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Towards dependently typed Haskell: System FC with kind equality (Extended Version)

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Abstract

System FC, the core language of the Glasgow Haskell Compiler, is an explicitly-typed variant of System F with first-class type equality proofs called *coercions*. This extensible proof system forms the foundation for type system extensions such as type families (type-level functions) and Generalized Algebraic Datatypes (GADTs). Such features, in conjunction with kind polymorphism and datatype promotion, support expressive compile-time reasoning.

However, the core language lacks explicit *kind* equality proofs. As a result, type-level computation does not have access to kind-level functions or promoted GADTs, the type-level analogues to expression-level features that have been so useful. In this paper, we eliminate such discrepancies by introducing kind equalities to System FC. Our approach is based on dependent type systems with heterogeneous equality and the "Type-in-Type" axiom, yet it preserves the metatheoretic properties of FC. In particular, type checking is simple, decidable and syntax directed. We prove the preservation and progress theorems for the extended language.

1. Introduction

Is Haskell a dependently typed programming language? For more than a decade, clever Haskellers have encoded many programs that were reputed to need dependent types. At the same time, the Glasgow Haskell Compiler (GHC), Haskell's primary implementation, has augmented its type system with many features inspired by dependently typed languages, such as Generalized Algebraic Datatypes (GADTs) [Peyton Jones et al. 2006; Schrijvers et al. 2009], type families [Chakravarty et al. 2005], and datatype promotion with kind polymorphism [Yorgey et al. 2012].

However, these extensions do not compose well. On one hand, GADTs allow the programmer to exploit type equalities to write richer *terms*. On the other, datatype promotion and kind polymorphism have opened the door to much more expressive *types*. However, GADTs cannot currently be promoted, so the useful *type equalities* available in *terms* cannot be lifted to useful *kind equalities* available in *types*. (We consider examples of the features that we want to enable in Section 2.)

Our goal in this paper is to eliminate such nonuniformities with a single blow, by unifying types and kinds. The challenge of this work is that enabling kind equalities introduces complexities similar to those found in dependently typed languages. However, our ultimate goal is to better support dependently typed programming in Haskell, so resolving these issues is an important step in this process. Specifically, we make the following contributions:

- We describe an explicitly-typed intermediate language with explicit equality proofs at the type and the kind level. The language is no toy: it is a modification of the System FC intermediate language used by GHC today [Sulzmann et al. 2007; Vytiniotis et al. 2012; Weirich et al. 2011; Yorgey et al. 2012]. We overview the existing design of FC in Section 3 before discussing the challenges of our version in Section 4. Note that as FC is the core language for a significant compiler, we cannot make sweeping changes. Our modifications must be compatible with the existing system.
- We extend the *type preservation* proof of FC to cover the new features (Section 5). The treatment of datatypes requires an important property—that the equational theory is *congruent*. In other words, we can derive a proof of equality for any form of type or kind, given equality proofs of subcomponents. The computational content of this theorem, called *lifting*, generalizes the standard substitution operation. This operation is required in the operational semantics for datatypes.
- We prove the *progress* theorem in the presence of kind coercions and dependent coercion abstraction, requiring significant changes to the current proof. We discuss these changes and their consequences in Section 6.
- We have implemented our extensions in a development branch of GHC. Our work is available online.[1]

Although our designs are inspired by the rich theory of dependent type systems, applying these ideas in the context of Haskell means that our language differs in many ways from existing work. We detail these comparisons in Section 7.

2. Why kind equalities?

Kind equalities enable new, useful features in GHC. In this section we review of some of the more recent extensions to the GHC type system (including GADTs, type families, kind polymorphism and datatype promotion) through examples. We also describe new features and examples that require our extensions. Below, code snippets that cannot be written in GHC 7.6 are highlighted in gray.

Below, we define "shallowly" and "deeply" indexed representations of types and show how they may be used for Generic Pro-

[1] This branch is available from `http://www.cis.upenn.edu/~eir/packages/nokinds/`.

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gramming (GP). The former use Haskell's types as indices [Crary et al. 1998; Yang 1998], whereas the latter use an algebraic datatype (also known as a *universe*) [Altenkirch and McBride 2002; Norell 2002]. Magalhães [2012] gives more details describing how extensions to Haskell, including the ones described in this paper, benefit generic programming.

Although the examples below are derived from generic programming, our extensions are by no means specific to this setting—we have been able to express dependently typed programs from McBride [2012] and Oury and Swierstra [2008], which we omit here for lack of space. Our overall goal is to make Agda and Coq programs more accessible in GHC.

Shallow indexing Consider a GADT for type representations:

```
data TyRep :: * → * where
 TyInt  :: TyRep Int
 TyBool :: TyRep Bool
```

GADTs differ from ordinary algebraic datatypes in that they allow each data constructor to constrain the type parameters to the datatype. For example, the `TyInt` constructor requires that the single parameter to `TyRep` be `Int`.

We can use type representations for type-indexed programming. A simple example is computing a default element for each type.

```
zero :: ∀ a. TyRep a → a
zero TyInt  = 0          -- ‘a’ must be Int
zero TyBool = False      -- ‘a’ must be Bool
```

This code pattern matches the type representation to determine what value to return. Because of the nonuniform type index, pattern matching recovers the identity of the type variable `a`. In the first case, because the data constructor is `TyInt`, this parameter must be `Int`, so `0` can be returned. Similarly, in the second case, the parameter `a` must be equal to `Bool`.

However, the GADT above can only be used to represent types of kind $\star$. Representing type constructors with kind $\star \to \star$, such as `Maybe` or `[]`, requires a separate data structure, perhaps called `TyRep1`. However, this approach is unsustainable—what about tuples? Do we need a `TyRep2`, `TyRep3`, and more?

Kind polymorphism [Yorgey et al. 2012] allows datatypes to be parameterized by kind variables as well as type variables. For example, the following type takes two phantom arguments, a kind variable κ and a type variable a of kind κ.

```
data Proxy (a :: k) = P
```

However, kind polymorphism is not enough to unify these representations—the type representation (shown below) should constrain its *kind* parameter.

```
data TyRep :: ∀ k. k → * where
  TyInt   :: TyRep Int
  TyBool  :: TyRep Bool
  TyMaybe :: TyRep Maybe
  TyApp   :: TyRep a → TyRep b → TyRep (a b)
```

Above, the `TyRep` type takes two parameters, a kind `k` and a type of that kind (not named in the kind annotation). The data constructors constrain `k` to a concrete kind. For the example to be well-formed, `TyInt` must constrain the *kind* parameter to $\star$. Similarly, `TyMaybe` requires the kind parameter to be $\star \to \star$. We call this example a *kind-indexed GADT* because the datatype is indexed by both kind and type information.

This extension is compatible with our prior uses of `TyRep`. For example, to extend `zero` with a default value of the `Maybe` type:

```
zero :: ∀ (a :: *). TyRep a → a
zero TyInt              = 0
zero TyBool             = False
zero (TyApp TyMaybe _) = Nothing
```

Here, pattern matching with this datatype can also refine kinds as well as types—determining whether a type is of the form `Maybe b` makes new kind and type equalities available.

```
isMaybe :: ∀ (a :: k). TyRep a → ...
isMaybe (TyApp TyMaybe _) =
     -- here we have k ∼ * and
     -- a ∼ Maybe b, for some b :: *
```

Deep indexing Kind equalities enable additional features besides kind-indexed GADTs. The previous example used Haskell types directly to index type representations. With datatype promotion, we can instead define a datatype (a universe) for type information.

```
data Ty = TInt | TBool
```

We can use this datatype to index the representation type.

```
data TyRep :: Ty → * where
 TyInt  :: TyRep TInt
 TyBool :: TyRep TBool
```

Note that the kind of the parameter to this datatype is `Ty` instead of `*`—promotion allows the type `Ty` to be used as a kind and allows its data constructors, `TyInt` and `TyBool` to appear in types.

To use these type representations, we describe their connection with Haskell types via a *type family* (a function at the type level).

```
type family I (t :: Ty) :: *
type instance I TInt  = Int
type instance I TBool = Bool
```

`I` is a function that maps the (promoted) data constructor `TInt` to the Haskell type `Int`, and similarly `TBool` to `Bool`.

We can use these type representations to define type-indexed operations, like before.

```
zero :: ∀ (a :: Ty). TyRep a → I a
zero TyInt  = 0
zero TyBool = False
```

Pattern matching `TyInt` refines `a` to `TInt`, which then uses the type family definition to show that the result type is equal to `Int`.

Dependently typed languages do not require an argument like `TyRep` to implement operations such as `zero`—they can match directly on the type of kind `Ty`. This is not allowed in Haskell, which maintains a separation between types and values. The `TyRep` argument is an example of a *singleton* type, a standard way of encoding dependently typed operations in Haskell.

Note that this representation is no better than the shallow version in one respect—`I` must produce a type of kind $\star$. What if we wanted to encode `TMaybe` with `Ty`?

To get around this issue, we use a GADT to represent different kinds of types. We first need a universe of kinds.

```
data Kind = Star | Arr Kind Kind
```

`Kind` is a normal datatype that, when promoted, can be used to index the `Ty` datatype, making it a (standard) GADT.

```
data Ty :: Kind → * where
  TInt   :: Ty Star
  TBool  :: Ty Star
  TMaybe :: Ty (Arr Star Star)
  TApp   :: Ty (Arr k1 k2) → Ty k1 → Ty k2
```

This indexing means that Ty can only represent well-kinded types. For example TMaybe has type Ty (Arr Star Star) and TApp TMaybe TBool has type Ty Star, while the value TApp TInt would be rejected.

Although this GADT can be expressed in GHC, the corresponding TyRep type requires two new extensions: *promoted GADTs* and *kind-families*.

With the current design of FC, only a subset of Haskell 98 datatypes can be promoted. In particular, GADTs cannot be used to index other GADTs. The extensions proposed in this work allow the GADT Ty above to be used as an index to TyRep or to be interpreted by the type family I, as shown below.

```
data TyRep (k :: Kind) (t :: Ty k) where
  TyInt   :: TyRep Star TInt
  TyBool  :: TyRep Star TBool
  TyMaybe :: TyRep (Arr Star Star) TMaybe
  TyApp   :: TyRep (Arr k1 k2) a → TyRep k1 b
          → TyRep k2 (TApp a b)
```

We now need to adapt the type family I to work with the new promoted GADT Ty. To do so, we must classify its return kind, and for that, we need a kind family. A kind family is a function that produces a kind. For example, we can use a kind family to interpret elements of the Kind datatype as a Haskell kind.

```
kind family IK (k :: Kind)
kind instance IK Star = *
kind instance IK (Arr k1 k2) = IK k1 → IK k2
```

This interpretation of kinds is necessary to define the interpretation of types—without it, this definition does not "kind-check":

```
type family I (t :: Ty k) :: IK k
type instance I TInt       = Int
type instance I TBool      = Bool
type instance I TMaybe     = Maybe
type instance I (TApp a b) = (I a) (I b)
```

However, once I has been defined, Ty and TyRep can be used in type-indexed operations as in previous versions.

```
zero :: ∀ (a :: Ty Star). TyRep Star a → I a
zero TyInt             = 0
zero TyBool            = False
zero (TyApp TyMaybe _) = Nothing
```

3. System FC

System FC is the typed intermediate language of GHC. GHC's advanced features, such as GADTs and type families, are compiled into FC using type equalities. This section reviews the current status of System FC, describes that compilation, and puts our work in context. FC has evolved over time, from its initial definition [Sulzmann et al. 2007], to several extensions FC_2 [Weirich et al. 2011], and $F_C^{\uparrow}$ [Yorgey et al. 2012]. In this paper, we use the name FC for the language and all of its variants. In the technical discussion below, we contrast our new extensions with the most recent prior version, $F_C^{\uparrow}$.

Along with the usual kinds (κ), types (τ) and expressions (e), FC contains coercions (γ) that are proofs of type equality. The judgement

$$\Gamma \vdash_{\mathsf{co}} \gamma \ :\ \tau_1 \sim \tau_2$$

checks that the coercion γ proves types τ_1 and τ_2 equal. These proofs appear in expressions and coerce their types. For example, if γ is a proof of $\tau_1 \sim \tau_2$, and the expression e has type τ_1, then the expression $e \triangleright \gamma$ (pronounced "e casted by γ") has type τ_2.

Making type conversion explicit ensures that the FC typing relation $\Gamma \vdash_{\mathsf{tm}} e \ :\ \tau$ is syntax-directed.[2] Straightforward type checking is an important sanity check on the internals of GHC—transformations and optimizations must preserve typability. Therefore, all information necessary for type checking is present in FC expressions. This information includes explicit type abstractions and applications (System FC is an extension of System F_ω [Girard 1972]) as well as explicit proofs of type equality.

For example, type family definitions are compiled to *axioms* about type equality that can be used in FC coercion proofs. A type family declaration and instance in source Haskell

```
type family F a :: *
type instance F Int = Bool
```

generates the following FC axiom declaration:

$$\mathsf{axF} : \mathsf{F}\ \mathsf{Int} \sim \mathsf{Bool}$$

When given a source language function of type

```
g :: ∀ a. a → F a → Char
```

the expression g 3 True translates to the FC expression

$$g\ \mathsf{Int}\ 3\ (\mathsf{True} \triangleright \mathbf{sym}\ \mathsf{axF})$$

that instantiates g at type Int and coerces True to have type $\mathsf{F}\ \mathsf{Int}$. The coercion $\mathbf{sym}\ \mathsf{axF}$ is a proof that $\mathsf{Bool} \sim \mathsf{F}\ \mathsf{Int}$.

Likewise, GADTs are compiled into FC so that pattern matching on their data constructors introduces *equality assumptions* into the context. For example, consider the following simple GADT.

```
data T :: * → * where
   TInt :: T Int
```

This declaration could have also been written as a normal datatype where the type parameter is constrained to be equal to Int.

```
data T a = (a ∼ Int) ⇒ TInt
```

In fact, all GADTs can be rewritten in this form using equality constraints. Pattern matching makes this constraint available to the type checker. For example, the type checker concludes below that 3 has type a because the type Int is known to be equal to a.

```
f :: T a → a
f Tint = 3
```

In the translation to FC, the TInt data constructor takes this equality constraint as an explicit argument.

$$\mathsf{TInt} : \forall\, a{:}\ \star\,.\,(a \sim \mathsf{Int}) \Rightarrow T\ a$$

When pattern matching on values of type $T\ a$, this proof is available for use in a cast.

$$\begin{array}{l} f = \Lambda a{:}\ \star\,.\lambda x{:}\ T\ a.\ \mathbf{case}\ x\ \mathbf{of} \\ \quad \mathsf{TInt}\ (c{:}\ a \sim \mathsf{Int}) \rightarrow (3 \triangleright \mathbf{sym}\ c) \end{array}$$

Coercion assumptions and axioms can be composed to form larger proofs. FC includes a number of forms in the coercion language γ that witness the reflexivity, symmetry and transitivity of type equality. Furthermore, equality is a congruent relation over types. For example, if we have proofs of $\tau_1 \sim \tau_2$ and $\tau_1' \sim \tau_2'$, then we can form a proof of the equality $\tau_1\,\tau_1' \sim \tau_2\,\tau_2'$. Finally, composite coercion proofs can be decomposed. For example, data

[2] This is not the case in the source language; type checking requires non-local reasoning, such as unification and type class resolution. Furthermore, in the presence of certain flags (such as UndecidableInstances), it may not terminate.

constructors T are injective, so given a proof of $T\,\tau_1 \sim T\,\tau_2$, a proof of $\tau_1 \sim \tau_2$ can be produced.

However, explicit coercion proofs are like explicit type arguments. They are erasable from expressions and do not interfere with the operational semantics. (We make this precise in Section 5.2.) Therefore, FC includes a number of "push rules" that ensure that coercions do not suspend computation. For example, when a coerced value is applied to an argument, the coercion must be "pushed" to the argument and result of the application so that β-reduction can occur.

$$\frac{\Gamma \vdash_{\mathsf{co}} \gamma \;:\; \sigma_1 \to \sigma_2 \sim \tau_1 \to \tau_2}{(v \rhd \gamma)\; e \longrightarrow (v\;(e \rhd \mathbf{sym}\,(\mathbf{nth}^1\;\gamma))) \rhd \mathbf{nth}^2\;\gamma} \quad \text{S\_PUSH}$$

In this rule, γ must be a proof of the equality $\sigma_1 \to \sigma_2 \sim \tau_1 \to \tau_2$. The coercions $\mathbf{sym}\,(\mathbf{nth}^1\;\gamma)$ and $\mathbf{nth}^2\;\gamma$ decompose this proof into proofs of $\tau_1 \sim \sigma_1$ and $\sigma_2 \sim \tau_2$ respectively.

4. System FC with kind equalities

The basic idea of this paper is to augment FC with proofs of equality between kinds and to use those proofs to explicitly coerce the kinds of types. We do so via new form of type, written $\tau \rhd \gamma$, that, when given a type τ, of kind κ_1, and a proof γ that kind κ_1 equals kind κ_2, produces a type of kind κ_2.

However, there are several novel challenges to this extension:

- *Language duplication.* A language with kind polymorphism, kind equalities, kind coercions, type polymorphism, type equalities and type coercions quickly becomes redundant (and somewhat overwhelming).

  Therefore, we follow pure type systems [Barendregt 1992] and unify the syntax of types and kinds, allowing us to reuse type coercions as kind coercions. Furthermore, GHC already uses a shared datatype for types and kinds, so this merger brings the formalism more in line with the actual implementation. Yet, we continue to use both of the words *type* and *kind* informally. We use the word *type* (metavariables τ and σ) for those members that classify runtime expressions, and *kind* (metavariable κ) for those members that classify expressions of the type language.

  The new syntax appears in Figure 1; forms that are new or modified in this paper are highlighted. These modifications are primarily in the type and coercion languages. Note that $\star$ is a new type constant and κ is a metavariable for types. The only difference in the grammar for expressions is that type abstractions and kind abstractions have been merged. In general, the type system and operational semantics for the expression language is the same here as in prior versions of FC.

- *Interaction with type equality.* Since kinds classify types, kind equality has nontrivial interactions with type equality.

  Because kind coercion is explicit, there are equivalent types that do not have syntactically identical kinds. Therefore, like McBride's "John Major" equality [2002], our definition of type equality $\tau_1 \sim \tau_2$ is heterogeneous—the types τ_1 and τ_2 could have kinds κ_1 and κ_2 that have no relation to each other. A proof γ of this proposition implies not only that τ_1 and τ_2 are equal, but also that their kinds are equal. The new coercion form $\mathbf{kind}\,\gamma$ extracts the proof of $\kappa_1 \sim \kappa_2$ from γ.

  Another difficulty comes from the need to equate polymorphic types that have coercible but not syntactically equal kinds for the bound variable. We discuss the modification to this coercion form in Section 4.3.2.

- *New type forms and associated coercions.* The examples in Section 2 motivate new features that require reasoning about kind equality, but some of these new features require additional ex-

H	$::=$		Type constants
		$\mid (\to)$	Arrow
		$\mid \star$	Type/Kind
		$\mid T$	Type constructor
		$\mid K$	Promoted data constructor
w	$::=$		Type-level names
		$\mid a$	Type variables
		$\mid F$	Type functions
		$\mid H$	Type constants
$\sigma,\ \tau,\ \kappa$	$::=$		Types and Kinds
		$\mid w$	Names
		$\mid \forall\, a{:}\,\kappa.\, \tau$	Polymorphic types
		$\mid \forall\, c{:}\,\phi.\, \tau$	Coercion abstr. type
		$\mid \tau_1\;\tau_2$	Type/kind application
		$\mid \tau \rhd \gamma$	Casting
		$\mid \tau\;\gamma$	Coercion application
ϕ	$::=$	$\sigma \sim \tau$	Propositions (coercion kinds)
$\gamma,\ \eta$	$::=$		Coercions
		$\mid c$	Variables
		$\mid C\,\overline{\rho}$	Axiom application
		$\mid \langle \tau \rangle$	Reflexivity
		$\mid \mathbf{sym}\,\gamma$	Symmetry
		$\mid \gamma_1 \mathbin{;} \gamma_2$	Transitivity
		$\mid \forall_{\eta}(a_1, a_2, c).\gamma$	Type/kind abstr. cong.
		$\mid \forall_{(\eta_1,\eta_2)}(c_1, c_2).\gamma$	Coercion abstr. cong.
		$\mid \gamma_1\;\gamma_2$	Type/kind app. cong.
		$\mid \gamma(\gamma_2, \gamma_2')$	Coercion app. cong.
		$\mid \gamma \rhd \gamma'$	Coherence
		$\mid \gamma@\gamma'$	Type/kind instantiation
		$\mid \gamma@(\gamma_1, \gamma_2)$	Coercion instantiation
		$\mid \mathbf{nth}^i\,\gamma$	nth argument projection
		$\mid \mathbf{kind}\,\gamma$	Kind equality extraction
ρ	$::=$	$\tau \mid \gamma$	Type or coercion
$e,\ u$	$::=$		Expressions
		$\mid x$	Variables
		$\mid \lambda x{:}\,\tau.\, e$	Abstraction
		$\mid e_1\;e_2$	Application
		$\mid \Lambda a{:}\,\kappa.\, e$	Type/kind abstraction
		$\mid e\;\tau$	Type/kind application
		$\mid \lambda c{:}\,\phi.\, e$	Coercion abstraction
		$\mid e\;\gamma$	Coercion application
		$\mid e \rhd \gamma$	Casting
		$\mid K$	Data constructors
		$\mid \mathbf{case}\; e\; \mathbf{of}\; \overline{p \to u}$	Case analysis
		$\mid \mathbf{contra}\,\gamma\;\tau$	Absurdity
p	$::=$	$K\;\Delta\;\overline{x{:}\tau}$	Patterns
Δ	$::=$		Telescopes
		$\mid \varnothing$	Empty
		$\mid \Delta,\ a{:}\,\kappa$	Type variable binding
		$\mid \Delta,\ c{:}\,\phi$	Coercion variable binding

Figure 1. Basic Grammar

tensions to the type language. Furthermore, each new addition to the type language requires new proof rules to create and decompose equality proofs concerning these new forms. We describe these new forms in Section 4.2 and their associated coercions in Section 4.3.

- *Semantics of datatypes.* The semantics of datatypes in FC is already complicated, both in the type checking rules and operational semantics. The version in this paper (covered in Section 4.4) simplifies this semantics by using telescopes Δ, which are nested bindings of type and coercion variables.

  However, the modifications to coercions, described above, add complexity to the S_KPUSH rule of the operational semantics. In particular, this rule relies on a "lifting" operation that uses a sequence of coercions that show $\tau_1 \sim \tau_1' \ldots \tau_n \sim \tau_n'$ to produce a coercion that shows $\sigma[\tau_1/a_1]\ldots[\tau_n/a_n] \sim \sigma[\tau_1'/a_1]\ldots[\tau_n'/a_n]$. Because of the interactions between kind and type equalities in this sequence, this lifting operation must be redefined.

4.1 Type system overview

In the next few subsections, we discuss our solutions to the challenges described above in more detail. However, we first give a quick overview of the type system to orient the discussion.

A context Γ is a list of assumptions for term variables (x), type variables/datatypes/data constructors (w), coercion variables (c), and coercion axioms (C).

$$\Gamma ::= \varnothing \mid \Gamma,\ x{:}\tau \mid \Gamma,\ w{:}\kappa \mid \Gamma,\ c{:}\phi \mid \Gamma,\ C{:}\ \forall\,\Delta.\,\phi$$

The type system includes the following judgements:

$\vdash_{\mathsf{wf}} \Gamma$	Context validity	(Figure 5)
$\Gamma \vdash_{\mathsf{ty}} \tau\ :\ \kappa$	Type/kind validity	(Figure 2)
$\Gamma \vdash_{\mathsf{pr}} \phi\ \mathsf{ok}$	Proposition validity	(Figure 3)
$\Gamma \vdash_{\mathsf{tm}} e\ :\ \tau$	Expression typing	(appendix)
$\Gamma \vdash_{\mathsf{co}} \gamma\ :\ \phi$	Coercion validity	(Figure 4)
$\Gamma \vdash_{\mathsf{tel}} \overline{\rho} : \Delta$	Telescope arg. validity	(appendix)

In this paper, we not only unify the grammars of types and kinds, but we also unify their semantics. This means that types and kinds share rules for validity and coercion checking. In fact, our type formation rules include the $\star{:}\star$ axiom which means that there is no real distinction between types and kinds. This choice simplifies many aspects of the language and has no cost to FC. Languages such as Coq and Agda avoid the $\star{:}\star$ axiom because it introduces inconsistency, but that is irrelevant here as the FC type language is already inconsistent (all kinds are inhabited). Inconsistency is not an issue because the type safety property of FC depends on the consistency of its *coercion* language, not its *type* language. See Section 6 and Section 7 for more discussion of this design decision.

Each of the judgements above is syntax directed. Given the information before the colon (if present) there is a simple algorithm that determines if the judgement holds (and produces the appropriate kind, type or proposition). For reasons of space, some of these judgements have been deferred to the appendix.

4.2 Type and kind formation

We next describe our extensions and modifications to the type language of FC. The rules for type formation appear in Figure 2. Some of these rules are unchanged or only slightly modified from prior versions of FC.

