly Logics, Fall 2015, Homework 1

Problem 1 (If you have seen this before, for example in Math 570, do Problem 1alt instead).

Give a definition of $\mathcal{A}, v \models \varphi$ by induction on the structure of φ.

Problem 1alt Let T be a nonempty set. Fix $a \in T$ and $e : T \to T$. Prove that there *exists* a function $f : \mathbb{N} \to T$ such that (1) $f(0) = a$ and (2) $\forall n \in \mathbb{N}\ f(n+1) = e(f(n))$. (It follows immediately by induction that f is uniquely determined by properties (1) and (2). But how do you know that such an f *exists*? It turns out that this also follows from the Induction Principle, but less immediately.)

Problem 2 Prove that if Σ is r.e then $Ded(\Sigma)$ is r.e.

Problem 3 Prove that for any set Σ of sentences the following are equivalent:

(i) $\Sigma \vdash \mathsf{false}$

(ii) $Ded(\Sigma)$ contains *all* sentences

(iii) $\Sigma \vdash \sigma$ and $\Sigma \vdash \neg\sigma$ for some sentence σ

(iv) $\Sigma \vdash p$ for some propositional constant p that does not appear in Σ

As you prove the equivalence, state clearly all the assumptions about the FOL proof system that you are using (e.g., that it is closed under *modus ponens*: if $\Sigma \vdash \sigma$ and $\Sigma \vdash \sigma \Rightarrow \rho$ then $\Sigma \vdash \rho$; **Hint:** to deal with (iv) you may have to state that the proof system is closed under a certain substitutivity property.).

Recall that we did not stipulate a specific FOL proof system. But you might find it helpful to take a look at a Hilbert-style proof system somewhere on the Web.

As stated in the lecture notes any of these four equivalent statements can be taken as the definition of the fact that the set Σ of sentences is **inconsistent**. (thus yielding a definition of **consistency**).

Problem 4 In this problem you are to prove equivalences between different statements of Soundness, Completeness, and Compactness of FOL found in the literature. Skip parts (a),(b), and/or (d) if you have done them before.)

(a) Prove that the statement of the Soundness Theorem (3.1) is *equivalent* to the following statement: for any Σ if Σ is satisfiable then it is consistent.

(b) Prove that the statement of the Compleness Theorem (3.3) is *equivalent* to the following statement: for any Σ if Σ is consistent then it is satisfiable.

(c) Prove that $VALID \subseteq PROV$ is *equivalent* to the following: each sentence is either *refutable* (i.e., its negation is provable) or satisfiable.

(d) Prove that the following two formulation of a statement known as the Compactness Theorem are *equivalent*:

(i) For all Σ, σ if $\Sigma \models \sigma$ then there exists a *finite* subset $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \sigma$.

(ii) For all Σ if all finite subsets of Σ are satisfiable then Σ is satisfiable.

Problem 5 In this problem it is important to make sure the formalism for FOL does *not* have propositional logic embedded in it (to see why read the footnotes). This can be done by restricting the proof system but since we didn't specify one let's just assume (locally) that the vocabulary does not contain propositional constants.

(a) Let Σ be a *consistent* set of sentences. Prove that the following are equivalent:

(i) For any $\sigma \notin Ded(\Sigma)$, $\Sigma \cup \{\sigma\}$ is inconsistent. (We say that Σ is **maximally** consistent.)

(ii) For any sentence σ, $\Sigma \vdash \sigma$ or $\Sigma \vdash \neg\sigma$.

Again, as you prove the equivalence, state clearly all the assumptions about the FOL proof system that you are using.

A consistent set of sentences is defined to be a **syntactically complete** (a.k.a. **Post-complete**) [1] axiomatization by any of the two equivalent properties above.

(b) Let Σ be a decidable, consistent, and syntactically complete set of sentences. Prove that $Th(\Sigma)(= Ded(\Sigma))$ is decidable.

(c) Let $\mathcal{A}$ be a model. Prove that any set Σ of sentences such that $Th(\Sigma) = Th(\mathcal{A})$ (i.e., Σ axiomatizes the theory of the single model $\mathcal{A}$) is consistent and syntactically complete. (**Note:** Putting (b) and (c) together we conclude that if the theory of a single model admits a decidable axiomatization then it is itself decidable.)

(d) Show that the statement in part (c) fails if instead of theories defined by a single model we consider theories defined by classes of two or more models.

(e) Conclude (in at least two different ways) that $\emptyset$ (while consistent) is not syntactically complete. [2]

Problem 6 We have seen that *VALID* is r.e. Moreover, the *proof* of the Church/Turing Theorem gives a reduction $SRP \leq_m VALID$. From the notes on computability we know that $K \leq_m SRP$. Putting all these facts together we conclude that *VALID* is r.e.-complete. Prove that

$$FO\text{-}SAT \stackrel{\text{def}}{=} \{\sigma \mid \sigma \text{ is satisfiable}\}$$

is co-r.e.-complete.

(Not to be confused with propositional (boolean) satisfiability which is decidable, in fact NP-complete.)

[1] Post and, independently, Hilbert and Bernays proved this property for propositional logic. (Can you prove it too? Hint: in Hilbert-Ackermann they use conjunctive normal forms)

[2] Therefore, while syntactic completeness holds for propositional logic, it fails for FOL.

Problem 7 Give an example of an FO vocabulary and an FOL sentence in that vocabulary that is finitely valid (i.e., true in all finite models of that vocabulary) but not valid. (This does not mean that finiteness is *definable* in FOL. In fact, it isn't, see the next problem. But the example that you construct here can be used to show that finiteness is definable in SOL—second-order logic.)

Problem 8

(a) Describe a *set* of FOL sentence Σ_{inf} such that for any model $\mathcal{A}$ we have that $\mathcal{A}$ is *infinite* iff $\mathcal{A} \models \Sigma_{inf}$.

(b) The Compactness Theorem (see Problem 4 part (d)) follows at once from the Completeness Theorem. Use a compactness argument to show that there is no set of FOL sentences Σ_{fin} such that for any model $\mathcal{A}$ we have that $\mathcal{A}$ is *finite* iff $\mathcal{A} \models \Sigma_{fin}$.

(c) Conclude that there is no *single* sentence σ_{inf} such that for any model $\mathcal{A}$ we have that $\mathcal{A}$ is *infinite* iff $\mathcal{A} \models \sigma_{inf}$.

Problem 9 Show that it follows from the *proof* of Trakhtenbrot's Theorem (pages from Libkin's book were attached to lecture notes 1 handout) that *FIN-VALID* is co-r.e.-complete.

Problem 10 We have seen that *VALID* and *FIN-SAT* are both r.e.-complete. Hence they are many-one reducible to each other. Describe as best you can two total computable functions that realize these two many-one reductions.