For example, rule K_VAR looks up the kind of a type-level name from the typing context. Unlike previous systems, this rule now covers the kinding of promoted constructors, since w ranges over them. Recall that datatype promotion allows data constructors, such as `TInt`, to appear in types and be the arguments of type

$$\boxed{\Gamma \vdash_{\mathsf{ty}} \tau\ :\ \kappa}$$

$$\frac{\vdash_{\mathsf{wf}} \Gamma \quad w{:}\kappa \in \Gamma}{\Gamma \vdash_{\mathsf{ty}} w\ :\ \kappa}\ \text{K\_VAR}$$

$$\frac{\vdash_{\mathsf{wf}} \Gamma}{\Gamma \vdash_{\mathsf{ty}} (\to)\ :\ \star \to \star \to \star}\ \text{K\_ARROW}$$

$$\frac{\Gamma \vdash_{\mathsf{ty}} \tau_1\ :\ \kappa_1 \to \kappa_2 \quad \Gamma \vdash_{\mathsf{ty}} \tau_2\ :\ \kappa_1}{\Gamma \vdash_{\mathsf{ty}} \tau_1\,\tau_2\ :\ \kappa_2}\ \text{K\_APP}$$

$$\frac{\Gamma \vdash_{\mathsf{ty}} \tau_1\ :\ \forall\,a{:}\kappa_1.\,\kappa_2 \quad \Gamma \vdash_{\mathsf{ty}} \tau_2\ :\ \kappa_1}{\Gamma \vdash_{\mathsf{ty}} \tau_1\,\tau_2\ :\ \kappa_2[\tau_2/a]}\ \text{K\_INST}$$

$$\frac{\Gamma,\ a{:}\kappa \vdash_{\mathsf{ty}} \tau\ :\ \star \quad \Gamma \vdash_{\mathsf{ty}} \kappa\ :\ \star}{\Gamma \vdash_{\mathsf{ty}} \forall\,a{:}\kappa.\,\tau\ :\ \star}\ \text{K\_ALLT}$$

$$\frac{\vdash_{\mathsf{wf}} \Gamma}{\Gamma \vdash_{\mathsf{ty}} \star\ :\ \star}\ \text{K\_STARINSTAR}$$

$$\frac{\Gamma \vdash_{\mathsf{ty}} \tau_1\ :\ \forall\,c{:}\phi.\,\kappa \quad \Gamma \vdash_{\mathsf{co}} \gamma_1\ :\ \phi}{\Gamma \vdash_{\mathsf{ty}} \tau_1\,\gamma_1\ :\ \kappa[\gamma_1/c]}\ \text{K\_CAPP}$$

$$\frac{\Gamma,\ c{:}\phi \vdash_{\mathsf{ty}} \tau\ :\ \star \quad \Gamma \vdash_{\mathsf{pr}} \phi\ \mathsf{ok}}{\Gamma \vdash_{\mathsf{ty}} \forall\,c{:}\phi.\,\tau\ :\ \star}\ \text{K\_ALLC}$$

$$\frac{\Gamma \vdash_{\mathsf{ty}} \tau\ :\ \kappa_1 \quad \Gamma \vdash_{\mathsf{co}} \eta\ :\ \kappa_1 \sim \kappa_2 \quad \Gamma \vdash_{\mathsf{ty}} \kappa_2\ :\ \star}{\Gamma \vdash_{\mathsf{ty}} \tau \rhd \eta\ :\ \kappa_2}\ \text{K\_CAST}$$

Figure 2. Kind and type formation rules

functions. Previously, the types of data constructors had to be explicitly promoted to kinds [Yorgey et al. 2012]. Now, *any* data constructor may freely be used as a type. When the constructor is used as a type, its kind is the same as the type of the constructor when used as a term.

Rule K_ARROW gives the expected kind for the arrow type constructor. We use the usual syntactic sugar for arrow types, writing $\tau_1 \to \tau_2$ for $(\to)\,\tau_1\,\tau_2$. Note that the kind of the arrow type constructor includes itself, but that does not cause difficulty.

The next two rules describe when type application is well-formed. Application is overloaded in these rules, but the system is still syntax-directed—the type of the first component determines which rule applies. We do not combine function types $\sigma_1 \to \sigma_2$ and polymorphic types $\forall\,a{:}\,\kappa.\,\sigma$ into a single form because of type erasure: term arguments are necessary at runtime, whereas type arguments may be erased. In kinds, the difference between nondependent and dependent arguments is not meaningful. However, when data constructors are promoted to the type level, their types retain this distinction.

Next, the rule K_ALLT describes when polymorphic types are well formed. This rule is almost the same as before, except that it checks the validity of κ, the kind of the bound variable.

The rules K_STARINSTAR, K_CAST and K_CAPP and K_ALLC check the new type forms. The first says that $\star$ has type $\star$ as discussed above.

To preserve the syntax-directed nature of FC, we must make the use of kind equality proofs explicit. We do so via new form $\tau \rhd \gamma$ of kind casts: when given a type τ of kind κ_1 and a proof γ that kind κ_1 equals kind κ_2, the cast produces a type of kind κ_2. Because equality is heterogeneous, the K_CAST rule requires a third premise which ensures that the new kind has the correct classification. This premise ensures that inhabited types have kind $\star$.

To promote GADTs we must be able to promote data constructors that take coercions as arguments. For example, the data constructor TInt must be applied to a type argument τ and a proof that $\tau \sim \mathsf{Int}$. Promoting this data constructor requires the new appli-

$\boxed{\Gamma \vdash_{\mathsf{pr}} \phi \ \mathsf{ok}}$

$$\frac{\begin{array}{c}\Gamma \vdash_{\mathsf{ty}} \sigma_1 : \kappa_1 \\ \Gamma \vdash_{\mathsf{ty}} \sigma_2 : \kappa_2\end{array}}{\Gamma \vdash_{\mathsf{pr}} \sigma_1 \sim \sigma_2 \ \mathsf{ok}} \quad \text{PROP\_EQUALITY}$$

Figure 3. Proposition formation rule

cation form $\tau\,\gamma$. Note that there is no type-level abstraction over coercion. The form $\tau\,\gamma$ can only appear when the head of τ is a promoted datatype constructor.

As in prior versions of FC, coercions can be passed as arguments (using coercion abstractions $\lambda c{:}\,\phi.\,e$) and stored in data structures (as the arguments to data constructors of GADTs). This system deviates from earlier versions in that the types for these objects, written $\forall\, c{:}\,\phi.\,\tau$, *bind* the abstracted proof with the coercion variable c and allow the body of the type τ to depend on this proof.

This is necessary for some kind-indexed GADTs. For example, consider the following datatype, which is polymorphic over a kind and type parameter. The single data constructor K constrains the kind to be $\star$ but does not otherwise constrain the type.

```
data T :: ∀ k. k → * where
   K :: all (b :: *). b → T b
```

After translation, the data constructor should be given the following FC type, where the abstracted kind coercion c is needed to cast the kind of the parameter a.

$$\mathsf{K} : \forall a{:}\star, b{:}\, a. \forall\, c{:}\,(a \sim \star).\,(b \rhd c) \to T\ b$$

4.3 Coercions

Coercions γ are proof terms witnessing the equality between types (and kinds), and are classified by propositions ϕ. The rules under which the proofs can be derived appear in Figure 4, with the validity rule for ϕ appearing in Figure 3. These rules establish properties of the type equality relation.

- Equality is an *equivalence relation*, as seen in rules CT_REFL, CT_SYM, and CT_TRANS.
- Equality is *congruent*—types with equal subcomponents are equal. Every type formation rule (except for the base cases like variables and constants) has an associated congruence rule. The exception is kind coercion $\tau \rhd \gamma$, where the congruence rule is derivable (see Section 4.3.1). The congruence rules are mostly straightforward; we discuss the rules for quantified types (rules CT_ALLT and CT_ALLC) in Section 4.3.2.
- Equality can be *assumed*. Coercion variables and axioms add assumptions about equality to the context and appear in proofs (using rules CT_VAR and CT_AXIOM respectively). These axioms for type equality are allowed to be *axiom schemes*—they may be parameterized and must be instantiated when used.

  The general form of the type of an axiom, $C{:}\ \forall\,\Delta.\,\phi$ gathers multiple parameters in a *telescope*, a context denoted with Δ of type and coercion variables, each of which scope over the remainder of the telescope as well as the body of the axiom. We specify the list of instantiations for a telescope with $\overline{\rho}$, a mixed list of types and coercions. When type checking an axiom application, we must type check its list of arguments $\overline{\rho}$ against the given telescope. The judgement form $\Gamma \vdash_{\mathsf{tel}} \overline{\rho} : \Delta$ (presented in the appendix) checks each argument ρ in turn against the binding in the telescope, scoping variables appropriately.
- Equality can be *decomposed* using the next six rules. For example, because we know that datatypes are injective type func-

$\boxed{\Gamma \vdash_{\mathsf{co}} \gamma : \phi}$

$$\frac{\Gamma \vdash_{\mathsf{ty}} \tau : \kappa}{\Gamma \vdash_{\mathsf{co}} \langle \tau \rangle : \tau \sim \tau} \quad \text{CT\_REFL}$$

$$\frac{\Gamma \vdash_{\mathsf{co}} \gamma : \tau_1 \sim \tau_2}{\Gamma \vdash_{\mathsf{co}} \mathbf{sym}\,\gamma : \tau_2 \sim \tau_1} \quad \text{CT\_SYM}$$

$$\frac{\Gamma \vdash_{\mathsf{co}} \gamma_1 : \tau_1 \sim \tau_2 \quad \Gamma \vdash_{\mathsf{co}} \gamma_2 : \tau_2 \sim \tau_3}{\Gamma \vdash_{\mathsf{co}} \gamma_1 \mathbin{⨾} \gamma_2 : \tau_1 \sim \tau_3} \quad \text{CT\_TRANS}$$

$$\frac{\begin{array}{c}\Gamma \vdash_{\mathsf{co}} \gamma_1 : \tau_1' \sim \tau_2' \quad \Gamma \vdash_{\mathsf{co}} \gamma_2 : \tau_1 \sim \tau_2 \\ \Gamma \vdash_{\mathsf{ty}} \tau_1'\,\tau_1 : \kappa_1 \quad \Gamma \vdash_{\mathsf{ty}} \tau_2'\,\tau_2 : \kappa_2\end{array}}{\Gamma \vdash_{\mathsf{co}} \gamma_1\,\gamma_2 : \tau_1'\,\tau_1 \sim \tau_2'\,\tau_2} \quad \text{CT\_APP}$$

$$\frac{\begin{array}{l}\Gamma \vdash_{\mathsf{co}} \gamma_1 : \tau_1 \sim \tau_1' \\ \Gamma \vdash_{\mathsf{ty}} \tau_1\,\gamma_2 : \kappa \quad \Gamma \vdash_{\mathsf{ty}} \tau_1'\,\gamma_2' : \kappa'\end{array}}{\Gamma \vdash_{\mathsf{co}} \gamma_1(\gamma_2, \gamma_2') : \tau_1\,\gamma_2 \sim \tau_1'\,\gamma_2'} \quad \text{CT\_CAPP}$$

$$\frac{\begin{array}{l}\Gamma \vdash_{\mathsf{co}} \eta : \kappa_1 \sim \kappa_2 \\ \Gamma,\ a_1{:}\,\kappa_1,\ a_2{:}\,\kappa_2,\ c{:}\ a_1 \sim a_2 \vdash_{\mathsf{co}} \gamma : \tau_1 \sim \tau_2 \\ \Gamma \vdash_{\mathsf{ty}} \forall\, a_1{:}\,\kappa_1.\,\tau_1 : \star \quad \Gamma \vdash_{\mathsf{ty}} \forall\, a_2{:}\,\kappa_2.\,\tau_2 : \star\end{array}}{\Gamma \vdash_{\mathsf{co}} \forall_\eta(a_1, a_2, c).\gamma : (\forall\, a_1{:}\,\kappa_1.\,\tau_1) \sim (\forall\, a_2{:}\,\kappa_2.\,\tau_2)} \quad \text{CT\_ALLT}$$

$$\frac{\begin{array}{l}\Gamma \vdash_{\mathsf{co}} \eta_1 : \sigma_1 \sim \sigma_1' \quad \phi_1 = \sigma_1 \sim \sigma_2 \\ \Gamma \vdash_{\mathsf{co}} \eta_2 : \sigma_2 \sim \sigma_2' \quad \phi_2 = \sigma_1' \sim \sigma_2' \\ c_1 \# |\gamma| \quad c_2 \# |\gamma| \\ \Gamma,\ c_1{:}\,\phi_1,\ c_2{:}\,\phi_2 \vdash_{\mathsf{co}} \gamma : \tau_1 \sim \tau_2 \\ \Gamma \vdash_{\mathsf{ty}} \forall\, c_1{:}\,\phi_1.\,\tau_1 : \star \quad \Gamma \vdash_{\mathsf{ty}} \forall\, c_2{:}\,\phi_2.\,\tau_2 : \star\end{array}}{\Gamma \vdash_{\mathsf{co}} \forall_{(\eta_1,\eta_2)}(c_1, c_2).\gamma : (\forall\, c_1{:}\,\phi_1.\,\tau_1) \sim (\forall\, c_2{:}\,\phi_2.\,\tau_2)} \quad \text{CT\_ALLC}$$

$$\frac{\Gamma \vdash_{\mathsf{co}} \gamma : \tau_1 \sim \tau_2 \quad \Gamma \vdash_{\mathsf{ty}} \tau_1 \rhd \gamma' : \kappa}{\Gamma \vdash_{\mathsf{co}} \gamma \rhd \gamma' : \tau_1 \rhd \gamma' \sim \tau_2} \quad \text{CT\_COH}$$

$$\frac{c{:}\,\phi \in \Gamma \quad \vdash_{\mathsf{wf}} \Gamma}{\Gamma \vdash_{\mathsf{co}} c : \phi} \quad \text{CT\_VAR}$$

$$\frac{C{:}\ \forall\,\Delta.\,(\tau_1 \sim \tau_2) \in \Gamma \quad \Gamma \vdash_{\mathsf{tel}} \overline{\rho} : \Delta}{\Gamma \vdash_{\mathsf{co}} C\,\overline{\rho} : \tau_1[\overline{\rho}/\Delta] \sim \tau_2[\overline{\rho}/\Delta]} \quad \text{CT\_AXIOM}$$

$$\frac{\begin{array}{c}\Gamma \vdash_{\mathsf{co}} \gamma : H\,\overline{\rho} \sim H\,\overline{\rho}' \\ \rho_i = \tau \quad \rho_i' = \tau'\end{array}}{\Gamma \vdash_{\mathsf{co}} \mathbf{nth}^i\,\gamma : \tau \sim \tau'} \quad \text{CT\_NTH}$$

$$\frac{\Gamma \vdash_{\mathsf{co}} \gamma_1 : (\forall\, a_1{:}\,\kappa_1.\,\tau_1) \sim (\forall\, a_2{:}\,\kappa_2.\,\tau_2)}{\Gamma \vdash_{\mathsf{co}} \mathbf{nth}^1\,\gamma_1 : \kappa_1 \sim \kappa_2} \quad \text{CT\_NTH1TA}$$

$$\frac{\begin{array}{l}\Gamma \vdash_{\mathsf{co}} \gamma_1 : (\forall\, a_1{:}\,\kappa_1.\,\tau_1) \sim (\forall\, a_2{:}\,\kappa_2.\,\tau_2) \\ \Gamma \vdash_{\mathsf{co}} \gamma_2 : \sigma_1 \sim \sigma_2 \\ \Gamma \vdash_{\mathsf{ty}} \sigma_1 : \kappa_1 \quad \Gamma \vdash_{\mathsf{ty}} \sigma_2 : \kappa_2\end{array}}{\Gamma \vdash_{\mathsf{co}} \gamma_1 @ \gamma_2 : \tau_1[\sigma_1/a_1] \sim \tau_2[\sigma_2/a_2]} \quad \text{CT\_INST}$$

$$\frac{\Gamma \vdash_{\mathsf{co}} \gamma : (\forall\, c{:}\,\kappa_1 \sim \kappa_2.\,\tau) \sim (\forall\, c'{:}\,\kappa_1' \sim \kappa_2'.\,\tau')}{\Gamma \vdash_{\mathsf{co}} \mathbf{nth}^1\,\gamma : \kappa_1 \sim \kappa_1'} \quad \text{CT\_NTH1CA}$$

$$\frac{\Gamma \vdash_{\mathsf{co}} \gamma : (\forall\, c{:}\,\kappa_1 \sim \kappa_2.\,\tau) \sim (\forall\, c'{:}\,\kappa_1' \sim \kappa_2'.\,\tau')}{\Gamma \vdash_{\mathsf{co}} \mathbf{nth}^2\,\gamma : \kappa_2 \sim \kappa_2'} \quad \text{CT\_NTH2CA}$$

$$\frac{\begin{array}{l}\Gamma \vdash_{\mathsf{co}} \gamma : (\forall\, c_1{:}\,\phi_1.\,\tau_1) \sim (\forall\, c_2{:}\,\phi_2.\,\tau_2) \\ \Gamma \vdash_{\mathsf{co}} \gamma_1 : \phi_1 \quad \Gamma \vdash_{\mathsf{co}} \gamma_2 : \phi_2\end{array}}{\Gamma \vdash_{\mathsf{co}} \gamma @ (\gamma_1, \gamma_2) : \tau_1[\gamma_1/c_1] \sim \tau_2[\gamma_2/c_2]} \quad \text{CT\_INSTC}$$

$$\frac{\Gamma \vdash_{\mathsf{co}} \gamma : \tau_1 \sim \tau_2 \quad \Gamma \vdash_{\mathsf{ty}} \tau_1 : \kappa_1 \quad \Gamma \vdash_{\mathsf{ty}} \tau_2 : \kappa_2}{\Gamma \vdash_{\mathsf{co}} \mathbf{kind}\,\gamma : \kappa_1 \sim \kappa_2} \quad \text{CT\_EXT}$$

Figure 4. Coercion formation rules

tions, we can decompose a proof of the equivalence of two datatypes into equivalence proofs for any pair of type parameters (CT_NTH). Furthermore, the equivalence of two polymorphic types means that the kinds of the bound variables are equivalent (CT_NTH1TA), and that all instantiations of the bound variables are equivalent (CT_INST). The same is true for coercion abstraction types (rules CT_NTH1CA, CT_NTH2CA, and CT_INSTC).

- Equality is *heterogeneous*. If γ is a proof of the equality $\tau_1 \sim \tau_2$, then $\mathbf{kind}\,\gamma$ extracts a proof of equality between the kinds of τ_1 and τ_2.

4.3.1 Coercion irrelevance and coherence

Although the type system includes a judgement for type equality, and types may include explicit coercion proofs, the system does not include a judgement that states when two coercions are equal. The reason is that this relation is trivial. All coercions between equivalent proofs can be considered equivalent, so coercion proofs are irrelevant to type equality. As a result, FC is open to extension by new, consistent coercion axioms.

This "proof irrelevance" is reflected in several of the coercion rules. Consider the congruence rule for coercion application, CT_CAPP: there are no restrictions on γ_2 and γ_2' other than well-formedness. Another example is rule CT_INSTC—no relation is required between the coercions γ_1 and γ_2.

Not only is the identity of coercion proofs irrelevant, but it is always possible to equate a type with a casted version of itself. The coherence rule, CT_COH, essentially says that the use of kind coercions can be ignored when proving type equalities. Although this rule seems limited, it is sufficient to derive the elimination and congruence rules for coerced types, as seen below.

$$\frac{\Gamma \vdash_{\mathsf{co}} \gamma \;:\; \tau_1 \sim \tau_2 \quad \Gamma \vdash_{\mathsf{ty}} \tau_1 \rhd \eta_1 \;:\; \kappa_1 \quad \Gamma \vdash_{\mathsf{ty}} \tau_2 \rhd \eta_2 \;:\; \kappa_2}{\Gamma \vdash_{\mathsf{co}} (\mathbf{sym}\,((\mathbf{sym}\,\gamma) \rhd \eta_2)) \rhd \eta_1 \;:\; \tau_1 \rhd \eta_1 \sim \tau_2 \rhd \eta_2}$$

(Note that there is no relation required between η_1 and η_2.) We will use the syntactic sugar $\gamma \rhd \eta_1 \sim \eta_2$ for this coercion.

Likewise, coherence derives a proof term for decomposing equalities between coerced types.

$$\frac{\Gamma \vdash_{\mathsf{co}} \gamma \;:\; \tau_1 \rhd \gamma_1 \sim \tau_2 \rhd \gamma_2}{\Gamma \vdash_{\mathsf{co}} \mathbf{sym}\,(\langle\tau_1\rangle \rhd \gamma_1) \,⨾\, \gamma \,⨾\, \langle\tau_2\rangle \rhd \gamma_2 \;:\; \tau_1 \sim \tau_2}$$

4.3.2 Congruence rules for quantified types

In prior versions of FC, the coercion $\forall a{:}\,\kappa.\gamma$ proved the equality proposition $\forall\, a{:}\,\kappa.\,\tau_1 \sim \forall\, a{:}\,\kappa.\,\tau_2$, using the following rule:

$$\frac{\Gamma \vdash_{\mathsf{ty}} \kappa \;:\; \star \quad \Gamma,\, a{:}\,\kappa \vdash_{\mathsf{co}} \gamma \;:\; \tau_1 \sim \tau_2}{\Gamma \vdash_{\mathsf{co}} \forall\, a{:}\,\kappa.\,\gamma \;:\; (\forall\, a{:}\,\kappa.\,\tau_1) \sim (\forall\, a{:}\,\kappa.\,\tau_2)} \quad \text{CT\_ALLTX}$$

This rule sufficed because the only quantified types that could be shown equal had the same syntactic kinds κ for the bound variable. However, we now have a nontrivial equality between kinds. We need to be able to show a more general proposition, $\forall\, a{:}\,\kappa_1.\,\tau_1 \sim \forall\, a{:}\,\kappa_2.\,\tau_2$, even when κ_1 is not syntactically equal to κ_2.

Without this generality, the language does not satisfy the preservation theorem, which requires that the equality relation be substitutive—given a valid type σ where a appears free, and a proof $\Gamma \vdash_{\mathsf{co}} \gamma \;:\; \tau_1 \sim \tau_2$, we must be able to derive a proof between $\sigma[\tau_1/a]$ and $\sigma[\tau_2/a]$. For this property to hold, if a occurs in the kind of a quantified type (or coercion) variable $\forall\, b{:}\,a.\,\tau$, then we must be able to derive $\forall\, b{:}\,\tau_1.\,\tau \sim \forall\, b{:}\,\tau_2.\,\tau$.

Rule CT_ALLT shows when two polytypes are equal. The first premise requires a proof η that the kinds of the bound variables are equal. But, these two kinds might not be *syntactically* equal, so we must have two type variables, a_1 and a_2, one of each kind. The second premise of the rule adds both bindings $a_1{:}\,\kappa_1$ and $a_2{:}\,\kappa_2$ to the context as well as an assertion c that a_1 and a_2 are equal. The polytypes themselves can only refer to their own variables, as verified by the last two premises of the rule.

$\boxed{\vdash_{\mathsf{wf}} \Gamma}$

$$\frac{\Gamma \vdash_{\mathsf{ty}} \forall\, \overline{a{:}\,\kappa}.\,\star \;:\; \star \qquad T \# \Gamma}{\vdash_{\mathsf{wf}} \Gamma,\; T{:}\,\forall\, \overline{a{:}\,\kappa}.\,\star} \quad \text{GWF\_TYDATA}$$

$$\frac{\Gamma \vdash_{\mathsf{ty}} \forall\, \overline{a{:}\,\kappa}.\,\forall\,\Delta.\,(\overline{\sigma} \to T\;\overline{a}) \;:\; \star \qquad K \# \Gamma}{\vdash_{\mathsf{wf}} \Gamma,\; K{:}\,\forall\, \overline{a{:}\,\kappa}.\,\forall\,\Delta.\,(\overline{\sigma} \to T\;\overline{a})} \quad \text{GWF\_CON}$$

$$\frac{\Gamma, \Delta \vdash_{\mathsf{pr}} \phi\ \mathsf{ok} \qquad C \# \Gamma}{\vdash_{\mathsf{wf}} \Gamma,\; C{:}\,\forall\,\Delta.\,\phi} \quad \text{GWF\_AX}$$

Figure 5. Context formation rules (excerpted)

The other type form that includes binding is the coercion abstractions, $\forall\, c{:}\,\phi.\,\tau$. The rule CT_ALLC constructs a proof that two such types of this form are equal. We can only construct such proofs when the abstracted propositions relate correspondingly equal types, as witnessed by proofs η_1 and η_2. The proof term introduces two coercion variables into the context, similar to the two type variables above. Due to proof irrelevance, there is no need for a proof of equality between coercions themselves. Note that the kind of c_1 is *not* that of η_1: the kind of c_1 is built from types in both η_1 and η_2.

The rule CT_ALLC also restricts how the variables c_1 and c_2 can be used in γ. The premises $c_1 \# |\gamma|$ and $c_2 \# |\gamma|$ prevent these variables from appearing in the relevant parts of γ. The reason for this restriction comes from our proof technique for the consistency of this proof system. We define the erasure operation $|\cdot|$ and discuss this issue in more detail in Section 6.

4.4 Datatypes

Because the focus of this paper is on the treatment of equality in the type language, we omit most of the discussion of the expression language and its operational semantics. However, because we have collapsed types and kinds, we must revise the treatment of datatypes, whose constructors can contain types and kinds as arguments. Previously, the arguments to datatype constructors could be ordered with all kind arguments occurring before all type arguments [Yorgey et al. 2012]. In this language, we cannot divide up the arguments in this way. Therefore, we again use the technique of telescopes to describe arbitrary dependency between arguments.

The validity rules for contexts (see Figure 5) restrict datatype constants T to have a kind $\forall\, \overline{a{:}\,\kappa}.\,\star$. We call the variables $\overline{a}$ the *parameters* of the datatype. For example, the kind of the datatype List is $\forall\, a{:}\,\star.\,\star$ and the kind of the datatype TyRep (from Section 2) is $\forall\, k{:}\,\star,\; t{:}\,k.\,\star$. Furthermore, datatypes can only be parameterized by types and kinds; no coercion parameters are allowed.

Likewise, the same validity rules force data constructors K to have types/kinds of the form

$$\forall\, \overline{a{:}\,\kappa}.\,\forall\,\Delta.\,(\overline{\sigma} \to T\;\overline{a}).$$

Each data constructor K must produce an element of T applied to all of its parameters $\overline{a{:}\,\kappa}$. Above, form $\forall\,\Delta.\,\tau$ is syntactic sugar for a list of nested quantified types. The scope of the bound variables includes both the remainder of the telescope Δ and the form within the quantification (in this case, $\overline{\sigma} \to T\;\overline{a}$).

The telescope Δ describes the *existential* arguments to the data constructor. These arguments may be either coercions or types, and because of the dependency, must be allowed to freely intermix. For example, the data constructor TyInt from Section 2 (a data

$$\frac{\begin{array}{l} K{:}\,\forall\,\overline{a{:}\,\kappa}.\,\forall\,\Delta.\,\overline{\sigma} \to (T\;\overline{a}) \;\in\; \Gamma \\ \Psi = \mathsf{extend}(\mathsf{context}(\gamma); \overline{\rho}; \Delta) \\ \overline{\tau'} = \Psi_2(\overline{a}) \\ \overline{\rho}' = \Psi_2(dom\,\Delta) \\ \text{for each } e_i \in \overline{e}, \\ \qquad e_i' = e_i \rhd \Psi(\sigma_i) \end{array}}{\begin{array}{l} \textbf{case}\;((K\;\overline{\tau}\;\overline{\rho}\;\overline{e}) \rhd \gamma)\;\textbf{of}\;\overline{p \to u} \longrightarrow \\ \quad \textbf{case}\;(K\;\overline{\tau'}\;\overline{\rho}'\;\overline{e'})\;\textbf{of}\;\overline{p \to u} \end{array}} \quad \text{S\_KPUSH}$$

Figure 6. The S_KPUSH rule

constructor belonging to $\mathsf{TyRep} \;:\; \forall\, k{:}\,\star,\, t{:}\,k.\,\star)$ includes two coercions in its telescope, one asserting that the kind parameter k is $\star$, the second asserting that the type parameter t is Int:

$$\mathsf{TyInt} : \forall\, k{:}\,\star\,.\,\forall\, t{:}\,k.\,\forall\, c_1{:}\,k \sim \star.\,\forall\, c_2{:}\,t \sim \mathsf{Int}.\;\mathsf{TyRep}\;\kappa\;\tau$$

Alternatively, the data constructor TyApp existentially binds k', a, b, and c—one kind and two type variables followed by a coercion.

$$\begin{array}{c} \mathsf{TyApp} : \forall k{:}\,\star, t{:}\,k.\forall k'{:}\,\star, a{:}\,k' \to k, b{:}\,k', c{:}\,t \sim a\;b. \\ \mathsf{TyRep}\;(k' \to k)\;a \to \mathsf{TyRep}\;k'\;b \to \mathsf{TyRep}\;k\;t \end{array}$$

A datatype value is of the form $K\;\overline{\tau}\;\overline{\rho}\;\overline{e}$, where the $\overline{\tau}$ denote the parameters (which cannot include coercions), the $\overline{\rho}$ instantiate the existential arguments, and $\overline{e}$ is the list of usual expression arguments to the data constructor.

5. The "push" rules and the preservation theorem

The most intricate part of the operational semantics of FC are the "push" rules, which ensure that coercions do not interfere with the small step semantics. Coercions are "pushed" into the subcomponents of values whenever a coerced value appears in an elimination context. System FC has four push rules, one for each such context: term application, type application, coercion application, and pattern matching on a datatype. The first three are straightforward and are detailed in previous work [Yorgey et al. 2012]. In this section, we shall focus on pattern matching and the S_KPUSH rule.

5.1 Pushing coercions through constructors

When pattern matching on a coerced datatype value of the form $K\;\overline{\tau}\;\overline{\rho}\;\overline{e} \rhd \gamma$, the coercion must be distributed over all of the arguments of the data constructor, producing a new scrutinee $K\;\overline{\tau'}\;\overline{\rho}'\;\overline{e'}$ as shown in Figure 6. In the rest of this section, we explain the rule by describing the formation of the *lifting context* Ψ and its use in the definition of $\overline{\tau'}$, $\overline{\rho}'$ and $\overline{e'}$.

The S_KPUSH rule uses a *lifting* operation $\Psi(\cdot)$ on expressions which coerces the type of its argument (es in Figure 6). For example, suppose we have a data constructor K of type $\forall\; a{:}\;\star\;.\,F\;a \to T\;a$ for some type function F and some type constructor T. Consider what happens when an expression $K\;\mathsf{Int}\;e \rhd \gamma$ where γ is a coercion of type $T\;\mathsf{Int} \sim T\;\tau'$ is a scrutinee of a case expression. The push rule should convert this expression to $K\;\tau'\;(e \rhd \gamma')$ for some new coercion γ' showing $F\;\mathsf{Int} \sim F\;\tau'$. To produce this γ', we need to lift the type $F\;a$ into a coercion with respect to the coercion $\mathbf{nth}^1\;\gamma$, which shows $\mathsf{Int} \sim \tau'$.

In previous work, lifting was written $\sigma[a \mapsto \gamma]$, defined by analogy with substitution. Because of the similar syntax of types and coercion proofs, we could think of lifting as replacing a type variable with a coercion to produce a new coercion. That intuition holds true here, but we require more machinery to describe precisely.

Lifting contexts We define lifting with respect to a *lifting context* Ψ, which maps type variables to triples (τ_1, τ_2, γ) and coercion variables to pairs (η_1, η_2). The forms τ_1 and η_1 refer to the original, uncoerced parameters to the data constructor (Int in our example). The forms τ_2 and η_2 refer to the new, coerced parameters to the data constructor (like τ' in our example). Finally, the coercion γ witnesses the equality of τ_1 and τ_2. No witness is needed for the equality between η_1 and η_2—equality on proofs is trivial.

The lifting operation is defined by structural recursion on its type argument. This operation is complicated by the two type forms that bind fresh variables: $\forall\; a{:}\;\kappa.\;\tau$ and $\forall\; c{:}\;\phi.\;\tau$. Lifting over these types introduces new mappings in the lifting context, marked with $\overset{\bullet}{\mapsto}$ to indicate that they create new bindings.

$$\begin{array}{rclcl} \Psi ::= \emptyset & | & \Psi, a{:}\,\kappa \mapsto (\tau_1, \tau_2, \gamma) & | & \Psi, c{:}\,\phi \mapsto (\gamma_1, \gamma_2) \\ & | & \Psi, a{:}\,\kappa \overset{\bullet}{\mapsto} (a_1, a_2, c) & | & \Psi, c{:}\,\phi \overset{\bullet}{\mapsto} (c_1, c_2) \end{array}$$

(We use the notation $\overset{?}{\mapsto}$ to refer to a mapping created either with $\mapsto$ or with $\overset{\bullet}{\mapsto}$.) A lifting context Ψ induces two multisubstitutions $\Psi_1(\cdot)$ and $\Psi_2(\cdot)$, as follows:

Definition 5.1 (Lifting context substitution). *$\Psi_1(\cdot)$ and $\Psi_2(\cdot)$ are multisubstitutions, applicable to types, coercions, telescopes, typing contexts, and even other lifting contexts.*

1. *For each $a{:}\,\kappa \overset{?}{\mapsto} (\tau_1, \tau_2, \gamma)$ in Ψ, $\Psi_1(\cdot)$ maps a to τ_1 and $\Psi_2(\cdot)$ maps a to τ_2.*
2. *For each $c{:}\,\phi \overset{?}{\mapsto} (\gamma_1, \gamma_2)$ in Ψ, $\Psi_1(\cdot)$ maps c to γ_1 and $\Psi_2(\cdot)$ maps c to γ_2.*

Now, we can now state the judgement form $\Gamma \vdash_{\mathsf{lc}} \Delta \leftrightsquigarrow \Psi$, shown in Figure 7, which defines when a lifting context is valid and compatible with a given telescope.

$\boxed{\Gamma \vdash_{\mathsf{lc}} \Delta \leftrightsquigarrow \Psi}$ Lifting context validity

$$\frac{\vdash_{\mathsf{wf}} \Gamma}{\Gamma \vdash_{\mathsf{lc}} \varnothing \leftrightsquigarrow \varnothing} \quad \text{LC\_EMPTY}$$

$$\frac{\begin{array}{l} \Gamma \vdash_{\mathsf{lc}} \Delta \leftrightsquigarrow \Psi \quad a \# \Gamma, \Delta \\ \Gamma \vdash_{\mathsf{ty}} \sigma_1 \;:\; \Psi_1(\kappa) \quad \Gamma \vdash_{\mathsf{ty}} \sigma_2 \;:\; \Psi_2(\kappa) \\ \Gamma \vdash_{\mathsf{co}} \gamma \;:\; \sigma_1 \sim \sigma_2 \end{array}}{\Gamma \vdash_{\mathsf{lc}} (\Delta,\; a{:}\,\kappa) \leftrightsquigarrow (\Psi, a{:}\,\kappa \mapsto (\sigma_1, \sigma_2, \gamma))} \quad \text{LC\_TY}$$

$$\frac{\begin{array}{l} \Gamma \vdash_{\mathsf{lc}} \Delta \leftrightsquigarrow \Psi \quad c \# \Gamma, \Delta \\ \Gamma \vdash_{\mathsf{co}} \eta_1 \;:\; \Psi_1(\phi) \quad \Gamma \vdash_{\mathsf{co}} \eta_2 \;:\; \Psi_2(\phi) \end{array}}{\Gamma \vdash_{\mathsf{lc}} (\Delta,\; c{:}\,\phi) \leftrightsquigarrow (\Psi, c{:}\,\phi \mapsto (\eta_1, \eta_2))} \quad \text{LC\_CO}$$

$$\frac{\begin{array}{l} a_1 \# \Gamma, \Delta \quad a_2 \# \Gamma, \Delta \quad c \# \Gamma, \Delta \\ \Gamma \vdash_{\mathsf{lc}} \Delta \leftrightsquigarrow \Psi \end{array}}{\Gamma \vdash_{\mathsf{lc}} (\Delta,\; a{:}\,\kappa) \leftrightsquigarrow (\Psi, a{:}\,\kappa \overset{\bullet}{\mapsto} (a_1, a_2, c))} \quad \text{LC\_TYFRESH}$$

$$\frac{\Gamma \vdash_{\mathsf{lc}} \Delta \leftrightsquigarrow \Psi \quad c_1 \# \Gamma, \Delta \quad c_2 \# \Gamma, \Delta}{\Gamma \vdash_{\mathsf{lc}} (\Delta,\; c{:}\,\phi) \leftrightsquigarrow (\Psi, c{:}\,\phi \overset{\bullet}{\mapsto} (c_1, c_2))} \quad \text{LC\_COFRESH}$$

Figure 7. Lifting context validity

The two substitution operations satisfy straightforward substitution lemmas, defined and proved in the appendix. The usual substitution lemmas, which substitute a single type or coercion, are a corollary of these lemmas.

We can now state the *lifting* algorithm:

Definition 5.2 (Lifting). *We define the lifting of types to coercions, written $\Psi(\tau)$, by induction on the type structure. The following equations, to be tried in order, define the operation. (Note that the* *last line uses the syntactic sugar introduced in Section 4.3.1.)*

$$\begin{array}{ll}
\Psi(a) & = \gamma \textit{ when} \\
 & \quad a{:}\,\kappa \mapsto (\tau_1, \tau_2, \gamma) \in \Psi \\
\Psi(a) & = c \textit{ when} \\
 & \quad a{:}\,\kappa \overset{\bullet}{\mapsto} (a_1, a_2, c) \in \Psi \\
\Psi(\tau) & = \langle\tau\rangle \textit{ when} \\
 & \quad \textit{dom}\,(\Psi) \;\#\; \mathbf{fv}\,(\tau) \\
\Psi(\tau_1\,\tau_2) & = \Psi(\tau_1)\,\Psi(\tau_2) \\
\Psi(\tau\,\gamma) & = \Psi(\tau)(\Psi_1(\gamma), \Psi_2(\gamma)) \\
\Psi(\forall\, a{:}\,\kappa.\,\tau) & = \forall_{\Psi(\kappa)}(a_1, a_2, c).\Psi'(\tau) \\
 & \quad \textit{where } \Psi' = \Psi, a{:}\,\kappa \overset{\bullet}{\mapsto} (a_1, a_2, c) \\
 & \quad \textit{and } a_1, a_2, c \textit{ are fresh} \\
\Psi(\forall\, c{:}\,\sigma_1 \sim \sigma_2.\,\tau) & = \forall_{(\Psi(\sigma_1), \Psi(\sigma_2))}(c_1, c_2).\Psi'(\tau) \\
 & \quad \textit{where } \Psi' = \Psi, c{:}\,\sigma_1 \sim \sigma_2 \overset{\bullet}{\mapsto} (c_1, c_2) \\
 & \quad \textit{and } c_1, c_2 \textit{ are fresh} \\
\Psi(\tau \rhd \gamma) & = \Psi(\tau) \rhd \Psi_1(\gamma) \sim \Psi_2(\gamma)
\end{array}$$

The lifting lemma establishes the correctness of the lifting operation and shows that equality is congruent.

Lemma 5.3 (Lifting). *If Ψ is a valid lifting context with respect to the context Γ and the telescope Δ, and $\Gamma, \Delta \vdash_{\mathsf{ty}} \tau : \kappa$, then*

$$\Gamma \vdash_{\mathsf{co}} \Psi(\tau) \;:\; \Psi_1(\tau) \sim \Psi_2(\tau)$$

Lifting context creation In the S_KPUSH rule, the actual context Ψ used for lifting is built in two stages. First, $\mathsf{context}(\gamma)$ defines a lifting context with coercions for the parameters to the datatype.

Definition 5.4 (Lifting context generation). *If $\Gamma \vdash_{\mathsf{co}} \gamma : T\,\overline{\sigma} \sim T\,\overline{\sigma'}$, and $T{:}\,\forall\,\overline{a{:}\,\kappa}.\star \in \Gamma$, where the lists $\overline{\sigma}$, $\overline{\sigma'}$, and $\overline{a{:}\,\kappa}$ are all of length n, then define $\mathsf{context}(\gamma)$ as*

$$\mathsf{context}(\gamma) = \overline{a_i{:}\,\kappa_i \mapsto (\sigma_i, \sigma'_i, \mathbf{nth}^i\,\gamma)}^{\,i \in 1..n}$$

Intuitively, $(\mathsf{context}(\gamma))_1(\tau)$ replaces all parameters a in τ with the corresponding type on the left of $\sim$ in the type of γ. Similarly, $(\mathsf{context}(\gamma))_2(\tau)$ replaces with the corresponding type on the right of $\sim$.

The lifting context that results from this coercion is compatible with the parameters of the datatype. More precisely:

Lemma 5.5 (Lifting context specification). *If $\Gamma \vdash_{\mathsf{co}} \gamma : T\,\overline{\sigma} \sim T\,\overline{\tau}$, and $T{:}\,\forall\,\overline{a{:}\,\kappa}.\star \in \Gamma$ then $\Gamma \vdash_{\mathsf{c}} \overline{a{:}\,\kappa} \leftrightsquigarrow \mathsf{context}(\gamma)$.*

Proof. Straightforward induction. ☐

Next, this initial lifting context is extended with coercions using the operation $\mathsf{extend}(\cdot)$, which extends a lifting context with mappings for the variables in Δ, the existential parameters to the data constructor K. Because these arguments are dependent, we define the operation recursively. The intuition presented before still holds: $(\mathsf{extend}(\Psi; \overline{\rho}; \Delta))_1(\tau)$ replaces any parameter or existential argument in τ with its corresponding "from" type and $(\mathsf{extend}(\Psi; \overline{\rho}; \Delta))_2(\tau)$ replaces a variable with its corresponding "to" type.

Definition 5.6 (Lifting context extension). *Define the operation of lifting context extension, written $\mathsf{extend}(\Psi; \overline{\rho}; \Delta)$, as:*

$$\begin{array}{lcl}
\mathsf{extend}(\Psi; \varnothing; \varnothing) & = & \Psi \\
\mathsf{extend}(\Psi; \overline{\rho}, \tau; \Delta,\; a{:}\,\kappa) & = & \\
\quad \Psi', a{:}\,\kappa \mapsto (\tau, \tau \rhd \Psi'(\kappa), \mathbf{sym}\,(\langle\tau\rangle \rhd \Psi'(\kappa))) && \\
\quad \textit{where } \Psi' = \mathsf{extend}(\Psi; \overline{\rho}; \Delta) && \\
\mathsf{extend}(\Psi; \overline{\rho}, \gamma; \Delta,\; c{:}\,\sigma_1 \sim \sigma_2) & = & \\
\quad \Psi', c{:}\,\sigma_1 \sim \sigma_2 \mapsto (\gamma, \mathbf{sym}\,(\Psi'(\sigma_1)) \fatsemi \gamma \fatsemi \Psi'(\sigma_2)) && \\
\quad \textit{where } \Psi' = \mathsf{extend}(\Psi; \overline{\rho}; \Delta) &&
\end{array}$$

5.2 Correctness of push rules: The type erasure theorem

In the appendix, we prove that the push rules in the operational semantics satisfy type preservation. But, do they do "the right thing"—that is, do these rules reduce to no-ops if we erase types and coercions? More generally, we would like to know that types and coercions do not change the semantics of an expression—we want to prove that type erasure works as expected.

To state this formally, we define an erasure operation $|\cdot|$ over expressions. This operation erases types, coercions, and equality propositions to trivial forms $\bullet_{\mathsf{ty}}$, $\bullet_{\mathsf{co}}$ and $\bullet_{\mathsf{prop}}$ and removes all casts. The full definition of this operation appears in the appendix, and we present only the interesting cases here:

$$|e\,\tau| = |e|\,\bullet_{\mathsf{ty}} \qquad |e\,\gamma| = |e|\,\bullet_{\mathsf{co}} \qquad |e \rhd \gamma| = |e|$$

We can then prove that erasing types, coercions and casts does not change how expressions evaluate e.

Theorem 5.7 (Type erasure). *If $e \longrightarrow e'$, then either $|e| = |e'|$ or $|e| \longrightarrow |e'|$.*

5.3 Type preservation

Now that we have explained the most novel part of the operational semantics, we can state and prove the usual preservation theorem.

Theorem 5.8 (Preservation). *If $\Gamma \vdash_{\mathsf{tm}} e : \tau$ and $e \longrightarrow e'$ then $\Gamma \vdash_{\mathsf{tm}} e' : \tau$.*

The proof of this theorem is by induction on the typing derivation, with a case analysis on the small-step. Most of the rules are straightforward, following directly by induction or by substitution. The "push" rules require reasoning about coercion propagation. We include the details of the rules that differ from previous work [Weirich et al. 2010] in the appendix.

6. Consistency and the progress theorem

The progress theorem holds only for *closed*, *consistent* contexts. A context is *closed* if it does not contain any expression variable bindings—as usual, open expressions could be stuck. We use the metavariable Σ to denote closed contexts.

The definition of consistent contexts is stated using the notions of uncoerced *values* and their types, *value types*. Formally, we define values v and value types ξ, with the following grammars:

$$\begin{array}{lcl}
v & ::= & \lambda x{:}\,\sigma.\,e \mid \Lambda a{:}\,\kappa.\,e \mid \lambda c{:}\,\phi.\,e \mid K\,\overline{\tau}\,\overline{\rho}\,\overline{e} \\
\xi & ::= & \sigma_1 \to \sigma_2 \mid \forall\, a{:}\,\kappa.\,\sigma \mid \forall\, c{:}\,\phi.\,\sigma \mid T\,\overline{\sigma}
\end{array}$$

The canonical forms lemma tells us that the shape of a value is determined by its type:

Lemma 6.1 (Canonical Forms). *Say $\Sigma \vdash_{\mathsf{tm}} v : \sigma$. Then σ is a value type. Furthermore,*

1. *If $\sigma = \sigma_1 \to \sigma_2$ then v is $\lambda x{:}\,\sigma_1.\,e$ or $K\,\overline{\tau}\,\overline{\rho}\,\overline{e}$.*
2. *If $\sigma = \forall\, a{:}\,\kappa.\,\sigma'$ then v is $\Lambda a{:}\,\kappa.\,e$ or $K\,\overline{\tau}\,\overline{\rho}\,\overline{e}$.*
3. *If $\sigma = \forall\, c{:}\,\phi.\,\sigma'$ then v is $\lambda c{:}\,\tau_1 \sim \tau_2.\,e$ or $K\,\overline{\tau}\,\overline{\rho}\,\overline{e}$.*
4. *If $\sigma = T\,\overline{\tau}$ then v is $K\,\overline{\tau}\,\overline{\rho}\,\overline{e}$.*

Definition 6.2 (Consistency). *A context Γ is consistent if ξ_1 and ξ_2 have the same head form whenever $\Gamma \vdash_{\mathsf{co}} \gamma : \xi_1 \sim \xi_2$.*

With this definition of consistency, we can now state and prove the progress theorem.

Theorem 6.3 (Progress). *Assume Σ is a closed, consistent context. If $\Sigma \vdash_{\mathsf{tm}} e_1 : \tau$ and e_1 is not a value v or a coerced value $v \rhd \gamma$, then there exists an e_2 such that $e_1 \longrightarrow e_2$.*

The proof for the progress theorem proceeds exactly as in previous work [Weirich et al. 2010].

However, the challenge is showing that the contexts produced by elaboration of source Haskell programs are consistent. Our consistency argument proceeds in four steps:

1. We define an *implicitly coerced* version of the language, where coercion proofs have been erased. By working with the implicit language, our consistency and progress results do not depend on specific proofs. Derivations in the explicit language can be erased to derivations in the implicit language. (Definition 6.4)
2. We define a *rewrite relation* that reduces types in the implicit system by firing axioms in the context. (Figure 8)
3. We specify a sufficient condition, which we write $\mathbf{Good}\,\Gamma$ (Definition 6.6), for a context to be consistent. This condition allows the axioms produced by type and kind family definitions.
4. We show that good contexts are consistent (Lemma 6.8), by arguing that the joinability of the rewrite relation is complete with respect to the implicit coercion proof system. Since the rewrite relation and erasure preserve the head form of value types, this gives consistency for both the implicit and explicit systems.

Similar to surface Haskell, the implicit language elides coercion proofs and casts from the type language. The implicit language judgements (denoted with a turnstile $\models$) are analogous to the explicit language but for a few key differences. First, kind coercions no longer show up in types: they are implicit.

$$\frac{\Gamma \models \tau \;:\; \kappa \quad \Gamma \models \gamma \;:\; \kappa \sim \kappa' \quad \Gamma \models \kappa' \;:\; \star}{\Gamma \models \tau \;:\; \kappa'} \quad \text{IT\_CAST}$$

Note that this system is no longer syntax directed—a type may have several syntactically different kinds. This is not a problem, as we use this system as a proof device for progress only.

Second, the coercion in an application is erased to $\bullet_{\mathsf{co}}$.

$$\frac{\Gamma \models \tau \;:\; \forall\, c{:}\,\phi.\, \kappa \quad \Gamma \models \gamma \;:\; \phi}{\Gamma \models \tau \bullet_{\mathsf{co}} \;:\; \kappa} \quad \text{IT\_CAPP}$$

In general, we use $\bullet_{\mathsf{co}}$ to represent an elided coercion proof.

We define coercion proofs between erased types in a similar fashion. Most of the rules carry over from the explicitly typed system, but there are three major differences.

First, the implicit language does not include a coherence rule. In the explicit language, given a coercion proof $\Gamma \vdash_{\mathsf{co}} \gamma \;:\; \tau \sim \tau'$, the coherence rule was used to construct a proof $\gamma \rhd \gamma'$ where the kind of the first type τ is changed, by applying a cast $\tau \rhd \gamma'$. However, we can accomplish this in the implicit language, by using IT_CAST to *implicitly* cast the kind of τ using coercion γ'.

Second, the coercion application congruence rule is modified:

$$\frac{\begin{array}{l}\Gamma \models \gamma \;:\; \tau \sim \tau' \\ \Gamma \models \tau \bullet_{\mathsf{co}} \;:\; \kappa \quad \Gamma \models \tau' \bullet_{\mathsf{co}} \;:\; \kappa'\end{array}}{\Gamma \models \gamma(\bullet_{\mathsf{co}}, \bullet_{\mathsf{co}}) \;:\; \tau \bullet_{\mathsf{co}} \sim \tau' \bullet_{\mathsf{co}}} \quad \text{ICT\_CAPP}$$

This rule says that if two erased coercion applications are well formed, then if the two erased coercion abstractions are equal, there is a proof that the two applications are equal.

The final difference is in the rule for coercion abstractions:

$$\frac{\begin{array}{l}\Gamma \models \eta_1 \;:\; \sigma_1 \sim \sigma_1' \quad \phi_1 = \sigma_1 \sim \sigma_2 \\ \Gamma \models \eta_2 \;:\; \sigma_2 \sim \sigma_2' \quad \phi_2 = \sigma_1' \sim \sigma_2' \\ c_1 \,\#\, \gamma \quad c_2 \,\#\, \gamma \\ \Gamma,\, c_1{:}\,\phi_1,\, c_2{:}\,\phi_2 \models \gamma \;:\; \tau_1 \sim \tau_2 \\ \Gamma \models \forall\, c_1{:}\,\phi_1.\, \tau_1 \;:\; \star \quad \Gamma \models \forall\, c_2{:}\,\phi_2.\, \tau_2 \;:\; \star\end{array}}{\Gamma \models \forall_{(\eta_1,\eta_2)}(c_1, c_2).\gamma \;:\; (\forall\, c_1{:}\,\phi_1.\, \tau_1) \sim (\forall\, c_2{:}\,\phi_2.\, \tau_2)}$$

The requirement that c_1 and c_2 not appear in the (erased) coercion proof γ is for purely technical reasons. To prove consistency, our proof technique requires that equalities in the premise are also between types with the same head form. This is only true if the context is consistent, but c_1 and c_2 may be inconsistent: perhaps $c{:}\,\mathsf{Int} \sim \mathsf{Bool}$. If we are allowed to introduce these into the context, induction will fail. This is the primary motivation for restricting the coercion abstraction equality rule in the explicit system as well.

$$\frac{\begin{array}{l}\Gamma \vdash_{\mathsf{co}} \eta_1 \;:\; \sigma_1 \sim \sigma_1' \quad \phi_1 = \sigma_1 \sim \sigma_2 \\ \Gamma \vdash_{\mathsf{co}} \eta_2 \;:\; \sigma_2 \sim \sigma_2' \quad \phi_2 = \sigma_1' \sim \sigma_2' \\ c_1 \,\#\, |\gamma| \quad c_2 \,\#\, |\gamma| \\ \Gamma,\, c_1{:}\,\phi_1,\, c_2{:}\,\phi_2 \vdash_{\mathsf{co}} \gamma \;:\; \tau_1 \sim \tau_2 \\ \Gamma \vdash_{\mathsf{ty}} \forall\, c_1{:}\,\phi_1.\, \tau_1 \;:\; \star \quad \Gamma \vdash_{\mathsf{ty}} \forall\, c_2{:}\,\phi_2.\, \tau_2 \;:\; \star\end{array}}{\Gamma \vdash_{\mathsf{co}} \forall_{(\eta_1,\eta_2)}(c_1, c_2).\gamma \;:\; (\forall\, c_1{:}\,\phi_1.\, \tau_1) \sim (\forall\, c_2{:}\,\phi_2.\, \tau_2)}$$

The consequence of these restrictions is that there are some types that cannot be shown equivalent. For example, there is no proof of equivalence between the types $\forall\, c_1{:}\,\mathsf{Int} \sim b.\, \mathsf{Int}$ and $\forall\, c_2{:}\,\mathsf{Int} \sim b.\, b$—a coercion between these two types would need to use c_1 or c_2. However, this lack of expressiveness is not significant. In source Haskell, it would show up only through uses of first-class polymorphism, which are rare. Furthermore, this restriction already exists in GHC—currently, coercions between the types $(\mathsf{Int} \sim b) \Rightarrow \mathsf{Int}$ and $(\mathsf{Int} \sim b) \Rightarrow b$ are disallowed. It is possible that a completely different consistency proof would validate a rule that does not restrict the use of these variables. However, we leave this possibility to future work.

To connect the explicit and implicit systems, we define an erasure operation:

Definition 6.4 (Coercion Erasure)**.** *Given an explicitly typed term τ or coercion γ, we define its* erasure*, denoted $|\tau|$ or $|\gamma|$, by induction on its structure. The interesting cases follow:*

$$\begin{array}{lcllcl} |\tau \rhd \gamma| & = & |\tau| & |\gamma(\gamma_1, \gamma_2)| & = & |\gamma|(\bullet_{\mathsf{co}}, \bullet_{\mathsf{co}}) \\ |\tau\, \gamma| & = & |\tau| \bullet_{\mathsf{co}} & |\gamma \rhd \gamma'| & = & |\gamma| \\ & & & |\gamma@(\gamma', \gamma'')| & = & |\gamma|@(\bullet_{\mathsf{co}}, \bullet_{\mathsf{co}}) \end{array}$$

All other cases follow simply propagate the $|\cdot|$ operation down the abstract syntax tree. (The full definition of this operation appears in the appendix.)

We further define the erasure *of a context Γ, denoted $|\Gamma|$, by erasing the types and equality propositions of each binding.*

Lemma 6.5 (Erasure is type preserving)**.** *If a judgement holds in the explicit system, the judgement with coercions erased throughout the context, types and coercions is derivable in the implicit system.*

1. *If $\vdash_{\mathsf{wf}} \Gamma$ then $\models |\Gamma|$.*
2. *If $\Gamma \vdash_{\mathsf{ty}} \tau \;:\; \kappa$ then $|\Gamma| \models |\tau| \;:\; |\kappa|$.*
3. *If $\Gamma \vdash_{\mathsf{pr}} \phi$ ok then $|\Gamma| \models |\phi|$ ok.*
4. *If $\Gamma \vdash_{\mathsf{co}} \gamma \;:\; \phi$ then $|\Gamma| \models |\gamma| \;:\; |\phi|$.*
5. *If $\Gamma \vdash_{\mathsf{tel}} \overline{\rho} : \Delta$ then $|\Gamma| \models |\overline{\rho}| \;:\; |\Delta|$.*

Next, we define a non-deterministic rewrite relation on open implicit types in Figure 8. We also define a joinability relation, written $\Gamma \models \sigma_1 \Leftrightarrow \sigma_2$, if both σ_1 and σ_2 can multi-rewrite to a common reduct.

Consistency does not hold in arbitrary contexts, and it is difficult in general to check whether a context is inconsistent. Therefore, like in previous work [Weirich et al. 2010], we give sufficient conditions for an *erased* context to be consistent, written $\mathbf{Good}\,\Gamma$.

Definition 6.6 (Good contexts)**.** *We have $\mathbf{Good}\,\Gamma$ when the following conditions hold:*

1. *All coercion assumptions and axioms in Γ are of the form $C{:}\, \forall\, \Delta.\, (F\, \overline{\tau} \sim \tau')$ or of the form $c{:}\, a_1 \sim a_2$. In the first form, the arguments to the type function must behave like patterns.*

$$\boxed{\Gamma \models \tau \leadsto \tau'}$$

$$\frac{}{\Gamma \models \tau \leadsto \tau} \quad \text{TS\_Refl}$$

$$\frac{\Gamma, \Gamma' \models \kappa \leadsto \kappa' \quad \Gamma,\, c{:}\, a_1 \sim a_2, \Gamma' \models \sigma \leadsto \sigma'}{\Gamma, \Gamma' \models \forall\, a_1{:}\, \kappa.\, \sigma \leadsto \forall\, a_2{:}\, \kappa'.\, \sigma'} \quad \text{TS\_AllT}$$

$$\frac{\Gamma \models \tau_1 \leadsto \tau_1' \quad \Gamma \models \tau_2 \leadsto \tau_2' \quad \Gamma \models \sigma \leadsto \sigma'}{\Gamma \models \forall\, c{:}\, \tau_1 \sim \tau_2.\, \sigma \leadsto \forall\, c{:}\, \tau_1' \sim \tau_2'.\, \sigma'} \quad \text{TS\_AllC}$$

$$\frac{\begin{array}{c} \underline{C{:}\, \forall\, \Delta.\, (F\, \overline{\tau} \sim \tau')} \in \Gamma \\ \sigma_1 = \tau[\overline{\rho}/\Delta] \quad \sigma_1' = \tau'[\overline{\rho}/\Delta] \end{array}}{\Gamma \models F\, \overline{\sigma_1} \leadsto \sigma_1'} \quad \text{TS\_Red}$$

$$\frac{c{:}\, a \sim \tau \in \Gamma}{\Gamma \models a \leadsto \tau} \quad \text{TS\_VarRed}$$

$$\frac{\Gamma \models \tau \leadsto \tau' \quad \Gamma \models \sigma \leadsto \sigma'}{\Gamma \models \tau\, \sigma \leadsto \tau'\, \sigma'} \quad \text{TS\_App}$$

$$\frac{\Gamma \models \tau \leadsto \tau'}{\Gamma \models \tau\, \bullet_{\mathsf{co}} \leadsto \tau'\, \bullet_{\mathsf{co}}} \quad \text{TS\_CApp}$$

Figure 8. Rewrite relation

For every well formed $\overline{\rho}$, every $\tau_i \in \overline{\tau}$ and every τ_i' such that $\Gamma \models \tau_i[\overline{\rho}/\Delta] \leadsto \tau_i'$, it must be $\tau_i' = \tau_i[\overline{\rho'}/\Delta]$ for some $\overline{\rho}'$ with $\Gamma \models \sigma_m \leadsto \sigma_m'$ for each $\sigma_m \in \overline{\rho}$ and $\sigma_m' \in \overline{\rho}'$.

2. *There is no overlap between axioms and coercion assumptions. For each $F\ \overline{\rho}$ there exists at most one prefix $\overline{\rho}_1$ of $\overline{\rho}$ such that there exist C and σ where $\Gamma \models C\, \overline{\rho}_1 \ :\ (F\ \overline{\rho_1} \sim \sigma)$. This C is unique for every matching $F\ \overline{\tau_1}$.*
3. *For each a, there is at most one assumption of the form $c{:}\ a \sim a'$ or $c{:}\ a' \sim a$, and $a \neq a'$.*
4. *Axioms equate types of the same kind. For each $C{:}\ \forall\, \Delta.\, (F\, \overline{\tau} \sim \tau')$ in Γ, the kinds of each side must equal: for some κ, $\Gamma, \Delta \models F\ \overline{\tau}\ :\ \kappa$ and $\Gamma, \Delta \models \tau'\ :\ \kappa$ and that kind must not mention bindings in the telescope, $\Gamma \models \kappa\ :\ \star$.*

In the rest of this section, we sketch the proof that good contexts are consistent. Our approach is similar to previous work [Weirich et al. 2010], but differs in two ways. First, the rewrite relation works on types in the implicit language. Second, the rewrite relation is not type directed: rewrite rules are guided by coercion axioms.

The main lemma required for consistency is the completeness of joinability. Here, we write $fcv(\gamma) \subseteq dom\, \Gamma'$ to indicate that all coercion variables and axioms used in γ are in the domain of Γ'. The proof appears in the appendix.

Lemma 6.7 (Completeness)**.** *Suppose that $\Gamma \models \gamma\ :\ \sigma_1 \sim \sigma_2$, and $fcv(\gamma) \subseteq dom\, \Gamma'$ for some subcontext Γ' satisfying $\mathbf{Good}\, \Gamma'$. Then $\Gamma \models \sigma_1 \Leftrightarrow \sigma_2$.*

Lemma 6.8 (Consistency)**.** *If $\mathbf{Good}\, |\Gamma|$ then Γ is consistent.*

7. Discussion and related work

Collapsing kinds and types Blurring the distinction between types and kinds is convenient, but is it wise? It is well known that type systems that include the $\Gamma \vdash_{\mathsf{ty}} \star\ :\ \star$ rule are inconsistent logics [Girard 1972]. Does that cause trouble? The answer is no—inconsistency here means that all kinds are inhabited. But the FC type language, even without our extensions, is already inconsistent as type families can be declared at any kind.

It is not clear whether adding the $\Gamma \vdash_{\mathsf{ty}} \star\ :\ \star$ rule to source Haskell would cause type inference to loop as the type language does not include anonymous abstractions. However, the presence of nonterminating type expressions only threatens the decidability of type inference—once a type equality has been decided by the constraint solver, it is elaborated to a finite equality proof. In FC, type checking is always decidable.

Other languages that adopt dependent types and the "type-in-type" axiom [Augustsson 1998; Cardelli 1986] do not have decidable type checking. These languages do not make the same distinctions that FC does between expressions, types and coercions, separating logically inconsistent types from logically consistent equality proofs. The FC coercion language is limited in expressive power compared to these other languages and the consistency of this language (i.e., that there are equalities that cannot be derived) is a consequence of this limitation. This interplay between an inconsistent programming language and a consistent metalogic is also the subject of current research in the Trellys project [Casinghino et al. 2012; Kimmell et al. 2012].

Heterogeneous equality Heterogeneous equality is an essential part of this system. It is primarily motivated by the presence of dependent application (such as rules K_Inst and K_CApp), where the kind of the result depends on the value of the argument. We would like type equivalence to be congruent with respect to application, as is demonstrated by rule CT_App. However, if all equalities are required to be homogeneous, then not all uses of the rule are valid because the result kinds may differ.

For example, consider the datatype TyRep of kind $\forall\, a{:}\, \star.\, \forall\, b{:}\, \star.\, \star$. If we have coercions $\Gamma \vdash_{\mathsf{co}} \gamma_1\ :\ \star \sim \kappa$ and $\Gamma \vdash_{\mathsf{co}} \gamma_2\ :\ \mathsf{Int} \sim \tau$, then we can construct the proof

$$\Gamma \vdash_{\mathsf{co}} \langle \mathsf{TyRep} \rangle\, \gamma_1\, \gamma_2\ :\ \mathsf{TyRep}\ \star\ \mathsf{Int} \sim \mathsf{TyRep}\, \kappa\, \tau$$

However, this proof requires heterogeneity because the first part ($\langle \mathsf{TyRep} \rangle\, \gamma_1$) creates an equality between types of different kinds: $\mathsf{TyRep}\, \star$ and $\mathsf{TyRep}\, \kappa$. The first has kind $\star \to \star$, whereas the second has kind $\kappa \to \star$.

The coherence rule (CT_Coh) also requires that equality be heterogeneous because it equates types that almost certainly have different kinds. This rule, inspired by Observational Type Theory [Altenkirch et al. 2007], provides a simple way of ensuring that proofs do not interfere with equality. Without it, we would need coercions analogous to the many "push" rules of the operational semantics.

There are several choices in the semantics of heterogeneous equality. We have chosen the most popular, where a proposition $\sigma_1 \sim \sigma_2$ is interpreted as a conjunction: "the types are equal and their kinds are equal". This semantics is similar to Epigram 1 [McBride 2002], the `HeterogeneousEquality` module in the Agda standard library,[3] and the treatment in Coq.[4] Epigram 2 [Altenkirch et al. 2007] uses an alternative semantics, interpreted as "if the kinds are equal than the types are equal". Guru [Stump et al. 2008] and Trellys [Kimmell et al. 2012; Sjöberg et al. 2012], use yet another interpretation which says nothing about the kinds. These differences arise from aspects of the overall type system—the syntax-directed type system of FC make the conjunctive interpretation the most reasonable, whereas the bidirectional type system of Epigram 2 makes the implicational version more convenient.

Unlike higher-dimensional type theory [Licata and Harper 2012], equality in this language has no computational content. Because of the separation between objects and proofs, FC is resolutely one-dimensional—we do not define what it means for proofs to be

[3] `http://wiki.portal.chalmers.se/agda/agda.php?n=Libraries.StandardLibrary`

[4] `http://coq.inria.fr/stdlib/Coq.Logic.JMeq.html`

equivalent. Instead, we ensure that in any context the identity of equality proofs is unimportant.

The implicit language Our proof technique for consistency, based on erasing explicit type conversions, is inspired by ICC [Miquel 2001]. Coercion proofs are irrelevant to the definition of type equality, so to reason about type equality it is convenient to eliminate them entirely. Following ICC* [Barras and Bernardo 2008], we could alternatively view the implicit language as the "real" semantics for FC, and then consider the language of this paper as an adaptation of that semantics with annotations to make typing decidable. Furthermore, the implicit language is interesting in its own right as it is closer to source Haskell, which also makes implicit use of type equalities.

However, although the implicit language allows type equality assumptions to be used implicitly, it is not extensional type theory (ETT) [Martin-Löf 1984]. Foremost, it separates proofs from programs so that it can weaken the former (ensuring consistency) while enriching the latter (with "type-in-type"). The proof language of FC is not as expressive as ETT. We have discussed the limitations on equalities between coercion abstractions in Section 6. Another difference is the lack of η-equivalence or extensional reasoning for type-level functions.

8. Conclusions and future work

This work provides the basis for the practical extension of a popular programming language implementation. It does so without sacrificing any important metatheoretic properties. This extension is a necessary step towards making Haskell more dependently typed. The next step in this research plan is to lift these extensions to the source language, incorporating these features within GHC's constraint solving algorithm. In particular, we plan future language extensions in support of type- and kind-level programming, such as datakinds (datatypes that exist only at the kind-level), kind synonyms and kind families. Although GHC already infers kinds, we will need to extend this mechanism to generate kind coercions and take advantage of these new features. We are also aware that Adam Gundry's forthcoming dissertation will include Π-types in a version of System FC,[5] and we will want to make this feature available in the source language as well.

Furthermore, even though this version of FC combines types and kinds, the Haskell source language need not do so—predictable type inference algorithms may require more traditional stratification. This gap would not be new—the desires of a simple core language have already lead FC to be more expressive than source Haskell.

Although the interaction between dependent types and type inference brings new research challenges, these challenges can be addressed in the context of a firm semantic basis.

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A. Additional semantics

Below, we list a few rules and definitions not included in the main discussion.

A.1 Context well-formedness

These rules ensure that all assumptions in the context are well formed and unique. They additionally constrain the form of the kinds of datatypes and the types of data constructors. $\boxed{\vdash_{\mathsf{wf}} \Gamma}$

$$\frac{}{\vdash_{\mathsf{wf}} \varnothing} \quad \text{GWF\_EMPTY}$$

$$\frac{\Gamma \vdash_{\mathsf{ty}} \kappa : \star \quad a \# \Gamma}{\vdash_{\mathsf{wf}} \Gamma,\ a{:}\,\kappa} \quad \text{GWF\_TYVAR}$$

$$\frac{\Gamma \vdash_{\mathsf{ty}} \kappa : \star \quad F \# \Gamma}{\vdash_{\mathsf{wf}} \Gamma,\ F{:}\,\kappa} \quad \text{GWF\_TYFUN}$$

$$\frac{\Gamma \vdash_{\mathsf{ty}} \forall\, \overline{a{:}\,\kappa}. \star : \star \quad T \# \Gamma}{\vdash_{\mathsf{wf}} \Gamma,\ T{:}\forall\, \overline{a{:}\,\kappa}. \star} \quad \text{GWF\_TYDATA}$$

$$\frac{\Gamma \vdash_{\mathsf{ty}} \tau : \kappa \quad x \# \Gamma}{\vdash_{\mathsf{wf}} \Gamma,\ x{:}\,\tau} \quad \text{GWF\_VAR}$$

$$\frac{\Gamma \vdash_{\mathsf{ty}} \forall\, \overline{a{:}\,\kappa}. \forall\, \Delta. (\overline{\sigma} \to T\ \overline{a}) : \star \quad K \# \Gamma}{\vdash_{\mathsf{wf}} \Gamma,\ K{:} \forall\, \overline{a{:}\,\kappa}. \forall\, \Delta. (\overline{\sigma} \to T\ \overline{a})} \quad \text{GWF\_CON}$$

$$\frac{\Gamma \vdash_{\mathsf{pr}} \phi\ \mathsf{ok} \quad c \# \Gamma}{\vdash_{\mathsf{wf}} \Gamma,\ c{:}\,\phi} \quad \text{GWF\_CVAR}$$

$$\frac{\Gamma, \Delta \vdash_{\mathsf{pr}} \phi\ \mathsf{ok} \quad C \# \Gamma}{\vdash_{\mathsf{wf}} \Gamma,\ C{:}\ \forall\, \Delta. \phi} \quad \text{GWF\_AX}$$

A.2 Telescope argument validity

$\boxed{\Gamma \vdash_{\mathsf{tel}} \overline{\rho} : \Delta}$

$$\frac{\vdash_{\mathsf{wf}} \Gamma}{\Gamma \vdash_{\mathsf{tel}} \varnothing : \varnothing} \quad \text{T2\_EMPTY}$$

$$\frac{\begin{array}{c}\Gamma, \Delta \vdash_{\mathsf{ty}} \kappa : \star \quad a \# \Gamma, \Delta \\ \Gamma \vdash_{\mathsf{ty}} \tau : \kappa[\overline{\rho}/\Delta] \quad \Gamma \vdash_{\mathsf{tel}} \overline{\rho} : \Delta\end{array}}{\Gamma \vdash_{\mathsf{tel}} \overline{\rho}, \tau : (\Delta,\ a{:}\,\kappa)} \quad \text{T2\_CONST}$$

$$\frac{\begin{array}{c}\Gamma, \Delta \vdash_{\mathsf{pr}} \phi\ \mathsf{ok} \quad c \# \Gamma, \Delta \\ \Gamma \vdash_{\mathsf{co}} \gamma : \phi[\overline{\rho}/\Delta] \quad \Gamma \vdash_{\mathsf{tel}} \overline{\rho} : \Delta\end{array}}{\Gamma \vdash_{\mathsf{tel}} \overline{\rho}, \gamma : (\Delta,\ c{:}\,\phi)} \quad \text{T2\_CONSG}$$

A.3 Expression typing and operational semantics

$\boxed{\Gamma \vdash_{\mathsf{tm}} e : \tau}$ Expression typing

$$\frac{\vdash_{\mathsf{wf}} \Gamma \quad x{:}\,\tau \in \Gamma}{\Gamma \vdash_{\mathsf{tm}} x : \tau} \quad \text{T\_VAR}$$

$$\frac{\Gamma,\ x{:}\,\tau_1 \vdash_{\mathsf{tm}} e : \tau_2}{\Gamma \vdash_{\mathsf{tm}} \lambda x{:}\,\tau_1.\, e : \tau_1 \to \tau_2} \quad \text{T\_ABS}$$

$$\frac{\Gamma \vdash_{\mathsf{tm}} e : \tau_1 \to \tau_2 \quad \Gamma \vdash_{\mathsf{tm}} u : \tau_1}{\Gamma \vdash_{\mathsf{tm}} e\, u : \tau_2} \quad \text{T\_APP}$$

$$\frac{\Gamma,\ c{:}\,\phi \vdash_{\mathsf{tm}} e : \tau \quad \Gamma \vdash_{\mathsf{pr}} \phi\ \mathsf{ok}}{\Gamma \vdash_{\mathsf{tm}} \lambda c{:}\,\phi.\, e : \forall\, c{:}\,\phi.\, \tau} \quad \text{T\_CABS}$$

$$\frac{\begin{array}{l}\Gamma \vdash_{\mathsf{tm}} e : \forall\, c{:}\,\phi.\, \tau \\ \Gamma \vdash_{\mathsf{co}} \gamma : \phi\end{array}}{\Gamma \vdash_{\mathsf{tm}} e\, \gamma : \tau[\gamma/c]} \quad \text{T\_CAPP}$$

$$\frac{\Gamma,\ a{:}\,\kappa \vdash_{\mathsf{tm}} e : \tau}{\Gamma \vdash_{\mathsf{tm}} \Lambda a{:}\,\kappa.\, e : \forall\, a{:}\,\kappa.\, \tau} \quad \text{T\_TABS}$$

$$\frac{\Gamma \vdash_{\mathsf{tm}} e : \forall\, a{:}\,\kappa.\, \tau \quad \Gamma \vdash_{\mathsf{ty}} \tau' : \kappa}{\Gamma \vdash_{\mathsf{tm}} e\, \tau' : \tau[\tau'/a]} \quad \text{T\_TAPP}$$

$$\frac{\Gamma \vdash_{\mathsf{tm}} e : \tau_1 \quad \Gamma \vdash_{\mathsf{co}} \gamma : \tau_1 \sim \tau_2 \quad \Gamma \vdash_{\mathsf{ty}} \tau_2 : \star}{\Gamma \vdash_{\mathsf{tm}} e \rhd \gamma : \tau_2} \quad \text{T\_CAST}$$

$$\frac{\vdash_{\mathsf{wf}} \Gamma \quad K{:}\,\tau \in \Gamma}{\Gamma \vdash_{\mathsf{tm}} K : \tau} \quad \text{T\_CON}$$

$$\frac{\begin{array}{l}\Gamma \vdash_{\mathsf{tm}} e : T\ \overline{\tau'} \\ \Gamma \vdash_{\mathsf{ty}} \tau : \star \\ \overline{K_i}\ \text{are exhaustive for}\ T \\ \text{for each}\ i \\ \quad K_i{:}\, \forall\, \overline{a{:}\,\kappa}. \forall\, \Delta_i.\, \overline{\sigma_i} \to (T\ \overline{a}) \in \Gamma \\ \quad \Delta'_i = \Delta_i[\overline{\tau'}/a] \\ \quad \sigma'_i = \sigma_i[\overline{\tau'}/\overline{a}] \\ \quad \Gamma, \Delta'_i, \overline{x_i{:}\,\sigma'_i} \vdash_{\mathsf{tm}} u_i : \tau\end{array}}{\Gamma \vdash_{\mathsf{tm}} \mathbf{case}\ e\ \mathbf{of}\ \overline{K_i\ \Delta'_i\ \overline{x_i{:}\,\sigma'_i} \to u_i} : \tau} \quad \text{T\_CASE}$$

$$\frac{\begin{array}{l}\Gamma \vdash_{\mathsf{co}} \gamma : H_1\, \overline{\rho}_1 \sim H_2\, \overline{\rho}_2 \quad H_1 \neq H_2 \\ \Gamma \vdash_{\mathsf{ty}} \tau : \star\end{array}}{\Gamma \vdash_{\mathsf{tm}} \mathbf{contra}\, \gamma\, \tau : \tau} \quad \text{T\_CONTRA}$$

$\boxed{e \longrightarrow e'}$ Step reduction, parameterized by toplevel context

$$\frac{}{(\lambda x{:}\,\tau.\, e)\ e' \longrightarrow e[e'/x]} \quad \text{S\_BETA}$$

$$\frac{e_1 \longrightarrow e'_1}{e_1\ e_2 \longrightarrow e'_1\ e_2} \quad \text{S\_EAPP}$$

$$\frac{\Gamma \vdash_{\mathsf{co}} \gamma \ :\ \sigma_1 \to \sigma_2 \sim \tau_1 \to \tau_2}{(v \rhd \gamma)\; e \longrightarrow (v\;(e \rhd \mathbf{sym}\,(\mathbf{nth}^1\,\gamma))) \rhd \mathbf{nth}^2\,\gamma}\ \textsc{S\_Push}$$

$$\frac{}{(\Lambda a{:}\,\kappa.\; e)\;\tau \longrightarrow e[\tau/a]}\ \textsc{S\_TBeta}$$

$$\frac{e_1 \longrightarrow e_1'}{e_1\;\sigma \longrightarrow e_1'\;\sigma}\ \textsc{S\_TApp}$$

$$\frac{\begin{array}{l}\Gamma \vdash_{\mathsf{co}} \gamma \ :\ \forall\, a{:}\,\kappa_1.\,\sigma_1 \sim \forall\, a{:}\,\kappa_2.\,\sigma_2\\ \gamma' = \mathbf{sym}\,(\mathbf{nth}^1\,\gamma)\\ \tau' = \tau \rhd \gamma'\end{array}}{(v \rhd \gamma)\;\tau \longrightarrow (v\;\tau') \rhd \gamma@(\langle\tau\rangle \rhd \gamma')}\ \textsc{S\_TPush}$$

$$\frac{}{(\lambda c{:}\,\sigma_1 \sim \sigma_2.\; e)\;\gamma \longrightarrow e[\gamma/c]}\ \textsc{S\_CBeta}$$

$$\frac{e_1 \longrightarrow e_1'}{e_1\;\gamma \longrightarrow e_1'\;\gamma}\ \textsc{S\_CApp}$$

$$\frac{\begin{array}{l}\Gamma \vdash_{\mathsf{co}} \gamma \ :\ (\forall\, c{:}\,\phi.\,\tau) \sim (\forall\, c'{:}\,\phi'.\,\tau')\\ \gamma'' = \mathbf{nth}^1\,\gamma \fatsemi \gamma' \fatsemi \mathbf{sym}\,(\mathbf{nth}^2\,\gamma)\end{array}}{(v \rhd \gamma)\;\gamma' \longrightarrow v\;\gamma'' \rhd \gamma@(\gamma'', \gamma')}\ \textsc{S\_CPush}$$

$$\frac{}{(v \rhd \gamma_1) \rhd \gamma_2 \longrightarrow v \rhd (\gamma_1 \fatsemi \gamma_2)}\ \textsc{S\_Comb}$$

$$\frac{e \longrightarrow e'}{e \rhd \gamma \longrightarrow e' \rhd \gamma}\ \textsc{S\_Coerce}$$

$$\frac{K_i\;\Delta_i\;\overline{x_i{:}\,\sigma_i} \to u_i \in \overline{p \to u}}{\mathbf{case}\;K_i\;\overline{\tau}\;\overline{\rho}\;\overline{e}\;\mathbf{of}\;\overline{p \to u} \longrightarrow u_i\,[\overline{e}/\overline{x_i}]\,[\overline{\rho}/\Delta_i]}\ \textsc{S\_CaseMatch}$$

$$\frac{e \longrightarrow e'}{\mathbf{case}\;e\;\mathbf{of}\;\overline{p \to u} \longrightarrow \mathbf{case}\;e'\;\mathbf{of}\;\overline{p \to u}}\ \textsc{S\_Case}$$

$$\frac{\begin{array}{l}K{:}\,\forall\,\overline{a{:}\,\kappa}.\,\forall\,\Delta.\,\overline{\sigma} \to (T\;\overline{a}) \in \Gamma\\ \Psi = \mathsf{extend}(\mathsf{context}(\gamma);\overline{\rho};\Delta)\\ \overline{\tau'} = \Psi_2(\overline{a})\\ \overline{\rho}' = \Psi_2(dom\,\Delta)\\ \text{for each } e_i \in \overline{e},\\ \qquad e_i' = e_i \rhd \Psi(\sigma_i)\end{array}}{\begin{array}{l}\mathbf{case}\;((K\;\overline{\tau}\;\overline{\rho}\;\overline{e}) \rhd \gamma)\;\mathbf{of}\;\overline{p \to u} \longrightarrow\\ \quad\mathbf{case}\;(K\;\overline{\tau'}\;\overline{\rho}'\;\overline{e'})\;\mathbf{of}\;\overline{p \to u}\end{array}}\ \textsc{S\_KPush}$$

A.4 Erasure operation

$$\begin{array}{lcl}
|x| & = & x\\
|\lambda x{:}\,\tau.\;e| & = & \lambda x{:}\,\bullet_{\mathsf{ty}}.\;|e|\\
|e_1\;e_2| & = & |e_1|\;|e_2|\\
|\Lambda a{:}\,\kappa.\;e| & = & \Lambda a{:}\,\bullet_{\mathsf{ty}}.\;|e|\\
|e\;\tau| & = & |e|\;\bullet_{\mathsf{ty}}\\
|\lambda c{:}\,\phi.\;e| & = & \lambda c{:}\,\bullet_{\mathsf{prop}}.\;|e|\\
|e\;\gamma| & = & |e|\;\bullet_{\mathsf{co}}\\
|e \rhd \gamma| & = & |e|\\
|K| & = & K\\
|\mathbf{case}\;e\;\mathbf{of}\;\overline{p \to u}| & = & \mathbf{case}\;|e|\;\mathbf{of}\;\overline{p \to |u|}\\
|\mathbf{contra}\;\gamma\;\tau| & = & \mathbf{contra}\;\bullet_{\mathsf{co}}\;\bullet_{\mathsf{ty}}\\
\\
|a| & = & a\\
|H| & = & H\\
|F| & = & F\\
|K| & = & K\\
|\forall\, a{:}\,\kappa.\;\tau| & = & \forall\, a{:}\,|\kappa|.\;|\tau|\\
|\forall\, c{:}\,\phi.\;\tau| & = & \forall\, c{:}\,|\phi|.\;|\tau|\\
|\tau_1\;\tau_2| & = & |\tau_1|\;|\tau_2|\\
|\tau_1 \rhd \gamma| & = & |\tau_1|\\
|\tau_1\;\gamma| & = & |\tau_1|\;\bullet_{\mathsf{co}}\\
\\
|\sigma_1 \sim \sigma_2| & = & |\sigma_1| \sim |\sigma_2|\\
\\
|c| & = & c\\
|C\;\overline{\rho}| & = & C\;|\overline{\rho}|\\
|\langle\tau\rangle| & = & \langle|\tau|\rangle\\
|\mathbf{sym}\;\gamma| & = & \mathbf{sym}\;|\gamma|\\
|\gamma_1 \fatsemi \gamma_2| & = & |\gamma_1| \fatsemi |\gamma_2|\\
|\forall_\eta(a_1, a_2, c).\gamma| & = & \forall_{|\eta|}(a_1, a_2, c).|\gamma|\\
|\forall_{(\eta_1,\eta_2)}(c_1, c_2).\gamma| & = & \forall_{(|\eta_1|,|\eta_2|)}(c_1, c_2).|\gamma|\\
|\gamma_1\;\gamma_2| & = & |\gamma_1|\;|\gamma_2|\\
|\gamma(\gamma_1, \gamma_2)| & = & |\gamma|(\bullet_{\mathsf{co}}, \bullet_{\mathsf{co}})\\
|\gamma \rhd \gamma'| & = & |\gamma|\\
|\gamma@\gamma'| & = & |\gamma|@|\gamma'|\\
|\gamma@(\gamma', \gamma'')| & = & |\gamma|@(\bullet_{\mathsf{co}}, \bullet_{\mathsf{co}})\\
|\mathbf{nth}^i\;\gamma| & = & \mathbf{nth}^i\;|\gamma|\\
|\mathbf{kind}\;\gamma| & = & \mathbf{kind}\;|\gamma|\\
\\
|\rho|\ (\text{where } \rho = \tau) & = & |\tau|\\
|\rho|\ (\text{where } \rho = \gamma) & = & \bullet_{\mathsf{co}}\\
\\
|\varnothing| & = & \varnothing\\
|\Gamma,\ a{:}\,\kappa| & = & |\Gamma|,\ a{:}\,|\kappa|\\
|\Gamma,\ c{:}\,\phi| & = & |\Gamma|,\ c{:}\,|\phi|\\
|\Gamma,\ C{:}\,\forall\,\Delta.\;\phi| & = & |\Gamma|,\ C{:}\,\forall\,|\Delta|.\;|\phi|
\end{array}$$

A.5 Implicit Language Typing

$\boxed{\models \Gamma}$ Implicit Validity

$$\frac{}{\models \varnothing}\ \textsc{IV\_Empty}$$

$$\frac{\Gamma \models \kappa \ :\ \star \quad a \mathrel{\#} \Gamma}{\models \Gamma,\ a{:}\,\kappa}\ \textsc{IV\_TyVar}$$

$$\frac{\Gamma \models \kappa \ :\ \star \quad F \mathrel{\#} \Gamma}{\models \Gamma,\ F{:}\,\kappa}\ \textsc{IV\_TyFun}$$

$$\frac{\Gamma \models \forall\,\overline{a{:}\,\kappa}.\star \ :\ \star \quad T \mathrel{\#} \Gamma}{\models \Gamma,\ T{:}\,\forall\,\overline{a{:}\,\kappa}.\star}\ \textsc{IV\_TyData}$$

$$\frac{\Gamma \models \tau \ :\ \kappa \quad x \mathrel{\#} \Gamma}{\models \Gamma,\ x{:}\,\tau}\ \textsc{IV\_Var}$$

$$\frac{\Gamma \models \forall\, \overline{a{:}\kappa}.\, \forall\, \Delta.\, (\overline{\sigma} \to T\ \overline{a})\ :\ \star \quad K \# \Gamma}{\models \Gamma,\ K{:}\forall\, \overline{a{:}\kappa}.\, \forall\, \Delta.\, (\overline{\sigma} \to T\ \overline{a})}\ \textsc{IV\_Con}$$

$$\frac{\Gamma \models \phi\ \mathsf{ok} \quad c \# \Gamma}{\models \Gamma,\ c{:}\,\phi}\ \textsc{IV\_CVar}$$

$$\frac{\Gamma, \Delta \models \phi\ \mathsf{ok} \quad C \# \Gamma}{\models \Gamma,\ C{:}\ \forall\, \Delta.\, \phi}\ \textsc{IV\_Ax}$$

$\boxed{\Gamma \models \phi\ \mathsf{ok}}$ Implicit coercion kind well-formedness

$$\frac{\begin{array}{c}\Gamma \models \sigma_1\ :\ \kappa_1 \\ \Gamma \models \sigma_2\ :\ \kappa_2\end{array}}{\Gamma \models \sigma_1 \sim \sigma_2\ \mathsf{ok}}\ \textsc{IP\_Equality}$$

$\boxed{\Gamma \models \tau\ :\ \kappa}$ Implicit Kinding

$$\frac{\models \Gamma}{\Gamma \models \star\ :\ \star}\ \textsc{IT\_StarInStar}$$

$$\frac{\models \Gamma}{\Gamma \models (\to)\ :\ \star \to \star \to \star}\ \textsc{IT\_Arrow}$$

$$\frac{\models \Gamma}{\Gamma \models \star \to \star \to \star\ :\ \star}\ \textsc{IT\_ArrowK}$$

$$\frac{\models \Gamma \quad w{:}\,\kappa \in \Gamma}{\Gamma \models w\ :\ \kappa}\ \textsc{IT\_Var}$$

$$\frac{\Gamma \models \tau_1\ :\ \kappa_1 \to \kappa_2 \quad \Gamma \models \tau_2\ :\ \kappa_1}{\Gamma \models \tau_1\, \tau_2\ :\ \kappa_2}\ \textsc{IT\_App}$$

$$\frac{\Gamma \models \tau_1\ :\ \forall\, a{:}\,\kappa_1.\, \kappa_2 \quad \Gamma \models \tau_2\ :\ \kappa_1}{\Gamma \models \tau_1\, \tau_2\ :\ \kappa_2[\tau_2/a]}\ \textsc{IT\_Inst}$$

$$\frac{\Gamma \models \tau\ :\ \forall\, c{:}\,\phi.\, \kappa \quad \Gamma \models \gamma\ :\ \phi}{\Gamma \models \tau\, \bullet_{\mathsf{co}}\ :\ \kappa}\ \textsc{IT\_CApp}$$

$$\frac{\Gamma,\ a{:}\,\kappa \models \tau\ :\ \star \quad \Gamma \models \kappa\ :\ \star}{\Gamma \models \forall\, a{:}\,\kappa.\, \tau\ :\ \star}\ \textsc{IT\_AllT}$$

$$\frac{\Gamma,\ c{:}\,\phi \models \tau\ :\ \star \quad \Gamma \models \phi\ \mathsf{ok}}{\Gamma \models \forall\, c{:}\,\phi.\, \tau\ :\ \star}\ \textsc{IT\_AllC}$$

$$\frac{\Gamma \models \tau\ :\ \kappa \quad \Gamma \models \gamma\ :\ \kappa \sim \kappa' \quad \Gamma \models \kappa'\ :\ \star}{\Gamma \models \tau\ :\ \kappa'}\ \textsc{IT\_Cast}$$

$\boxed{\Gamma \models \gamma\ :\ \phi}$ Implicit Coercion Typing

$$\frac{\begin{array}{l}\Gamma \models \gamma\ :\ \tau \sim \tau' \\ \Gamma \models \tau\, \bullet_{\mathsf{co}}\ :\ \kappa \quad \Gamma \models \tau'\, \bullet_{\mathsf{co}}\ :\ \kappa'\end{array}}{\Gamma \models \gamma(\bullet_{\mathsf{co}}, \bullet_{\mathsf{co}})\ :\ \tau\, \bullet_{\mathsf{co}} \sim \tau'\, \bullet_{\mathsf{co}}}\ \textsc{ICT\_CApp}$$

$$\frac{\begin{array}{l}\Gamma \models \eta_1\ :\ \sigma_1 \sim \sigma_1' \quad \phi_1 = \sigma_1 \sim \sigma_2 \\ \Gamma \models \eta_2\ :\ \sigma_2 \sim \sigma_2' \quad \phi_2 = \sigma_1' \sim \sigma_2' \\ c_1 \# \gamma \quad c_2 \# \gamma \\ \Gamma,\ c_1{:}\,\phi_1,\ c_2{:}\,\phi_2 \models \gamma\ :\ \tau_1 \sim \tau_2 \\ \Gamma \models \forall\, c_1{:}\,\phi_1.\, \tau_1\ :\ \star \quad \Gamma \models \forall\, c_2{:}\,\phi_2.\, \tau_2\ :\ \star\end{array}}{\Gamma \models \forall_{(\eta_1,\eta_2)}(c_1, c_2).\gamma\ :\ (\forall\, c_1{:}\,\phi_1.\, \tau_1) \sim (\forall\, c_2{:}\,\phi_2.\, \tau_2)}\ \textsc{ICT\_AllC}$$

$$\frac{\begin{array}{l}\Gamma \models \gamma_1\ :\ (\forall\, a_1{:}\,\kappa_1.\, \tau_1) \sim (\forall\, a_2{:}\,\kappa_2.\, \tau_2) \\ \Gamma \models \gamma_2\ :\ \sigma_1 \sim \sigma_2 \\ \Gamma \models \sigma_1\ :\ \kappa_1 \quad \Gamma \models \sigma_2\ :\ \kappa_2\end{array}}{\Gamma \models \gamma_1 @ \gamma_2\ :\ \tau_1[\sigma_1/a_1] \sim \tau_2[\sigma_2/a_2]}\ \textsc{ICT\_Inst}$$

$$\frac{\begin{array}{l}\Gamma \models \gamma\ :\ (\forall\, c{:}\,\sigma_1 \sim \sigma_2.\, \tau) \sim (\forall\, c'{:}\,\sigma_1' \sim \sigma_2'.\, \tau') \\ \Gamma \models \gamma_1\ :\ \sigma_1 \sim \sigma_2 \quad \Gamma \models \gamma_2\ :\ \sigma_1' \sim \sigma_2'\end{array}}{\Gamma \models \gamma @ (\bullet_{\mathsf{co}}, \bullet_{\mathsf{co}})\ :\ \tau \sim \tau'}\ \textsc{ICT\_InstC}$$

$$\frac{\Gamma \models \tau\ :\ \kappa}{\Gamma \models \langle \tau \rangle\ :\ \tau \sim \tau}\ \textsc{ICT\_Refl}$$

$$\frac{\Gamma \models \gamma\ :\ \tau_1 \sim \tau_2}{\Gamma \models \mathbf{sym}\, \gamma\ :\ \tau_2 \sim \tau_1}\ \textsc{ICT\_Sym}$$

$$\frac{\Gamma \models \gamma_1\ :\ \tau_1 \sim \tau_2 \quad \Gamma \models \gamma_2\ :\ \tau_2 \sim \tau_3}{\Gamma \models \gamma_1 \mathbin{;} \gamma_2\ :\ \tau_1 \sim \tau_3}\ \textsc{ICT\_Trans}$$

$$\frac{\begin{array}{c}\Gamma \models \gamma_1\ :\ \tau_1' \sim \tau_2' \quad \Gamma \models \gamma_2\ :\ \tau_1 \sim \tau_2 \\ \Gamma \models \tau_1'\, \tau_1\ :\ \kappa_1 \quad \Gamma \models \tau_2'\, \tau_2\ :\ \kappa_2\end{array}}{\Gamma \models \gamma_1\, \gamma_2\ :\ \tau_1'\, \tau_1 \sim \tau_2'\, \tau_2}\ \textsc{ICT\_App}$$

$$\frac{\begin{array}{l}\Gamma \models \eta\ :\ \kappa_1 \sim \kappa_2 \\ \Gamma,\ a_1{:}\,\kappa_1,\ a_2{:}\,\kappa_2,\ c{:}\,a_1 \sim a_2 \models \gamma\ :\ \tau_1 \sim \tau_2 \\ \Gamma \models \forall\, a_1{:}\,\kappa_1.\, \tau_1\ :\ \star \quad \Gamma \models \forall\, a_2{:}\,\kappa_2.\, \tau_2\ :\ \star\end{array}}{\Gamma \models \forall_\eta(a_1, a_2, c).\gamma\ :\ (\forall\, a_1{:}\,\kappa_1.\, \tau_1) \sim (\forall\, a_2{:}\,\kappa_2.\, \tau_2)}\ \textsc{ICT\_AllT}$$

$$\frac{c{:}\,\phi\ \in\ \Gamma \quad \models \Gamma}{\Gamma \models c\ :\ \phi}\ \textsc{ICT\_Var}$$

$$\frac{C{:}\ \forall\, \Delta.\, (\tau_1 \sim \tau_2)\ \in\ \Gamma \quad \Gamma \models \overline{\rho}\ :\ \Delta}{\Gamma \models C\, \overline{\rho}\ :\ \tau_1[\overline{\rho}/\Delta] \sim \tau_2[\overline{\rho}/\Delta]}\ \textsc{ICT\_Axiom}$$

$$\frac{\Gamma \models \gamma\ :\ H\, \overline{\tau} \sim H\, \overline{\tau'}}{\Gamma \models \mathbf{nth}^i\, \gamma\ :\ \tau_i \sim \tau_i'}\ \textsc{ICT\_Nth}$$

$$\frac{\Gamma \models \gamma_1\ :\ (\forall\, a_1{:}\,\kappa_1.\, \tau_1) \sim (\forall\, a_2{:}\,\kappa_2.\, \tau_2)}{\Gamma \models \mathbf{nth}^1\, \gamma_1\ :\ \kappa_1 \sim \kappa_2}\ \textsc{ICT\_Nth1TA}$$

$$\frac{\Gamma \models \gamma\ :\ (\forall\, c{:}\,\kappa_1 \sim \kappa_2.\, \tau) \sim (\forall\, c'{:}\,\kappa_1' \sim \kappa_2'.\, \tau')}{\Gamma \models \mathbf{nth}^1\, \gamma\ :\ \kappa_1 \sim \kappa_1'}\ \textsc{ICT\_Nth1CA}$$

$$\frac{\Gamma \models \gamma\ :\ (\forall\, c{:}\,\kappa_1 \sim \kappa_2.\, \tau) \sim (\forall\, c'{:}\,\kappa_1' \sim \kappa_2'.\, \tau')}{\Gamma \models \mathbf{nth}^2\, \gamma\ :\ \kappa_2 \sim \kappa_2'}\ \textsc{ICT\_Nth2CA}$$

$$\frac{\Gamma \models \gamma\ :\ \tau_1 \sim \tau_2 \quad \Gamma \models \tau_1\ :\ \kappa_2 \quad \Gamma \models \tau_2\ :\ \kappa_2}{\Gamma \models \mathbf{kind}\, \gamma\ :\ \kappa_1 \sim \kappa_2}\ \textsc{ICT\_Ext}$$

$\boxed{\Gamma \models \overline{\rho}\ :\ \Delta}$ Implicit telescope argument validity

$$\frac{\models \Gamma}{\Gamma \models \varnothing\ :\ \varnothing}\ \textsc{It2\_empty}$$

$$\frac{\begin{array}{l}\Gamma, \Delta \models \kappa\ :\ \star \quad a \# \Gamma, \Delta \\ \Gamma \models \tau\ :\ \kappa[\overline{\rho}/\Delta] \quad \Gamma \models \overline{\rho}\ :\ \Delta\end{array}}{\Gamma \models \overline{\rho}, \tau\ :\ (\Delta,\ a{:}\,\kappa)}\ \textsc{It2\_const}$$

$$\frac{\begin{array}{l}\Gamma, \Delta \models \phi\ \mathsf{ok} \quad c \# \Gamma, \Delta \\ \Gamma \models \gamma\ :\ \phi[\overline{\rho}/\Delta] \quad \Gamma \models \overline{\rho}\ :\ \Delta\end{array}}{\Gamma \models \overline{\rho}, \bullet_{\mathsf{co}}\ :\ (\Delta,\ c{:}\,\phi)}\ \textsc{It2\_consg}$$

A.6 Telescope reduction

$\boxed{\Gamma \models \overline{\rho} \rightsquigarrow \overline{\rho}'}$

$$\frac{}{\Gamma \models \varnothing \rightsquigarrow \varnothing}\ \textsc{RNil}$$

$$\frac{\Gamma \models \tau \rightsquigarrow \tau' \quad \Gamma \models \overline{\rho} \rightsquigarrow \overline{\rho}'}{\Gamma \models \overline{\rho}, \tau \rightsquigarrow \overline{\rho}', \tau'}\ \textsc{RCons}$$

B. Regularity

These typing judgements are designed to satisfy the following generation properties that ensure that the subcomponents of each judgement are valid.

Lemma B.1 (Regularity/Generation)**.**

1. *If* $\Gamma \vdash_{\mathsf{ty}} \tau\ :\ \kappa$ *then* $\Gamma \vdash_{\mathsf{ty}} \kappa\ :\ \star$ *and* $\vdash_{\mathsf{wf}} \Gamma$.
2. *If* $\Gamma \vdash_{\mathsf{tm}} e\ :\ \tau$ *then* $\Gamma \vdash_{\mathsf{ty}} \tau\ :\ \star$ *and* $\vdash_{\mathsf{wf}} \Gamma$.
3. *If* $\Gamma \vdash_{\mathsf{co}} \gamma\ :\ \sigma_1 \sim \sigma_2$ *then* $\Gamma \vdash_{\mathsf{pr}} \sigma_1 \sim \sigma_2$ *ok and* $\vdash_{\mathsf{wf}} \Gamma$.

Proof. The proof of this lemma is a straightforward induction on typing derivations, appealing to substitution Lemma C.4. □

C. Preservation

This section presents the necessary details for the proof of the preservation theorem. The theorem itself is proved by induction on the typing derivation with a case analysis of the rule used in the operational semantics. Below, we present only three cases, those for S_KPUSH, S_TPUSH, and S_CPUSH. These are the only cases that differ from the proof described in previous work [Weirich et al. 2010]. We also present many supporting lemmas needed for these cases, particularly regarding the treatment of lifting contexts.

C.1 Lifting contexts

A lifting context contains two types of mappings: those labeled with $\mapsto$ that denote a mapping from type or coercion variables to existing types and coercions, and those labeled with $\overset{\bullet}{\mapsto}$ that denote a mapping from type or coercion variables to fresh type and coercion variables.

A lifting context is considered valid with respect to a context Γ and telescope Δ when the following judgement holds:

$\boxed{\Gamma \vdash_{\mathsf{lc}} \Delta \leftrightsquigarrow \Psi}$ Lifting context validity

$$\frac{\vdash_{\mathsf{wf}} \Gamma}{\Gamma \vdash_{\mathsf{lc}} \varnothing \leftrightsquigarrow \varnothing}\ \text{LC\_EMPTY}$$

$$\frac{\begin{array}{l}\Gamma \vdash_{\mathsf{lc}} \Delta \leftrightsquigarrow \Psi \quad a \# \Gamma, \Delta \\ \Gamma \vdash_{\mathsf{ty}} \sigma_1 : \Psi_1(\kappa) \quad \Gamma \vdash_{\mathsf{ty}} \sigma_2 : \Psi_2(\kappa) \\ \Gamma \vdash_{\mathsf{co}} \gamma : \sigma_1 \sim \sigma_2\end{array}}{\Gamma \vdash_{\mathsf{lc}} (\Delta,\, a{:}\,\kappa) \leftrightsquigarrow (\Psi, a{:}\,\kappa \mapsto (\sigma_1, \sigma_2, \gamma))}\ \text{LC\_TY}$$

$$\frac{\begin{array}{l}\Gamma \vdash_{\mathsf{lc}} \Delta \leftrightsquigarrow \Psi \quad c \# \Gamma, \Delta \\ \Gamma \vdash_{\mathsf{co}} \eta_1 : \Psi_1(\phi) \quad \Gamma \vdash_{\mathsf{co}} \eta_2 : \Psi_2(\phi)\end{array}}{\Gamma \vdash_{\mathsf{lc}} (\Delta,\, c{:}\,\phi) \leftrightsquigarrow (\Psi, c{:}\,\phi \mapsto (\eta_1, \eta_2))}\ \text{LC\_CO}$$

$$\frac{\begin{array}{l}a_1 \# \Gamma, \Delta \quad a_2 \# \Gamma, \Delta \quad c \# \Gamma, \Delta \\ \Gamma \vdash_{\mathsf{lc}} \Delta \leftrightsquigarrow \Psi\end{array}}{\Gamma \vdash_{\mathsf{lc}} (\Delta,\, a{:}\,\kappa) \leftrightsquigarrow (\Psi, a{:}\,\kappa \overset{\bullet}{\mapsto} (a_1, a_2, c))}\ \text{LC\_TYFRESH}$$

$$\frac{\Gamma \vdash_{\mathsf{lc}} \Delta \leftrightsquigarrow \Psi \quad c_1 \# \Gamma, \Delta \quad c_2 \# \Gamma, \Delta}{\Gamma \vdash_{\mathsf{lc}} (\Delta,\, c{:}\,\phi) \leftrightsquigarrow (\Psi, c{:}\,\phi \overset{\bullet}{\mapsto} (c_1, c_2))}\ \text{LC\_COFRESH}$$

We can view the fresh bindings of a lifting context as a typing context:

Definition C.1 (Single flattening). *The operation $\dot\Psi_j$ turns a lifting context into a typing context.*

1. *For each $a{:}\,\kappa \overset{\bullet}{\mapsto} (a_1, a_2, c) \in \Psi$, the context $\dot\Psi_j$ includes the binding $a_j{:}\,\Psi_j(\kappa)$.*
2. *For each $c{:}\,\phi \overset{\bullet}{\mapsto} (c_1, c_2)$, the context includes the binding $c_j : \Psi_j(\phi)$.*

Definition C.2 (Flattening). *The operation $\dot\Psi$ turns a lifting context into a typing context.*

1. *For each $a{:}\,\kappa \overset{\bullet}{\mapsto} (a_1, a_2, c) \in \Psi$, the context $\dot\Psi$ includes the bindings $a_1{:}\,\Psi_1(\kappa)$, $a_2{:}\,\Psi_2(\kappa)$, $c{:}\,a_1 \sim a_2$.*
2. *For each $c{:}\,\phi \overset{\bullet}{\mapsto} (c_1, c_2)$, the context includes the bindings $c_1{:}\,\Psi_1(\phi)$, $c_2{:}\,\Psi_2(\phi)$.*

Lemma C.3 (Lifting context domains). *If $\Gamma \vdash_{\mathsf{lc}} \Delta \leftrightsquigarrow \Psi$, then the domain of Δ equals the set of variables mapped in Ψ.*

Proof. Straightforward induction on the derivation of $\Gamma \vdash_{\mathsf{lc}} \Delta \leftrightsquigarrow \Psi$. □

Lemma C.4 (Lifting context substitution). *Suppose $\Gamma \vdash_{\mathsf{lc}} \Delta \leftrightsquigarrow \Psi$.*

1. *If $\Gamma, \Delta, \Gamma' \vdash_{\mathsf{ty}} \tau : \kappa$ then $\Gamma, \dot\Psi_j, \Psi_j(\Gamma') \vdash_{\mathsf{ty}} \Psi_j(\tau) : \Psi_j(\kappa)$*
2. *If $\Gamma, \Delta, \Gamma' \vdash_{\mathsf{pr}} \phi$ ok then $\Gamma, \dot\Psi_j, \Psi_j(\Gamma') \vdash_{\mathsf{pr}} \Psi_j(\phi)$ ok*
3. *If $\Gamma, \Delta, \Gamma' \vdash_{\mathsf{co}} \gamma : \phi$ then $\Gamma, \dot\Psi_j, \Psi_j(\Gamma') \vdash_{\mathsf{co}} \Psi_j(\gamma) : \Psi_j(\phi)$*
4. *If $\Gamma, \Delta, \Gamma' \vdash_{\mathsf{lc}} \Delta' \leftrightsquigarrow \Psi'$ then $\Gamma, \dot\Psi_j, \Psi_j(\Gamma') \vdash_{\mathsf{lc}} \Psi_j(\Delta') \leftrightsquigarrow \Psi_j(\Psi')$*
5. *If $\Gamma, \Delta, \Gamma' \vdash_{\mathsf{tel}} \overline{\rho} : \Delta'$ then $\Gamma, \dot\Psi_j, \Psi_j(\Gamma') \vdash_{\mathsf{tel}} \Psi_j(\overline{\rho}) : \Psi_j(\Delta')$*
6. *If $\vdash_{\mathsf{wf}} \Gamma, \Delta, \Gamma'$, then $\vdash_{\mathsf{wf}} \Gamma, \dot\Psi_j, \Psi_j(\Gamma')$*

Proof. We proceed by mutual induction.

There are many cases to consider. We consider the interesting ones here:

Case K_VAR: We know $\Gamma, \Delta, \Gamma' \vdash_{\mathsf{ty}} w : \kappa$, and by inversion, $\vdash_{\mathsf{wf}} \Gamma, \Delta, \Gamma'$ and $w{:}\,\kappa \in \Gamma, \Delta, \Gamma'$. We must show $\Gamma, \dot\Psi_j, \Psi_j(\Gamma') \vdash_{\mathsf{ty}} \Psi_j(w) : \Psi_j(\kappa)$.

We have two cases:

$w \in \mathit{dom}\,\Gamma$**:** Because $w \notin \mathit{dom}\,\Delta$, $\Psi_j(w) = w$. Furthermore, because κ appears in Γ, Δ, Γ' before any element in Δ is declared, we know that κ cannot refer to any variable declared in Δ. Therefore, $\Psi_j(\kappa) = \kappa$. By the induction hypothesis, $\vdash_{\mathsf{wf}} \Gamma, \dot\Psi_j, \Psi_j(\Gamma')$, and we can use rule K_VAR to get $\Gamma, \dot\Psi_j, \Psi_j(\Gamma') \vdash_{\mathsf{ty}} w : \kappa$ as desired.

$w \in \mathit{dom}\,\Delta$**:** By Lemma C.3, a mapping $w{:}\,\kappa \overset{?}{\mapsto} (\tau_1, \tau_2, \gamma)$ must exist in Ψ. Here, we have two further cases, depending on the nature of the mapping:

$\mapsto$**:** Inverting $\Gamma \vdash_{\mathsf{lc}} \Delta \leftrightsquigarrow \Psi$ eventually gives us $\Gamma \vdash_{\mathsf{ty}} \Psi_j(w) : \Psi_j(\kappa)$ (from rule LC_TY). Weakening then gives us $\Gamma, \dot\Psi_j, \Psi_j(\Gamma') \vdash_{\mathsf{ty}} \Psi_j(w) : \Psi_j(\kappa)$ as desired.

$\overset{\bullet}{\mapsto}$**:** By the definition of $\dot\Psi_j$, $w{:}\,\Psi_j(\kappa) \in \dot\Psi_j$. By the induction hypothesis, we can derive $\vdash_{\mathsf{wf}} \Gamma, \dot\Psi_j, \Psi_j(\Gamma')$. Then, we apply rule K_VAR to get $\Gamma, \dot\Psi_j, \Psi_j(\Gamma') \vdash_{\mathsf{ty}} \Psi_j(w) : \Psi_j(\kappa)$ as desired.

$w \in \mathit{dom}\,\Gamma'$**:** Because $w \notin \mathit{dom}\,\Delta$, $\Psi_j(w) = w$. Furthermore, we know $w{:}\,\kappa \in \Gamma'$ and therefore $w{:}\,\Psi_j(\kappa) \in \Psi_j(\Gamma')$. The induction hypothesis gives us $\vdash_{\mathsf{wf}} \Gamma, \dot\Psi_j, \Psi_j(\Gamma')$ and we can use rule K_VAR to derive $\Gamma, \dot\Psi_j, \Psi_j(\Gamma') \vdash_{\mathsf{ty}} \Psi_j(w) : \Psi_j(\kappa)$ as desired.

$w \notin \mathit{dom}\,\Delta$**:**

Case K_ALLC: We know $\Gamma, \Delta, \Gamma' \vdash_{\mathsf{ty}} \forall\, c{:}\,\phi.\, \tau : \star$, and by inversion, $\Gamma, \Delta, \Gamma',\ c{:}\,\phi \vdash_{\mathsf{ty}} \tau : \star$ and $\Gamma, \Delta, \Gamma' \vdash_{\mathsf{pr}} \phi$ ok. We must show $\Gamma, \dot\Psi_j, \Psi_j(\Gamma') \vdash_{\mathsf{ty}} \forall\, c{:}\,\Psi_j(\phi).\, \Psi_j(\tau) : \star$.

The induction hypothesis gives us $\Gamma, \dot\Psi_j, \Psi_j(\Gamma'),\ c{:}\,\Psi_j(\phi) \vdash_{\mathsf{ty}} \Psi_j(\tau) : \star$ and $\Gamma, \dot\Psi_j, \Psi_j(\Gamma') \vdash_{\mathsf{pr}} \Psi_j(\phi)$ ok. Thus, by rule K_ALLC, $\Gamma, \dot\Psi_j, \Psi_j(\Gamma') \vdash_{\mathsf{ty}} \forall\, c{:}\,\Psi_j(\phi).\, \Psi_j(\tau) : \star$ and we are done.

Case CT_ALLC: We know

$$\Gamma, \Delta, \Gamma' \vdash_{\mathsf{co}} \forall_{(\eta_1, \eta_2)}(c_1, c_2).\gamma : (\forall\, c_1{:}\,\phi_1.\, \tau_1) \sim (\forall\, c_2{:}\,\phi_2.\, \tau_2)$$

and by inversion

$$\begin{array}{l}\Gamma, \Delta, \Gamma' \vdash_{\mathsf{co}} \eta_1 : \sigma_1 \sim \sigma'_1 \\ \Gamma, \Delta, \Gamma' \vdash_{\mathsf{co}} \eta_2 : \sigma_2 \sim \sigma'_2 \\ \phi_1 = \sigma_1 \sim \sigma_2 \\ \phi_2 = \sigma'_1 \sim \sigma'_2 \\ c_1 \# |\gamma| \\ c_2 \# |\gamma| \\ \Gamma, \Delta, \Gamma',\ c_1{:}\,\phi_1,\ c_2{:}\,\phi_2 \vdash_{\mathsf{co}} \gamma : \tau_1 \sim \tau_2 \\ \Gamma, \Delta, \Gamma' \vdash_{\mathsf{ty}} \forall\, c_1{:}\,\phi_1.\, \tau_1 : \star \\ \Gamma, \Delta, \Gamma' \vdash_{\mathsf{ty}} \forall\, c_2{:}\,\phi_2.\, \tau_2 : \star\end{array}$$

We wish to show

$$\Gamma, \dot{\Psi}_j, \Psi_j(\Gamma') \vdash \forall_{(\Psi_j(\eta_1), \Psi_j(\eta_2))}(c_1, c_2).\Psi_j(\gamma) : \\ (\forall\, c_1{:}\, \Psi_j(\phi_1).\, \Psi_j(\tau_1)) \sim (\forall\, c_2{:}\, \Psi_j(\phi_2).\, \Psi_j(\tau_2)).$$

To use rule CT_ALLC, we need to show

$$\Gamma, \dot{\Psi}_j, \Psi_j(\Gamma') \vdash_{\mathsf{co}} \Psi_j(\eta_1) \ :\ \Psi_j(\sigma_1) \sim \Psi_j(\sigma_1') \qquad (1)$$
$$\Gamma, \dot{\Psi}_j, \Psi_j(\Gamma') \vdash_{\mathsf{co}} \Psi_j(\eta_2) \ :\ \Psi_j(\sigma_2) \sim \Psi_j(\sigma_2') \qquad (2)$$
$$c_1 \ \#\ |\Psi_j(\gamma)| \qquad (3)$$
$$c_2 \ \#\ |\Psi_j(\gamma)| \qquad (4)$$
$$\Gamma, \dot{\Psi}_j, \Psi_j(\Gamma'),\ c_1{:}\, \Psi_j(\phi_1),\ c_2{:}\, \Psi_j(\phi_2) \vdash_{\mathsf{co}} \\ \Psi_j(\gamma) : \Psi_j(\tau_1) \sim \Psi_j(\tau_2) \qquad (5)$$
$$\Gamma, \dot{\Psi}_j, \Psi_j(\Gamma') \vdash_{\mathsf{ty}} \forall\, c_1{:}\, \Psi_j(\phi_1).\, \Psi_j(\tau_1) \ :\ \star \qquad (6)$$
$$\Gamma, \dot{\Psi}_j, \Psi_j(\Gamma') \vdash_{\mathsf{ty}} \forall\, c_2{:}\, \Psi_j(\phi_2).\, \Psi_j(\tau_2) \ :\ \star \qquad (7)$$

We know (1), (2), (5), (6), and (7) by the induction hypothesis. We can derive (3) and (4) by noting that $\Psi_j(\cdot)$ and $|\cdot|$ commute with each other and that c_1, c_2 do not appear in Ψ. Therefore, if $c_1 \ \#\ |\gamma|$, then $c_1 \ \#\ |\Psi_j(\gamma)|$ and likewise for c_2. Now, we can apply CT_ALLC and we are done.

Case CT_AXIOM: We know $\Gamma, \Delta, \Gamma' \vdash_{\mathsf{co}} C\,\overline{\rho} \ :\ \tau_1[\overline{\rho}/\Delta'] \sim \tau_2[\overline{\rho}/\Delta']$, and by inversion, $C{:}\ \forall\, \Delta'.\, (\tau_1 \sim \tau_2) \in \Gamma, \Delta, \Gamma'$ and $\Gamma, \Delta, \Gamma' \vdash_{\mathsf{tel}} \overline{\rho} : \Delta'$. We must show

$$\Gamma, \dot{\Psi}_j, \Psi_j(\Gamma') \vdash_{\mathsf{co}} \Psi_j(C\,\overline{\rho}) \ :\ \Psi_j(\tau_1[\overline{\rho}/\Delta'] \sim \tau_2[\overline{\rho}/\Delta'])$$

or, equivalently,

$$\Gamma, \dot{\Psi}_j, \Psi_j(\Gamma') \vdash_{\mathsf{co}} C\, \Psi_j(\overline{\rho}) \ : \\ \Psi_j(\tau_1)[\Psi_j(\overline{\rho})/\Psi_j(\Delta')] \sim \Psi_j(\tau_2)[\Psi_j(\overline{\rho})/\Psi_j(\Delta')]$$

To use CT_AXIOM to prove this fact, we need, in turn

$$C{:}\ \forall\, \Psi_j(\Delta').\, (\Psi_j(\tau_1) \sim \Psi_j(\tau_2)) \in \Gamma, \dot{\Psi}_j, \Psi_j(\Gamma')$$
$$\Gamma, \dot{\Psi}_j, \Psi_j(\Gamma') \vdash_{\mathsf{tel}} \Psi_j(\overline{\rho}) : \Psi_j(\Delta')$$

The induction hypothesis gives us the second fact above.
By construction, a lifting context cannot map an axiom variable C. Thus, appealing to Lemma C.3, we know that $C \notin dom\, \Delta$. We then have two cases:

$C \in dom\, \Gamma$**:** Because C is well-formed in Γ, the type of C cannot mention any variables in Δ. Thus, $\Psi_j(\Delta') = \Delta'$, $\Psi_j(\tau_1) = \tau_1$ and $\Psi_j(\tau_2) = \tau_2$. Then, we can conclude that $C{:}\ \forall\, \Delta'.\, (\tau_1 \sim \tau_2) \in \Gamma$ and we are done.

$C \in dom\, \Gamma'$**:** In this case, we can conclude that

$$C{:}\ \forall\, \Psi_j(\Delta').\, (\Psi_j(\tau_1) \sim \Psi_j(\tau_2)) \in \Psi_j(\Gamma')$$

and we are done.

□

We will need the following lemmas to prove the lifting lemma:

Lemma C.5 (Lifting context coercions)**.** *If* $\Gamma \vdash_{\mathsf{c}} \Delta \leftrightsquigarrow \Psi$ *and* Ψ *contains the mapping* $a{:}\,\kappa \overset{?}{\mapsto} (\tau_1, \tau_2, \gamma)$*, then* $\Gamma, \dot{\Psi} \vdash_{\mathsf{co}} \gamma \ :\ \tau_1 \sim \tau_2$.

Proof. Straightforward induction on $\Gamma \vdash_{\mathsf{c}} \Delta \leftrightsquigarrow \Psi$. □

Lemma C.6 (Weakened lifting context substitution)**.** *Suppose* $\Gamma \vdash_{\mathsf{c}} \Delta \leftrightsquigarrow \Psi$.

1. *If* $\Gamma, \Delta \vdash_{\mathsf{ty}} \tau \ :\ \kappa$ *then* $\Gamma, \dot{\Psi} \vdash_{\mathsf{ty}} \Psi_j(\tau) \ :\ \Psi_j(\kappa)$
2. *If* $\Gamma, \Delta \vdash_{\mathsf{co}} \gamma \ :\ \phi$ *then* $\Gamma, \dot{\Psi} \vdash_{\mathsf{co}} \Psi_j(\gamma) \ :\ \Psi_j(\phi)$
3. *If* $\Gamma, \Delta \vdash_{\mathsf{c}} \Delta' \leftrightsquigarrow \Psi'$ *then* $\Gamma, \dot{\Psi} \vdash_{\mathsf{c}} \Psi_j(\Delta') \leftrightsquigarrow \Psi_j(\Psi')$
4. *If* $\Gamma, \Delta \vdash_{\mathsf{tel}} \overline{\rho} : \Delta'$ *then* $\Gamma, \dot{\Psi} \vdash_{\mathsf{tel}} \Psi_j(\overline{\rho}) : \Psi_j(\Delta')$

(This lemma is the same as Lemma C.4, except the j subscripts in the conclusion contexts are removed.)

Proof. Immediate from Lemma C.4 and weakening, noting that any difference between $\dot{\Psi}_j$ and $\dot{\Psi}$ are guaranteed to be fresh bindings. □

Lemma C.7 (Erased lifted coercions)**.** *Let* Ψ *contain the mapping* $c{:}\,\phi \overset{\bullet}{\mapsto} (c_1, c_2)$*. If* $\Gamma \vdash_{\mathsf{c}} \Delta \leftrightsquigarrow \Psi$ *and* $\Gamma, \Delta \vdash_{\mathsf{ty}} \tau \ :\ \kappa$*, then* $c_1 \ \#\ |\Psi(\tau)|$ *and* $c_2 \ \#\ |\Psi(\tau)|$.

Proof. We proceed by induction on the typing derivation for τ.

Cases K_STARINSTAR and K_ARROW: Trivial.

Case K_VAR: $\tau = w$, and we know $\vdash_{\mathsf{wf}} \Gamma, \Delta$ and $w{:}\,\kappa \in \Gamma, \Delta$. Here we have two cases:

$w \in dom\, \Delta$**:** We know w is a type variable, so $w \neq c$. Thus, w appears either after or before c in Ψ. If w appears after c, then, by the fact that all mappings with $\mapsto$ precede all mappings with $\overset{\bullet}{\mapsto}$ in Ψ, $\Psi(w)$ is some fresh variable c', and thus $c_1 \ \#\ |\Psi(w)|$ and $c_2 \ \#\ |\Psi(w)|$ as desired. Going forward, we can assume w occurs before c in Ψ. Now, the mapping from w may be built with $\mapsto$ or $\overset{\bullet}{\mapsto}$. We have already handled the latter case, so going forward, we can assume that the mapping is built with $\mapsto$. $\Psi(w) = \gamma$ for some γ. However, this γ is built from components all of which are out of scope of c, c_1, and c_2. Thus, neither c_1 nor c_2 appear in γ and thus do not appear in $|\gamma|$. Thus, $c_1 \ \#\ |\Psi(w)|$ and $c_2 \ \#\ |\Psi(w)|$ as desired.

$w \notin dom\, \Delta$**:** In this case $\Psi(w) = \langle w \rangle$. Because the spaces of type variables and coercion variables are distinct, we know that $w \neq c_1$ and $w \neq c_2$, as desired.

Case K_APP: $\tau = \tau_1\, \tau_2$, and we know $\Gamma, \Delta \vdash_{\mathsf{ty}} \tau_1 \ :\ \kappa_1 \to \kappa_2$ and $\Gamma, \Delta \vdash_{\mathsf{ty}} \tau_2 \ :\ \kappa_1$. Here, $\Psi(\tau_1\, \tau_2) = \Psi(\tau_1)\, \Psi(\tau_2)$. The induction hypothesis tells us that c_1, c_2 do not appear in $|\Psi(\tau_1)|, |\Psi(\tau_2)|$. Since $|\gamma_1\, \gamma_2| = |\gamma_1|\, |\gamma_2|$, the desired result follows directly from this result.

Case K_INST: Analogous to K_APP.

Case K_CAPP: $\tau = \tau_1\, \gamma_1$, and we know

$$\Gamma, \Delta \vdash_{\mathsf{ty}} \tau_1 \ :\ \forall\, c{:}\,\phi.\, \kappa$$
$$\Gamma, \Delta \vdash_{\mathsf{co}} \gamma_1 \ :\ \phi$$

We have $\Psi(\tau_1\, \gamma_1) = \Psi(\tau_1)(\Psi_1(\gamma_1), \Psi_2(\gamma_1))$. The induction hypothesis tells us that c_1, c_2 do not appear in $|\Psi(\tau_1)|$. By the definition of $|\cdot|$, $|\Psi(\tau_1\, \gamma_1)| = |\Psi(\tau_1)|(\bullet_{\mathsf{co}}, \bullet_{\mathsf{co}})$. Thus, c_1, c_2 do not appear in $|\Psi(\tau_1\, \gamma_1)|$ as desired.

Case K_ALLT: $\tau = \forall\, a{:}\,\kappa.\, \tau'$, and we know $\Gamma, \Delta,\ a{:}\,\kappa \vdash_{\mathsf{ty}} \tau' \ :\ \star$ and $\Gamma, \Delta \vdash_{\mathsf{ty}} \kappa \ :\ \star$. Letting $\Psi' = \Psi, a{:}\,\kappa \overset{\bullet}{\mapsto} (a_1, a_2, c)$ (for fresh a_1, a_2, c), we have $|\Psi(\forall\, a{:}\,\kappa.\, \tau')| = |\forall_{\Psi(\kappa)}(a_1, a_2, c).\Psi'(\tau')| = \forall_{|\Psi(\kappa)|}(a_1, a_2, c).|\Psi'(\tau')|$. The induction hypothesis tells us that c_1, c_2 do not appear in $|\Psi(\kappa)|$ and $|\Psi'(\tau')|$, so we are done.

Case K_ALLC: Analogous to K_ALLT.

Case K_CAST: $\tau = \tau' \rhd \eta$ and we know $\Gamma, \Delta \vdash_{\mathsf{ty}} \tau' \ :\ \kappa_1, \Gamma, \Delta \vdash_{\mathsf{co}} \eta \ :\ \kappa_1 \sim \kappa_2$, and $\Gamma, \Delta \vdash_{\mathsf{ty}} \kappa_2 \ :\ \star$. We have $|\Psi(\tau' \rhd \eta)| = |\Psi(\tau') \rhd \Psi_1(\eta) \sim \Psi_2(\eta)| = |\mathbf{sym}\,((\mathbf{sym}\, \Psi(\tau')) \rhd \Psi_2(\eta)) \rhd \Psi_1(\eta)| = \mathbf{sym}\,(\mathbf{sym}\, |\Psi(\tau')|)$. The induction hypothesis tells us that c_1, c_2 do not appear in $|\Psi(\tau')|$, so we are done.

□

Proof of Lemma 5.3 (Lifting): If $\Gamma \vdash_{\mathsf{c}} \Delta \leftrightsquigarrow \Psi$ and $\Gamma, \Delta \vdash_{\mathsf{ty}} \tau : \kappa$, then

$$\Gamma, \dot{\Psi} \vdash_{\mathsf{co}} \Psi(\tau) : \Psi_1(\tau) \sim \Psi_2(\tau)$$

Proof. We proceed by induction on the typing derivation for τ.

Case K_STARINSTAR: Trivial: $\Gamma, \dot{\Psi} \vdash_{\mathsf{co}} \langle\star\rangle : \star \sim \star$.

Case K_ARROW: Trivial: $\Gamma, \dot{\Psi} \vdash_{\mathsf{co}} \langle(\rightarrow)\rangle : (\rightarrow) \sim (\rightarrow)$.

Case K_VAR: $\tau = w$, and we know $\vdash_{\mathsf{wf}} \Gamma, \Delta$ and $w{:}\,\kappa \in \Gamma, \Delta$. Here we have two cases:

$w \in dom\,\Delta$**:** By the definition of Δ, w must be a type variable a. Using Lemma C.3, there must exist a mapping $a{:}\,\kappa \overset{?}{\mapsto} (\tau_1, \tau_2, \gamma)$ in Ψ. Then, we know $\Psi(w) = \gamma$, $\Psi_1(w) = \tau_1$, and $\Psi_2(w) = \tau_2$. By Lemma C.5, we can get $\Gamma, \dot{\Psi} \vdash_{\mathsf{co}} \gamma : \tau_1 \sim \tau_2$, and thus $\Gamma, \dot{\Psi} \vdash_{\mathsf{co}} \Psi(w) : \Psi_1(w) \sim \Psi_2(w)$ as desired.

$w \notin dom\,\Delta$**:** Trivial: $\Gamma, \dot{\Psi} \vdash_{\mathsf{co}} \langle w\rangle : w \sim w$.

Case K_APP: $\tau = \tau_1\,\tau_2$, and we know $\Gamma, \Delta \vdash_{\mathsf{ty}} \tau_1 : \kappa_1 \rightarrow \kappa_2$ and $\Gamma, \Delta \vdash_{\mathsf{ty}} \tau_2 : \kappa_1$. $\Psi(\tau_1\,\tau_2) = \Psi(\tau_1)\,\Psi(\tau_2)$. The induction hypothesis gives us

$$\Gamma, \dot{\Psi} \vdash_{\mathsf{co}} \Psi(\tau_1) : \Psi_1(\tau_1) \sim \Psi_2(\tau_1)$$
$$\Gamma, \dot{\Psi} \vdash_{\mathsf{co}} \Psi(\tau_2) : \Psi_1(\tau_2) \sim \Psi_2(\tau_2).$$

We now wish to use rule CT_APP, but we need to know

$$\Gamma, \dot{\Psi} \vdash_{\mathsf{ty}} \Psi_1(\tau_1)\,\Psi_1(\tau_2) : \sigma_1$$
$$\Gamma, \dot{\Psi} \vdash_{\mathsf{ty}} \Psi_2(\tau_1)\,\Psi_2(\tau_2) : \sigma_2$$

for some types σ_1 and σ_2. Lemma C.6 applied to the types of τ_1 and τ_2, along with straightforward typing rule applications, gives us exactly these facts. Thus,

$$\Gamma, \dot{\Psi} \vdash_{\mathsf{co}} \Psi(\tau_1)\,\Psi(\tau_2) : \Psi_1(\tau_1)\,\Psi_1(\tau_2) \sim \Psi_2(\tau_1)\,\Psi_2(\tau_2)$$

or

$$\Gamma, \dot{\Psi} \vdash_{\mathsf{co}} \Psi(\tau_1\,\tau_2) : \Psi_1(\tau_1\,\tau_2) \sim \Psi_2(\tau_1\,\tau_2)$$

as desired.

Case K_INST: $\tau = \tau_1\,\tau_2$, and we know $\Gamma, \Delta \vdash_{\mathsf{ty}} \tau_1 : \forall\, a{:}\,\kappa_1.\,\kappa_2$ and $\Gamma, \Delta \vdash_{\mathsf{ty}} \tau_2 : \kappa_1$. This case then proceeds identically to the previous case.

Case K_CAPP: $\tau = \tau_1\,\gamma_1$, and we know $\Gamma, \Delta \vdash_{\mathsf{ty}} \tau_1 : \forall\, c{:}\,\phi.\,\kappa$ and $\Gamma, \Delta \vdash_{\mathsf{co}} \gamma_1 : \phi$. $\Psi(\tau_1\,\gamma_1) = \Psi(\tau_1)(\Psi_1(\gamma_1), \Psi_2(\gamma_1))$. The induction hypothesis gives us

$$\Gamma, \dot{\Psi} \vdash_{\mathsf{co}} \Psi(\tau_1) : \Psi_1(\tau_1) \sim \Psi_2(\tau_1).$$

We now wish to use rule CT_CAPP, but we need to know

$$\Gamma, \Delta \vdash_{\mathsf{ty}} \Psi_1(\tau_1)\,\Psi_1(\gamma_1) : \kappa$$
$$\Gamma, \Delta \vdash_{\mathsf{ty}} \Psi_2(\tau_1)\,\Psi_2(\gamma_1) : \kappa'$$

for some types κ and κ'. Lemma C.6 applied to the types of τ_1 and γ_1, along with straightforward typing rule applications, gives us exactly these facts. Thus,

$$\Gamma, \dot{\Psi} \vdash_{\mathsf{co}} \Psi(\tau_1\,\gamma_1) : \Psi_1(\tau_1)\,\Psi_1(\gamma_1) \sim \Psi_2(\tau_1)\,\Psi_2(\gamma_1)$$

or

$$\Gamma, \dot{\Psi} \vdash_{\mathsf{co}} \Psi(\tau_1\,\gamma_1) : \Psi_1(\tau_1\,\gamma_1) \sim \Psi_2(\tau_1\,\gamma_1)$$

as desired.

Case K_ALLT: $\tau = \forall\, a{:}\,\kappa.\,\tau'$, and we know $\Gamma, \Delta,\; a{:}\,\kappa \vdash_{\mathsf{ty}} \tau' : \star$ and $\Gamma, \Delta \vdash_{\mathsf{ty}} \kappa : \star$. We can use LC_TYFRESH to derive $\Gamma \vdash_{\mathsf{c}} \Delta,\, a{:}\,\kappa \leftrightsquigarrow \Psi, a{:}\,\kappa \overset{\bullet}{\mapsto} (a_1, a_2, c)$ for fresh a_1, a_2, c. Write Ψ' for this extended lifting context.

We wish to show

$$\Gamma, \dot{\Psi} \vdash_{\mathsf{co}} \Psi(\forall\, a{:}\,\kappa.\,\tau') : \Psi_1(\forall\, a{:}\,\kappa.\,\tau') \sim \Psi_2(\forall\, a{:}\,\kappa.\,\tau')$$

or, equivalently,

$$\Gamma, \dot{\Psi} \vdash_{\mathsf{co}} \forall_{\Psi(\kappa)}(a_1, a_2, c).\Psi'(\tau') : \forall\, a_1{:}\,\Psi_1(\kappa).\,\Psi'_1(\tau') \sim \forall\, a_2{:}\,\Psi_2(\kappa).\,\Psi'_2(\tau')$$

By the induction hypothesis, we have

$$\Gamma, \dot{\Psi} \vdash_{\mathsf{co}} \Psi(\kappa) : \Psi_1(\kappa) \sim \Psi_2(\kappa)$$
$$\Gamma, \dot{\Psi}' \vdash_{\mathsf{co}} \Psi'(\tau') : \Psi'_1(\tau') \sim \Psi'_2(\tau')$$

We wish to use CT_ALLT. The first two premises are already satisfied, noting that $\dot{\Psi}'$ contains the extra bindings for the second premise. We must show

$$\Gamma, \dot{\Psi} \vdash_{\mathsf{ty}} \forall a_j : \Psi_j(\kappa).\Psi_j(\tau') : \star$$

This fact comes directly from the use of Lemma C.6 applied to the type of $\forall\, a{:}\,\kappa.\,\tau'$.

Thus, we can apply CT_ALLT, and we are done.

Case K_ALLC: $\tau = \forall\, c{:}\,\phi.\,\tau'$, and we know $\Gamma, \Delta,\; c{:}\,\phi \vdash_{\mathsf{ty}} \tau' : \star$ and $\Gamma, \Delta \vdash_{\mathsf{pr}} \phi$ ok. We can use LC_COFRESH to derive $\Gamma \vdash_{\mathsf{c}} \Delta,\, c{:}\,\phi \leftrightsquigarrow \Psi, c{:}\,\phi \overset{\bullet}{\mapsto} (c_1, c_2)$ for fresh c_1, c_2. Write Ψ' for this extended lifting context, and let $\phi = \sigma_1 \sim \sigma_2$.

We wish to show

$$\Gamma, \dot{\Psi} \vdash_{\mathsf{co}} \Psi(\forall\, c{:}\,\phi.\,\tau') : \Psi_1(\forall\, c{:}\,\phi.\,\tau') \sim \Psi_2(\forall\, c{:}\,\phi.\,\tau')$$

or, equivalently,

$$\Gamma, \dot{\Psi} \vdash_{\mathsf{co}} \forall_{(\Psi(\sigma_1),\Psi(\sigma_2))}(c_1, c_2).\Psi'(\tau') : \forall\, c_1{:}\,\Psi_1(\sigma_1) \sim \Psi_1(\sigma_2).\,\Psi'_1(\tau') \sim \forall\, c_2{:}\,\Psi_2(\sigma_1) \sim \Psi_2(\sigma_2).\,\Psi'_2(\tau')$$

We use inversion on $\Gamma, \Delta \vdash_{\mathsf{pr}} \sigma_1 \sim \sigma_2$ ok to get

$$\Gamma, \Delta \vdash_{\mathsf{ty}} \sigma_1 : \kappa_1$$
$$\Gamma, \Delta \vdash_{\mathsf{ty}} \sigma_2 : \kappa_2$$

By the induction hypothesis, we have

$$\Gamma, \dot{\Psi} \vdash_{\mathsf{co}} \Psi(\sigma_1) : \Psi_1(\sigma_1) \sim \Psi_2(\sigma_1)$$
$$\Gamma, \dot{\Psi} \vdash_{\mathsf{co}} \Psi(\sigma_2) : \Psi_1(\sigma_1) \sim \Psi_2(\sigma_2)$$
$$\Gamma, \dot{\Psi}' \vdash_{\mathsf{co}} \Psi'(\tau') : \Psi'_1(\tau') \sim \Psi'_2(\tau')$$

We wish to use CT_ALLC. The first, second, third, fourth, and seventh premises are already satisfied. The fifth and sixth premises are $c_1 \;\#\; |\Psi'(\tau')|$ and $c_2 \;\#\; |\Psi'(\tau')|$, respectively. We use Lemma C.7 to get these conditions. Now, it remains only to show

$$\Gamma, \dot{\Psi} \vdash_{\mathsf{ty}} \forall c_j : \Psi_j(\phi).\Psi'_j(\tau') : \star$$

This fact comes directly from the use of Lemma C.6 applied to the type of $\forall\, c{:}\,\phi.\,\tau'$.

Thus, we can apply CT_ALLC, and we are done.

Case K_CAST: $\tau = \tau' \triangleright \eta$, and we know $\Gamma, \Delta \vdash_{\mathsf{ty}} \tau' : \kappa_1$, $\Gamma, \Delta \vdash_{\mathsf{co}} \eta : \kappa_1 \sim \kappa_2$, and $\Gamma, \Delta \vdash_{\mathsf{ty}} \kappa_2 : \star$. We wish to show

$$\Gamma, \dot{\Psi} \vdash_{\mathsf{co}} \Psi(\tau' \triangleright \eta) : \Psi_1(\tau' \triangleright \eta) \sim \Psi_2(\tau' \triangleright \eta)$$

or, equivalently,

$$\Gamma, \dot{\Psi} \vdash_{\mathsf{co}} \mathbf{sym}\,((\mathbf{sym}\,\Psi(\tau')) \triangleright \Psi_2(\eta)) \triangleright \Psi_1(\eta) : \Psi_1(\tau') \triangleright \Psi_1(\eta) \sim \Psi_2(\tau') \triangleright \Psi_2(\eta).$$

By the induction hypothesis, we have

$$\Gamma, \dot{\Psi} \vdash_{\mathsf{co}} \Psi(\tau') : \Psi_1(\tau') \sim \Psi_2(\tau').$$

Using this fact with straightforward application of typing rules gives us the desired result.

□

C.2 Metatheory for S_KPUSH Preservation

Having proved the lifting lemma, we still must present and prove a number of other lemmas before proving that the types are preserved in the S_KPUSH case.

Lemma C.8 (Telescope substitution)**.** *If* $\Gamma \vdash_{\mathsf{c}} \Delta \leftrightsquigarrow \Psi$ *and* $\vdash_{\mathsf{wf}} \Delta$*, then* $\Psi_j(\Delta) = \Delta$.

Proof. By Lemma C.3, the domain of Ψ equals the domain of Δ. Furthermore, $\vdash_{\mathsf{wf}} \Delta$ implies that all kinds in Δ (constructs to the right of a colon) are well-scoped—that is, no variable is mentioned before it is declared. Because the $\Psi_j(\Delta)$ operation is defined only to substitute in kinds and to not substitute a variable after it is locally bound, it is impossible for the substitution to change Δ. Thus, $\Psi_j(\Delta) = \Delta$, as desired. □

Lemma C.9 (Ψ_j-consistency)**.** *If* $\Gamma \vdash_{\mathsf{c}} \Delta \leftrightsquigarrow \Psi$*, then* $\Gamma \vdash_{\mathsf{tel}} \Psi_j(\mathit{dom}\,\Delta) : \Delta$.

Proof. We wish to use clause 5 of the lifting context substitution lemma (Lemma C.4), with $\overline{\rho} = \mathit{dom}\,\Delta$ and $\Delta' = \Delta$. We must show $\Gamma, \Delta \vdash_{\mathsf{tel}} \mathit{dom}\,\Delta : \Delta$. This is true by straightforward induction on the length of Δ. Then, we apply Lemma C.4 to get $\Gamma \vdash_{\mathsf{tel}} \Psi_j(\mathit{dom}\,\Delta) : \Psi_j(\Delta)$. By Lemma C.8, this can be rewritten as $\Gamma \vdash_{\mathsf{tel}} \Psi_j(\mathit{dom}\,\Delta) : \Delta$, as desired. □

Lemma C.10 (Lifting context extension consistency)**.** *If* $\Gamma \vdash_{\mathsf{c}} \Delta_1 \leftrightsquigarrow \Psi$*,* $\vdash_{\mathsf{wf}} \Gamma, \Delta_1, \Delta_2$*,* $\Gamma \vdash_{\mathsf{tel}} \overline{\rho}_2 : \Psi_1(\Delta_2)$*, and* $\Psi' = \mathsf{extend}(\Psi; \overline{\rho}_2; \Delta_2)$*, then* $\Gamma \vdash_{\mathsf{c}} \Delta_1, \Delta_2 \leftrightsquigarrow \Psi'$.

Proof. We proceed by induction on the derivation of $\Gamma \vdash_{\mathsf{tel}} \overline{\rho}_2 : \Psi_1(\Delta_2)$.

- Case $\overline{\rho}_2 = \varnothing$; $\Delta_2 = \varnothing$: In this case $\Psi' = \Psi$, and thus we must show $\Gamma \vdash_{\mathsf{c}} \Delta_1 \leftrightsquigarrow \Psi$, which we know by assumption.
- Case $\overline{\rho}_2 = \overline{\rho}'_2, \tau$; $\Delta_2 = \Delta'_2,\ a{:}\,\kappa$:
  The inductive hypothesis is: if $\vdash_{\mathsf{wf}} \Gamma, \Delta_1, \Delta'_2$, $\Gamma \vdash_{\mathsf{tel}} \overline{\rho}'_2 : \Psi_1(\Delta'_2)$, and $\Psi'' = \mathsf{extend}(\Psi; \overline{\rho}'_2; \Delta'_2)$, then $\Gamma \vdash_{\mathsf{c}} \Delta_1, \Delta'_2 \leftrightsquigarrow \Psi''$. We must show $\Gamma \vdash_{\mathsf{c}} \Delta_1, \Delta'_2,\ a{:}\,\kappa \leftrightsquigarrow \Psi'$, where $\Psi' = \mathsf{extend}(\Psi; \overline{\rho}'_2, \tau; \Delta'_2,\ a{:}\,\kappa)$.
  By the definition of extend, we know we will have to use rule LC_TY. It is easy to see from the definition of extend that Ψ' is Ψ'' with an additional mapping from a. Thus, $\Gamma \vdash_{\mathsf{c}} \Delta_1, \Delta'_2 \leftrightsquigarrow \Psi''$ fulfills the first premise of LC_TY. To use LC_TY, we must show the following:
  1. $\Gamma \vdash_{\mathsf{ty}} \tau \;:\; \Psi''_1(\kappa)$
     We know $\Gamma \vdash_{\mathsf{tel}} \overline{\rho}'_2, \tau : \Psi_1(\Delta'_2,\ a{:}\,\kappa)$. Inverting gives us $\Gamma \vdash_{\mathsf{ty}} \tau \;:\; \Psi_1(\kappa)[\overline{\rho}'_2/\Psi_1(\Delta'_2)]$. Because we care only about the names of the variables in the substitution expression, we can rewrite this as $\Gamma \vdash_{\mathsf{ty}} \tau \;:\; \Psi_1(\kappa)[\overline{\rho}'_2/\Delta'_2]$. From the definition of extend, we can see that all of the substitutions performed by $\Psi''_1(\cdot)$ that are not in Ψ map a domain element of Δ'_2 to its corresponding $\rho \in \overline{\rho}'_2$. Thus, we can rewrite the judgement above as $\Gamma \vdash_{\mathsf{ty}} \tau \;:\; \Psi''_1(\kappa)$ as desired.
  2. $\Gamma \vdash_{\mathsf{ty}} \tau \rhd \Psi''(\kappa) \;:\; \Psi''_2(\kappa)$
     We wish to use the lifting lemma (Lemma 5.3). We know $\Gamma \vdash_{\mathsf{c}} \Delta_1, \Delta'_2 \leftrightsquigarrow \Psi''$. We must show $\Gamma, \Delta_1, \Delta'_2 \vdash_{\mathsf{ty}} \kappa : \sigma$ for some σ. This fact, for $\sigma = \star$, comes directly from inversion on $\vdash_{\mathsf{wf}} \Gamma, \Delta_1, \Delta'_2,\ a{:}\,\kappa$.
     Now, we apply the lifting lemma to get $\Gamma \vdash_{\mathsf{co}} \Psi''(\kappa) \;:\; \Psi''_1(\kappa) \sim \Psi''_2(\kappa)$. As shown in the previous case, $\Gamma \vdash_{\mathsf{ty}} \tau \;:\; \Psi''_1(\kappa)$. Therefore, by simple application of typing rules, we can derive $\Gamma \vdash_{\mathsf{ty}} \tau \rhd \Psi''(\kappa) \;:\; \Psi''_2(\kappa)$ as desired.
  3. $\Gamma \vdash_{\mathsf{co}} \mathbf{sym}\,(\langle \tau \rangle \rhd \Psi''(\kappa)) \;:\; \tau \sim (\tau \rhd \Psi''(\kappa))$
     Straightforward application of typing rules.
- Case $\overline{\rho}_2 = \overline{\rho}'_2, \gamma$; $\Delta_2 = \Delta'_2,\ c{:}\,\phi$:
  The inductive hypothesis is the same as in the previous case. We must show $\Gamma \vdash_{\mathsf{c}} \Delta_1, \Delta'_2,\ c{:}\,\phi \leftrightsquigarrow \Psi'$, where $\Psi' = \mathsf{extend}(\Psi; \overline{\rho}'_2, \gamma; \Delta'_2,\ c{:}\,\phi)$.
  By the definition of the extend operation, we know we will have to use rule LC_CO. It is easy to see from the definition of extend that Ψ' is Ψ'' with an additional mapping from c. Thus, $\Gamma \vdash_{\mathsf{c}} \Delta_1, \Delta'_2 \leftrightsquigarrow \Psi''$ fulfills the first premise of LC_CO. To use LC_CO, we must show the following:
  1. $\Gamma \vdash_{\mathsf{co}} \gamma \;:\; \Psi''_1(\phi)$
     We know $\Gamma \vdash_{\mathsf{tel}} \overline{\rho}'_2, \gamma : \Psi_1(\Delta'_2,\ c{:}\,\phi)$. Inverting gives us $\Gamma \vdash_{\mathsf{co}} \gamma \;:\; \Psi_1(\phi)[\overline{\rho}'_2/\Psi_1(\Delta'_2)]$. Because we care only about the names of the variables in the substitution expression, we can rewrite this as $\Gamma \vdash_{\mathsf{co}} \gamma \;:\; \Psi_1(\phi)[\overline{\rho}'_2/\Delta'_2]$. From the definition of extend, we can see that all of the substitutions performed by $\Psi''_1(\cdot)$ that are not in Ψ map a domain element of Δ'_2 to its corresponding $\rho \in \overline{\rho}'_2$. Thus, we can rewrite the judgement above as $\Gamma \vdash_{\mathsf{co}} \gamma \;:\; \Psi''_1(\phi)$, as desired.
  2. $\Gamma \vdash_{\mathsf{co}} \mathbf{sym}\,(\Psi''(\sigma_1)) \fatsemi \gamma \fatsemi \Psi''(\sigma_2) \;:\; \Psi''_2(\sigma_1) \sim \Psi''_2(\sigma_2)$, where $\phi = \sigma_1 \sim \sigma_2$
     We wish to use the lifting lemma (Lemma 5.3) twice to get the types of $\Psi''(\sigma_1)$ and $\Psi''(\sigma_2)$. We know $\Gamma \vdash_{\mathsf{c}} \Delta_1, \Delta_2 \leftrightsquigarrow \Psi''$. We must show $\Gamma, \Delta_1, \Delta_2 \vdash_{\mathsf{ty}} \sigma_1 \;:\; \kappa_1$ for some κ_1 and $\Gamma, \Delta_1, \Delta_2 \vdash_{\mathsf{ty}} \sigma_2 \;:\; \kappa_2$ for some κ_2. Inversion on $\vdash_{\mathsf{wf}} \Gamma, \Delta_1, \Delta_2,\ c{:}\,\sigma_1 \sim \sigma_2$ gives us $\Gamma, \Delta_1, \Delta_2 \vdash_{\mathsf{pr}} \sigma_1 \sim \sigma_2$ ok, and further inversion gives us $\Gamma, \Delta_1, \Delta_2 \vdash_{\mathsf{ty}} \sigma_1 \;:\; \kappa_1$ and $\Gamma, \Delta_1, \Delta_2 \vdash_{\mathsf{ty}} \sigma_2 \;:\; \kappa_2$ for some κ_1 and κ_2, exactly what we needed.
     Now, we apply the lifting lemma to get $\Gamma \vdash_{\mathsf{co}} \Psi''(\sigma_i) \;:\; \Psi''_1(\sigma_i) \sim \Psi''_2(\sigma_i)$. As shown in the previous case, $\Gamma \vdash_{\mathsf{co}} \gamma \;:\; \Psi''_1(\sigma_1) \sim \Psi''_1(\sigma_2)$. Therefore, by simple application of typing rules, we can derive $\Gamma \vdash_{\mathsf{co}} \mathbf{sym}\,(\Psi''(\sigma_1)) \fatsemi \gamma \fatsemi \Psi''(\sigma_2) \;:\; \Psi''_2(\sigma_1) \sim \Psi''_2(\sigma_2)$ as desired.

□

Lemma C.11 (Telescope composition)**.** *If* $\Gamma \vdash_{\mathsf{tel}} \overline{\rho}_1 \;:\; \Delta_1$ *and* $\Gamma \vdash_{\mathsf{tel}} \overline{\rho}_1, \overline{\rho}_2 : \Delta_1, \Delta_2$*, then* $\Gamma \vdash_{\mathsf{tel}} \overline{\rho}_2 : \Delta_2[\overline{\rho}_1/\Delta_1]$.

Proof Sketch. By induction on the length of $\overline{\rho}_2$. □

Lemma C.12 (S_KPUSH preservation)**.** *If*

1. $\Gamma \vdash_{\mathsf{tm}} \mathbf{case}\ (K\ \overline{\tau}\ \overline{\rho}\ \overline{e} \rhd \gamma)\ \mathbf{of}\ \overline{p \to u} \;:\; \sigma$ *and*
2. $\mathbf{case}\ (K\ \overline{\tau}\ \overline{\rho}\ \overline{e} \rhd \gamma)\ \mathbf{of}\ \overline{p \to u} \longrightarrow \mathbf{case}\ (K\ \overline{\tau'}\ \overline{\rho'}\ \overline{e'})\ \mathbf{of}\ \overline{p \to u}$*, then*

$\Gamma \vdash_{\mathsf{tm}} \mathbf{case}\ (K\ \overline{\tau'}\ \overline{\rho'}\ \overline{e'})\ \mathbf{of}\ \overline{p \to u} \;:\; \sigma$

Proof. By inversion we know that:

- $K{:}\,\forall\, \overline{a{:}\,\kappa}.\, \forall\, \Delta.\, \overline{\sigma} \to (T\ \overline{a})$
- $\Psi = \mathsf{extend}(\mathsf{context}(\gamma); \overline{\rho}; \Delta)$
- $\overline{\tau'} = \Psi_2(\overline{a})$
- $\overline{\rho}' = \Psi_2(\mathit{dom}\,\Delta)$
- $e'_i = e_i \rhd \Psi(\sigma_i)$
- $\Gamma \vdash_{\mathsf{tm}} e_i \;:\; \sigma_i[\overline{\tau}/\overline{a}][\overline{\rho}/\Delta]$
- $\Gamma \vdash_{\mathsf{tm}} (K\ \overline{\tau}\ \overline{\rho}\ \overline{e}) \rhd \gamma \;:\; T\ \overline{\tau'}$.
- $\Gamma \vdash_{\mathsf{tm}} K\ \overline{\tau}\ \overline{\rho}\ \overline{e} \;:\; T\ \overline{\tau}$.

We will have to use rule T_CASE to get the desired result. Because the patterns are not changing, we need only show that $\Gamma \vdash_{\mathsf{tm}} K\ \overline{\tau'}\ \overline{\rho}'\ \overline{e'} \;:\; T\ \overline{\tau'}$.

By convention, we have chosen the length of the list $\overline{\tau}$ to be the same as that of the list $\overline{a{:}\kappa}$ in the type of K. Thus, we know that $\Gamma \vdash_{\mathsf{ty}} K\ \overline{\tau'}\ :\ \forall \Delta[\overline{\tau'/a}].\,(\overline{\sigma}[\overline{\tau'/a}] \to T\ \overline{\tau'})$.

Now, we must show that $\Gamma \vdash_{\mathsf{tel}} \overline{\rho'}\ :\ \Delta[\overline{\tau'/a}]$. This can be rewritten as $\Gamma \vdash_{\mathsf{tel}} \Psi_2(dom\,\Delta) : \Delta[\overline{\tau'/a}]$.

We know from Lemma C.10 that $\Gamma \vdash_{\mathsf{c}} \overline{a{:}\kappa}, \Delta \leftrightsquigarrow \Psi$ (using Lemma 5.5 to get $\Gamma \vdash_{\mathsf{c}} \overline{a{:}\kappa} \leftrightsquigarrow \mathsf{context}(\gamma)$). Lemma C.9 then gives us $\Gamma \vdash_{\mathsf{tel}} \Psi_2(\overline{a}, dom\,\Delta)\ :\ \overline{a{:}\kappa}, \Delta$. Invoking Lemma C.11 gives us $\Gamma \vdash_{\mathsf{tel}} \Psi_2(dom\,\Delta) : \Delta[\overline{\tau'/a}]$ as desired.

We have now shown that $\Gamma \vdash_{\mathsf{ty}} K\ \overline{\tau'}\,\overline{\rho'}\ :\ \Psi_2(\overline{\sigma}) \to T\ \overline{\tau'}$. We need to show that $\Gamma \vdash_{\mathsf{tm}} e_i'\ :\ \Psi_2(\sigma_i)$, or equivalently, $\Gamma \vdash_{\mathsf{tm}} e_i \rhd \Psi(\sigma_i)\ :\ \Psi_2(\sigma_i)$. We will need the lifting lemma (Lemma 5.3). We have already shown $\Gamma \vdash_{\mathsf{c}} \overline{a{:}\kappa}, \Delta \leftrightsquigarrow \Psi$; we must show $\Gamma, \overline{a{:}\kappa}, \Delta \vdash_{\mathsf{ty}} \sigma_i\ :\ \kappa_i$ for some type κ_i. By repeated inversion on the typing judgement for K, we will get $\Gamma, \overline{a{:}\kappa}, \Delta \vdash_{\mathsf{ty}} \sigma_i\ :\ \kappa_i$ as desired. Thus, the lifting lemma gives us $\Gamma \vdash_{\mathsf{co}} \Psi(\sigma_i)\ :\ \Psi_1(\sigma_i) \sim \Psi_2(\sigma_i)$. We note that, by construction, $\Psi_1(\cdot)$ maps $\overline{a}$ to $\overline{\tau}$ and $dom\,\Delta$ to $\overline{\rho}$. Thus, $\sigma_i[\overline{\tau/a}][\overline{\rho/\Delta}] = \Psi_1(\sigma_i)$. Now, by straightforward application of typing rules, we can see that $\Gamma \vdash_{\mathsf{tm}} e_i \rhd \Psi(\sigma_i)\ :\ \Psi_2(\sigma_i)$ as desired.

Thus, $\Gamma \vdash_{\mathsf{tm}} K\ \overline{\tau'}\ \overline{\rho'}\ \overline{e'}\ :\ T\ \overline{\tau'}$ as desired, and we are done. □

C.3 Other preservation cases

Lemma C.13 (S_TPUSH Preservation). *If*

1. $\Gamma \vdash_{\mathsf{tm}} (v \rhd \gamma)\,\tau\ :\ \sigma_2[\tau/a_2]$ *and*
2. $\Gamma \vdash_{\mathsf{co}} \gamma\ :\ \forall\, a_1{:}\kappa_1.\,\sigma_1 \sim \forall\, a_2{:}\kappa_2.\,\sigma_2$
3. $(v \rhd \gamma)\,\tau \longrightarrow e'$ *where*
4. $e' = v\ (\tau \rhd \gamma') \rhd \gamma@(\langle\tau\rangle \rhd \gamma')$ *and*
5. $\gamma' = \mathbf{sym}\,(\mathbf{nth}^1\,\gamma)$,

then $\Gamma \vdash_{\mathsf{tm}} e'\ :\ \sigma_2[\tau/a_2]$.

Proof. By inversion of the typing derivation we know that $\Gamma \vdash_{\mathsf{tm}} v \rhd \gamma\ :\ \forall\, a_2{:}\kappa_2.\,\sigma_2$ and $\Gamma \vdash_{\mathsf{ty}} \tau\ :\ \kappa_2$. An additional inversion gives us $\Gamma \vdash_{\mathsf{tm}} v\ :\ \forall\, a_1{:}\kappa_1.\,\sigma_1$. Therefore we can show that

- $\Gamma \vdash_{\mathsf{co}} \gamma'\ :\ \kappa_2 \sim \kappa_1$, by the rules for symmetry and nth and
- $\Gamma \vdash_{\mathsf{ty}} \tau \rhd \gamma'\ :\ \kappa_1$, by casting and
- $\Gamma \vdash_{\mathsf{tm}} v\ (\tau \rhd \gamma')\ :\ \sigma_1[\tau \rhd \gamma'/a_1]$, by type application.

Furthermore, we have

- $\Gamma \vdash_{\mathsf{co}} \langle\tau\rangle \rhd \gamma'\ :\ \tau \rhd \gamma' \sim \tau$, by reflexivity and coherence and
- $\Gamma \vdash_{\mathsf{co}} \gamma@(\langle\tau\rangle \rhd \gamma')\ :\ \sigma_1[\tau \rhd \gamma'/a_1] \sim \sigma_2[\tau/a_2]$, by instantiation.

Thus the final term has the desired type by casting. □

Lemma C.14 (S_CPUSH Preservation). *If*

1. $\Gamma \vdash_{\mathsf{tm}} (v \rhd \gamma)\,\gamma'\ :\ \sigma$ *and*
2. $(v \rhd \gamma)\,\gamma' \longrightarrow v\,\gamma'' \rhd \gamma@(\gamma'', \gamma')$, *where*
3. $\gamma'' = \mathbf{nth}^1\,\gamma \fatsemi \gamma' \fatsemi \mathbf{sym}\,(\mathbf{nth}^2\,\gamma)$ *and*
4. $\Gamma \vdash_{\mathsf{co}} \gamma\ :\ (\forall\, c{:}\,\phi.\,\tau) \sim (\forall\, c'{:}\,\phi'.\,\tau')$,

then $\Gamma \vdash_{\mathsf{tm}} v\,\gamma'' \rhd \gamma@(\gamma'', \gamma')\ :\ \sigma$.

Proof. By inversion, we have

- $\Gamma \vdash_{\mathsf{tm}} v \rhd \gamma\ :\ \forall\, c'{:}\,\phi'.\,\tau'$
- $\Gamma \vdash_{\mathsf{tm}} v\ :\ \forall\, c{:}\,\phi.\,\tau$
- $\Gamma \vdash_{\mathsf{co}} \gamma'\ :\ \phi'$
- $\sigma = \tau'[\gamma'/c']$.

Let $\phi = \sigma_1 \sim \sigma_2$ and $\phi' = \sigma_1' \sim \sigma_2'$. We can show

- $\Gamma \vdash_{\mathsf{co}} \mathbf{nth}^1\,\gamma\ :\ \sigma_1 \sim \sigma_1'$, by CT_NTH1CA
- $\Gamma \vdash_{\mathsf{co}} \mathbf{nth}^2\,\gamma\ :\ \sigma_2 \sim \sigma_2'$, by CT_NTH2CA
- $\Gamma \vdash_{\mathsf{co}} \mathbf{sym}\,(\mathbf{nth}^2\,\gamma)\ :\ \sigma_2' \sim \sigma_2$, by symmetry
- $\Gamma \vdash_{\mathsf{co}} \gamma''\ :\ \sigma_1 \sim \sigma_2$, by definition of transitivity
- $\Gamma \vdash_{\mathsf{tm}} v\,\gamma''\ :\ \tau[\gamma''/c]$, by coercion instantiation
- $\Gamma \vdash_{\mathsf{co}} \gamma@(\gamma'', \gamma')\ :\ \tau[\gamma''/c] \sim \tau'[\gamma'/c']$, by CT_INSTC.

The final term has the desired type by casting. □

D. Type Erasure

We need the following small lemma before we can prove type erasure:

Lemma D.1 (Type erasure erases types). *No type variable a or coercion variable c appears free in $|e|$, for any expression e.*

Proof. Straightforward inspection of definition of $|e|$. □

Corollary D.2 (Substitution in erased expressions). *For all expressions e, types τ, coercions γ, and variables a and c,*

1. $|e[\tau/a]| = |e|$
2. $|e[\gamma/c]| = |e|$

We can now present the proof for the type erasure theorem: If $e \longrightarrow e'$, then either $|e| = |e'|$ or $|e| \longrightarrow |e'|$.

Proof. We proceed by induction on $e \longrightarrow e'$.

Cases S_BETA, S_TBETA, S_CBETA, S_CASEMATCH: The erased expression steps by the same rule as the unerased expression, appealing to Corollary D.2 in the S_TBETA and S_CBETA cases.

Cases S_EAPP, S_TAPP, S_CAPP, S_CASE: We appeal to the induction hypothesis. If $|e| = |e'|$, we are done. If $|e| \longrightarrow |e'|$, then we use the same stepping rule as the unerased expression used.

Cases S_PUSH, S_TPUSH, S_CPUSH, S_COMB: Straightforward application of the definition of erasure yields $|e| = |e'|$.

Case S_COERCE:

$$\frac{e \longrightarrow e'}{e \rhd \gamma \longrightarrow e' \rhd \gamma}\ \text{S\_COERCE}$$

By the definition of erasure, $|e \rhd \gamma| = |e|$ and $|e' \rhd \gamma| = |e'|$. By the induction hypothesis, either $|e| = |e'|$ or $|e| \longrightarrow |e'|$. In either case, we are done.

Case S_KPUSH:

$$\frac{\begin{array}{l} K{:}\,\forall\,\overline{a{:}\kappa}.\,\forall\,\Delta.\,\overline{\sigma} \to (T\ \overline{a}) \in \Gamma \\ \Psi = \mathsf{extend}(\mathsf{context}(\gamma); \overline{\rho}; \Delta) \\ \overline{\tau'} = \Psi_2(\overline{a}) \\ \overline{\rho'} = \Psi_2(dom\,\Delta) \\ \text{for each } e_i \in \overline{e}, \\ \qquad e_i' = e_i \rhd \Psi(\sigma_i) \end{array}}{\begin{array}{l}\mathbf{case}\ ((K\ \overline{\tau}\ \overline{\rho}\ \overline{e}) \rhd \gamma)\ \mathbf{of}\ \overline{p \to u} \longrightarrow \\ \quad \mathbf{case}\ (K\ \overline{\tau'}\ \overline{\rho'}\ \overline{e'})\ \mathbf{of}\ \overline{p \to u}\end{array}}\ \text{S\_KPUSH}$$

Here, $e = \mathbf{case}\ ((K\ \overline{\tau}\ \overline{\rho}\ \overline{e}) \rhd \gamma)\ \mathbf{of}\ \overline{p \to u}$ and $e' = \mathbf{case}\ (K\ \overline{\tau'}\ \overline{\rho'}\ \overline{e'})\ \mathbf{of}\ \overline{p \to u}$. Thus, by the definition of erasure, $|e| = \mathbf{case}\ (K\ \bar{\bullet}\ \overline{|e|})\ \mathbf{of}\ \overline{p \to |u|}$ and $|e'| = \mathbf{case}\ (K\ \bar{\bullet}\ \overline{|e'|})\ \mathbf{of}\ \overline{p \to |u|}$. To show that $|e| = |e'|$, we must show only that $\overline{|e|} = \overline{|e'|}$. From the definition of S_KPUSH, $e_i' = e_i \rhd \Psi(\sigma_i)$ and thus $|e_i'| = |e_i|$ as desired.

□

E. Metatheory for Consistency

In this section, we show that good contexts are consistent contexts following the plan laid out in Section 6. Recall the conditions of a good context:

We have $\mathbf{Good}\,\Gamma$ when the following conditions hold:

1. All coercion assumptions and axioms in Γ are of the form $C{:}\,\forall\,\Delta.\,(F\,\overline{\tau} \sim \tau')$ or of the form $c{:}\,a_1 \sim a_2$. In the first form, the arguments to the type function must behave like patterns. For every well formed $\overline{\rho}$, every $\tau_i \in \overline{\tau}$ and every τ'_i such that $\Gamma \models \tau_i[\overline{\rho}/\Delta] \rightsquigarrow \tau'_i$, it must be $\tau'_i = \tau_i[\overline{\rho'}/\Delta]$ for some $\overline{\rho}'$ with $\Gamma \models \sigma_m \rightsquigarrow \sigma'_m$ for each $\sigma_m \in \overline{\rho}$ and $\sigma'_m \in \overline{\rho}'$.
2. There is no overlap between axioms and coercion assumptions. For each $F\,\overline{\rho}$ there exists at most one prefix $\overline{\rho}_1$ of $\overline{\rho}$ such that there exist C and σ where $\Gamma \models C\,\overline{\rho}_1 \;:\; (F\,\overline{\rho_1} \sim \sigma)$. This C is unique for every matching $F\,\overline{\tau_1}$.
3. For each a, there is at most one assumption of the form $c{:}\,a \sim a'$ or $c{:}\,a' \sim a$, and $a \neq a'$.
4. Axioms equate types of the same kind. For each $C{:}\,\forall\,\Delta.\,(F\,\overline{\tau} \sim \tau')$ in Γ, the kinds of each side must equal: for some κ, $\Gamma, \Delta \models F\,\overline{\tau} \;:\; \kappa$ and $\Gamma, \Delta \models \tau' \;:\; \kappa$ and that kind must not mention bindings in the telescope, $\Gamma \models \kappa \;:\; \star$.

Showing that these conditions ensure that the context cannot prove two value types equal requires a number of auxiliary lemmas.

Lemma E.1 (No free coercion variables in erased types)**.** *If* $\Gamma \vdash_{\mathsf{ty}} \tau \;:\; \kappa$*, then* $c \# |\tau|$.

Proof. Proof is by inspection of the erasure function. All coercions are removed from types. ☐

Proof of Lemma 6.5 (Erasure is type preserving) If a judgement holds in the explicit system, the judgement with coercions erased throughout the context, types and coercions is derivable in the implicit system.

1. If $\vdash_{\mathsf{wf}} \Gamma$ then $\models |\Gamma|$.
2. If $\Gamma \vdash_{\mathsf{ty}} \tau \;:\; \kappa$ then $|\Gamma| \models |\tau| \;:\; |\kappa|$.
3. If $\Gamma \vdash_{\mathsf{pr}} \phi$ ok then $|\Gamma| \models |\phi|$ ok.
4. If $\Gamma \vdash_{\mathsf{co}} \gamma \;:\; \phi$ then $|\Gamma| \models |\gamma| \;:\; |\phi|$.
5. If $\Gamma \vdash_{\mathsf{tel}} \overline{\rho} : \Delta$ then $|\Gamma| \models |\overline{\rho}| \;:\; |\Delta|$.

Proof. By simultaneous induction on the length of the explicit typing derivation. We present a few representative cases.

Case K_CAST: Given rule:

$$\frac{\Gamma \vdash_{\mathsf{ty}} \tau \;:\; \kappa_1 \quad \Gamma \vdash_{\mathsf{co}} \eta \;:\; \kappa_1 \sim \kappa_2 \quad \Gamma \vdash_{\mathsf{ty}} \kappa_2 \;:\; \star}{\Gamma \vdash_{\mathsf{ty}} \tau \rhd \eta \;:\; \kappa_2}\ \text{K\_CAST}$$

By induction, we have $|\Gamma| \models |\tau| \;:\; |\kappa_1|$ and $|\Gamma| \models |\eta| \;:\; |\kappa_1| \sim |\kappa_2|$ and $|\Gamma| \models |\kappa_2| \;:\; |\star|$. By the rule IT_CAST, we have $|\Gamma| \models |\tau| \;:\; |\kappa_2|$. Finally, by definition of erasure, we have $|\tau \rhd \eta| = |\tau|$, and we are done.

Case K_CAPP: Given rule:

$$\frac{\Gamma \vdash_{\mathsf{ty}} \tau_1 \;:\; \forall\, c{:}\,\phi.\,\kappa \quad \Gamma \vdash_{\mathsf{co}} \gamma_1 \;:\; \phi}{\Gamma \vdash_{\mathsf{ty}} \tau_1\,\gamma_1 \;:\; \kappa[\gamma_1/c]}\ \text{K\_CAPP}$$

By induction and definition of erasure, we have $|\Gamma| \models |\tau_1| \;:\; \forall\, c{:}\,|\phi|.\,|\kappa|$, and $|\Gamma| \models |\gamma_1| \;:\; |\phi|$. Hence, by rule IT_CAPP, we have $|\Gamma| \models |\tau_1|\,\bullet_{\mathsf{co}} \;:\; |\kappa|$, and by erasure $|\tau_1\,\gamma_1| = |\tau_1|\,\bullet_{\mathsf{co}}$. Finally, we have $|\kappa[\gamma/c]| = |\kappa|$, as the erasure operation erases all coercions within κ.

Case CT_COH: Given rule:

$$\frac{\Gamma \vdash_{\mathsf{co}} \gamma \;:\; \tau_1 \sim \tau_2 \quad \Gamma \vdash_{\mathsf{ty}} \tau_1 \rhd \gamma' \;:\; \kappa}{\Gamma \vdash_{\mathsf{co}} \gamma \rhd \gamma' \;:\; \tau_1 \rhd \gamma' \sim \tau_2}\ \text{CT\_COH}$$

By induction and erasure, we have $|\Gamma| \models |\gamma| \;:\; |\tau_1| \sim |\tau_2|$. But also by erasure, we have $|\gamma \rhd \gamma'| = |\gamma|$ and $|\tau_1 \rhd \gamma' \sim \tau_2| = |\tau_1| \sim |\tau_2|$, so we are done.

Case CT_CAPP: Given rule:

$$\frac{\begin{array}{l}\Gamma \vdash_{\mathsf{co}} \gamma_1 \;:\; \tau_1 \sim \tau'_1 \\ \Gamma \vdash_{\mathsf{ty}} \tau_1\,\gamma_2 \;:\; \kappa \quad \Gamma \vdash_{\mathsf{ty}} \tau'_1\,\gamma'_2 \;:\; \kappa'\end{array}}{\Gamma \vdash_{\mathsf{co}} \gamma_1(\gamma_2, \gamma'_2) \;:\; \tau_1\,\gamma_2 \sim \tau'_1\,\gamma'_2}\ \text{CT\_CAPP}$$

By induction and definition of erasure, we have $|\Gamma| \models |\tau_1|\,\bullet_{\mathsf{co}} \;:\; |\kappa|$, $|\Gamma| \models |\tau'_1|\,\bullet_{\mathsf{co}} \;:\; |\kappa'|$, and $|\Gamma| \models |\gamma| \;:\; |\tau_1| \sim |\tau'_1|$. Hence, by rule ICT_CAPP, we have $|\Gamma| \models |\gamma|(\bullet_{\mathsf{co}}, \bullet_{\mathsf{co}}) \;:\; |\tau_1|\,\bullet_{\mathsf{co}} \sim |\tau'_1|\,\bullet_{\mathsf{co}}$, and we are done by erasure.

Case CT_ALLC: Given rule:

$$\frac{\begin{array}{l}\Gamma \vdash_{\mathsf{co}} \eta_1 \;:\; \sigma_1 \sim \sigma'_1 \quad \phi_1 = \sigma_1 \sim \sigma_2 \\ \Gamma \vdash_{\mathsf{co}} \eta_2 \;:\; \sigma_2 \sim \sigma'_2 \quad \phi_2 = \sigma'_1 \sim \sigma'_2 \\ c_1 \# |\gamma| \quad c_2 \# |\gamma| \\ \Gamma,\ c_1{:}\,\phi_1,\ c_2{:}\,\phi_2 \vdash_{\mathsf{co}} \gamma \;:\; \tau_1 \sim \tau_2 \\ \Gamma \vdash_{\mathsf{ty}} \forall\, c_1{:}\,\phi_1.\,\tau_1 \;:\; \star \quad \Gamma \vdash_{\mathsf{ty}} \forall\, c_2{:}\,\phi_2.\,\tau_2 \;:\; \star\end{array}}{\Gamma \vdash_{\mathsf{co}} \forall_{(\eta_1,\eta_2)}(c_1, c_2).\gamma \;:\; (\forall\, c_1{:}\,\phi_1.\,\tau_1) \sim (\forall\, c_2{:}\,\phi_2.\,\tau_2)}\ \text{CT\_ALLC}$$

By induction and definition of erasure, we have

- $|\Gamma| \models |\eta_1| \;:\; |\sigma_1| \sim |\sigma'_1|$,
- $|\Gamma| \models |\eta_2| \;:\; |\sigma_2| \sim |\sigma'_2|$,
- $|\phi_1| = |\sigma_1| \sim |\sigma_2|$,
- $|\phi_2| = |\sigma'_1| \sim |\sigma'_2|$,
- $|\Gamma|,\ c_1{:}\,|\phi_1|,\ c_2{:}\,|\phi_2| \models |\gamma| \;:\; |\tau_1| \sim |\tau_2|$,
- $|\Gamma| \models \forall\, c_1{:}\,|\phi_1|.\,|\tau_1| \;:\; \star$, and
- $|\Gamma| \models \forall\, c_2{:}\,|\phi_2|.\,|\tau_2| \;:\; \star$.

Furthermore, the original rule restricted c_1 and c_2 from appearing in $|\gamma|$. Hence by, ICT_ALLC, we have $|\Gamma| \models \forall_{(|\eta_1|,|\eta_2|)}(c_1, c_2).|\gamma| \;:\; (\forall\, c_1{:}\,|\phi_1|.\,|\tau_1|) \sim (\forall\, c_2{:}\,|\phi_2|.\,|\tau_2|)$ and we are done by erasure.

☐

Lemma E.2 (Application)**.** *If* $\mathbf{Good}\,\Gamma$ *and* $\Gamma \models \sigma_1 \Leftrightarrow \sigma'_1$ *and* $\Gamma \models \sigma_2 \Leftrightarrow \sigma'_2$ *then* $\Gamma \models \sigma_1\,\sigma_2 \Leftrightarrow \sigma'_1\,\sigma'_2$.

Proof. Let τ_1 be a join point of σ_1, σ'_1, and τ_2 a join point for σ_2, σ'_2. By repeatedly applying rule TS_APP and reflexivity of rewriting, we find that $\tau_1\,\tau_2$ is a join point for $\sigma_1\,\sigma_2$ and $\sigma'_1\,\sigma'_2$. ☐

Lemma E.3 (Type function preservation)**.** *Suppose that* $C{:}\,\forall\,\Delta.\,(F\,\overline{\tau} \sim \tau') \in \Gamma$*, and* $\mathbf{Good}\,\Gamma$*. Now, suppose that* $\Gamma \models F\,\overline{\rho} \rightsquigarrow \sigma$*, where* $\overline{\rho}$ *has length strictly smaller than the size of the telescope* $\overline{\tau}$*. Then,* $\sigma = F\,\overline{\rho}'$*, such that* $\Gamma \models \overline{\rho} \rightsquigarrow \overline{\rho}'$.

Proof. By induction on the length of the telescope $\overline{\rho}$. The base case is trivial. For the induction step, note that the rule that applies in the given reduction cannot be TS_RED, as there aren't enough terms in the telescope to reduce. Thus, it must be TS_APP that applies. Therefore, if $\overline{\rho} = \overline{\rho}', \sigma'$, then $\Gamma \models F\,\overline{\rho}' \rightsquigarrow \sigma'$, and $\Gamma \models \sigma' \rightsquigarrow \sigma''$. By induction, $\sigma' = F\,\overline{\rho}''$, and by rule TS_CONS, $\Gamma \models \overline{\rho} \rightsquigarrow \overline{\rho}'', \sigma''$ as desired. ☐

Here, we prove completeness of the rewrite reduction with respect to the coercion relation. The two key lemmas of the completeness proof are that joinability is preserved under substitution, and a local diamond property of rewriting.

Lemma E.4 (Local diamond property)**.** *If* $\mathbf{Good}\,\Gamma$, $\Gamma \models \sigma \rightsquigarrow \sigma_1$, *and* $\Gamma \models \sigma \rightsquigarrow \sigma_2$ *then there exists a* σ_3 *such that* $\Gamma \models \sigma_1 \rightsquigarrow \sigma_3$ *and* $\Gamma \models \sigma_2 \rightsquigarrow \sigma_3$.

Proof. Induction on lengths of the two step derivations with a case analysis on the last rule used in each.

The overlapping cases are TS_REFL and anything else, TS_APP-TS_RED (and symmetric), and all instances with the same final rule on both sides. The reflexivity overlaps are trivial. All other pairs of rules apply to types with different head forms. Of the same-same overlaps, most follow by induction. (We demonstrate an example of this pattern with case TS_APP-TS_APP below.) The exception is TS_RED-TS_RED and TS_VARRED-TS_VARRED which are both deterministic. Below, we complete the proof with the TS_APP-TS_RED case.

Case TS_APP-TS_APP Concretely, we have a type $\tau\,\sigma$ with reductions:

$$\Gamma \models \tau\,\sigma \rightsquigarrow \tau'\,\sigma', \quad \Gamma \models \tau\,\sigma \rightsquigarrow \tau''\,\sigma''$$

Now, we can deduce:

$$\Gamma \models \sigma \rightsquigarrow \sigma', \quad \Gamma \models \sigma \rightsquigarrow \sigma''$$

So by induction, we can find σ''' that is a common reduct. We also know

$$\Gamma \models \tau \rightsquigarrow \tau', \quad \Gamma \models \tau \rightsquigarrow \tau''$$

So, also by induction, we can find τ''' that is a common reduct of the two. Hence, by TS_TAPP,

$$\Gamma \models \tau'\,\sigma' \rightsquigarrow \tau'''\,\sigma''' \qquad \Gamma \models \tau''\,\sigma'' \rightsquigarrow \tau'''\,\sigma'''$$

Case TS_RED-TS_APP Concretely, we have a type $F\,\overline{\rho}$, with reductions:

$$\Gamma \models (F\,\overline{\rho}) \rightsquigarrow \sigma_1, \quad \Gamma \models (F\,\overline{\rho}) \rightsquigarrow \sigma_2'\,\sigma'$$

where the first reduction is a type function reduction. Now note that, since context is good, type functions axioms are non-overlapping. Now say that $\overline{\rho} = \overline{\rho}_0, \sigma$ We have by inversion, $\Gamma \models F\,\overline{\rho}_0 \rightsquigarrow \sigma_2'$. By Lemma E.3, we have that $\sigma_2' = F\,\overline{\rho}_0'$, such that $\Gamma \models \overline{\rho}_0 \rightsquigarrow \overline{\rho}_0'$, and so that $\Gamma \models \overline{\rho}_0, \sigma \rightsquigarrow \overline{\rho}_0', \sigma'$. We have that if the coercion for F is $C{:}\ \forall\,\Delta.\,(F\,\overline{\tau} \sim \tau')$, then we have $\overline{\rho}_0, \sigma = \overline{\tau}[\overline{\rho_1}/\Delta]$, and now by the second condition of good contexts, we have a $\overline{\rho}_1'$, such that

$$\overline{\rho}_0', \sigma' = \overline{\tau}[\overline{\rho_1'}/\Delta] \qquad \Gamma \models \overline{\rho}_1 \rightsquigarrow \overline{\rho}_1'$$

In which case we have a reduction $\Gamma \models F\,\overline{\rho}_0'\,\sigma' \rightsquigarrow \tau'[\overline{\rho_1'}/\Delta]$. But, by an extension of Appendix E for telescopes, we have that

$$\sigma_1 = \tau'[\overline{\rho_1}/\Delta] \qquad \Gamma \models \sigma_1 \rightsquigarrow \tau'[\overline{\rho_1'}/\Delta]$$

as desired.

Case TS_RED-TS_RED Concretely, we have a type $F\ \overline{\sigma_1}\ \overline{\sigma_2}$, which can also be written as $F\ \overline{\sigma_3}\ \overline{\sigma_4}$, such that we have reductions:

$$\Gamma \models F\,\overline{\sigma} \rightsquigarrow \sigma', \quad \Gamma \models F\,\overline{\sigma} \rightsquigarrow \sigma''$$

But since good contexts have non-overlapping axioms, we have that only one axiom applies. Hence, we are done: $\sigma' = \sigma''$.

□

Lemma E.5 (Transitivity of Rewriting)**.** *If* $\mathbf{Good}\,\Gamma$ *and* $\Gamma \models \sigma_1 \Leftrightarrow \sigma_2$ *and* $\Gamma \models \sigma_2 \Leftrightarrow \sigma_3$*, then* $\Gamma \models \sigma_1 \Leftrightarrow \sigma_3$.

Proof. Appeal to the local diamond property. Suppose σ_{12} is a join point for σ_1, σ_2 and σ_{23} is a join point for σ_2, σ_3. By Lemma E.4, there is a join point σ_0 for σ_{12}, σ_{23}, and hence is a join point for σ_1, σ_3. □

Lemma E.6 (Single Step Substitution)**.** *If* $\Gamma \models \tau \rightsquigarrow \tau'$*,* σ *well-formed and* a *not free in* Γ*, then* $\Gamma \models \sigma[\tau/a] \rightsquigarrow \sigma[\tau'/a]$.

Proof. By induction on σ. For instance, if $\sigma = \forall\,c{:}\,\sigma_1 \sim \sigma_2.\,\sigma'$, by induction,

$$\Gamma \models \sigma'[\tau/a] \rightsquigarrow \sigma'[\tau'/a],$$

and also

$$\Gamma \models \sigma_i[\tau/a] \rightsquigarrow \sigma_i[\tau'/a].$$

Thus, by rule TS_ALLC, we conclude

$$\Gamma \models (\forall\,c{:}\,\sigma_1 \sim \sigma_2.\,\sigma')[\tau/a] \rightsquigarrow (\forall\,c{:}\,\sigma_1 \sim \sigma_2.\,\sigma')[\tau'/a].$$

The other cases are similar. □

Lemma E.7 (Multistep Substitution)**.** *If* $\Gamma \models \tau \rightsquigarrow^* \tau'$*,* σ *well-formed and* a *not free in* Γ*, then* $\Gamma \models \sigma[\tau/a] \rightsquigarrow^* \sigma[\tau'/a]$.

Proof. By induction on the length of the reduction $\Gamma \models \tau \rightsquigarrow^* \tau'$. The base case is trivial, and the inductive step follows by Lemma E.6. □

Lemma E.8 (Single Substitution)**.** *If* $\Gamma \models \tau \Leftrightarrow \tau'$*, and* $\Gamma \models \sigma \rightsquigarrow \sigma'$*, with* σ, σ' *well-formed and* a *not free in* Γ*, then* $\Gamma \models \sigma[\tau/a] \Leftrightarrow \sigma'[\tau'/a]$.

Proof. There is a join point τ'' of τ and τ', we can apply Lemma E.7 to the reductions $\Gamma \models \tau \rightsquigarrow^* \tau''$ and $\Gamma \models \tau' \rightsquigarrow^* \tau''$, and connect two reductions with Lemma E.5. □

Lemma E.9 (Single Step Double Substitution)**.** *Suppose* $\mathbf{Good}\,\Gamma$ *and* $\Gamma \models \sigma \rightsquigarrow \sigma'$*, with* a *free in* σ *and* a' *free in* σ' *well-formed, and* $\Gamma = \Gamma', c{:}\,a \sim a', \Gamma''$ *or* $\Gamma = \Gamma', c{:}\,a' \sim a, \Gamma''$*. Then* $\mathbf{Good}\,(\Gamma', \Gamma'')[\tau/a][\tau'/a']$*, and if* $(\Gamma', \Gamma'')[\tau/a][\tau'/a'] \models \tau \Leftrightarrow \tau'$*, then* $(\Gamma', \Gamma'')[\tau/a][\tau'/a'] \models \sigma[\tau/a] \Leftrightarrow \sigma'[\tau'/a']$.

Proof. Note first that since $\mathbf{Good}\,\Gamma$, the only axiom mentioning a, a' is c. Hence, $\mathbf{Good}\,(\Gamma', \Gamma'')[\tau/a][\tau'/a']$ is immediate. The rest follows by induction on the derivation of $\Gamma \models \sigma \rightsquigarrow \sigma'$.

Case TS_REFL: Follows from Lemma E.8.

Case TS_ALLT: The rule is

$$\frac{\Gamma, \Gamma' \models \kappa \rightsquigarrow \kappa' \quad \Gamma, c{:}\,a_1 \sim a_2, \Gamma' \models \sigma \rightsquigarrow \sigma'}{\Gamma, \Gamma' \models \forall\,a_1{:}\,\kappa.\,\sigma \rightsquigarrow \forall\,a_2{:}\,\kappa'.\,\sigma'}\ \text{TS\_ALLT}$$

where $a_1 \neq a_2 \neq a \neq a'$, and $\Gamma, \Gamma' = \Gamma'', c'{:}\,a \sim a', \Gamma'''$. By induction, we have both $(\Gamma, \Gamma')[\tau/a][\tau'/a'] \models \kappa[\tau/a] \Leftrightarrow \kappa[\tau'/a']$ and $\Gamma[\tau/a][\tau'/a'], c{:}\,a_1 \sim a_2, \Gamma'[\tau/a][\tau'/a'] \models \sigma[\tau/a] \Leftrightarrow \sigma'[\tau'/a']$. But now, we can put these transitions together with rule TS_ALLT: first, we have

$$(\Gamma, \Gamma')[\tau/a][\tau'/a'] \models$$
$$\forall\,a_1{:}\,\kappa[\tau/a].\,\sigma[\tau/a] \Leftrightarrow \forall\,a_1{:}\,\kappa'[\tau'/a'].\,\sigma[\tau/a]$$

By α-renaming, the right hand side is $\forall\,a_2{:}\,\kappa'[\tau'/a'].\,\sigma[\tau/a]$. Now, we have

$$(\Gamma, \Gamma')[\tau/a][\tau'/a'] \models$$
$$\forall\,a_2{:}\,\kappa'[\tau'/a'].\,\sigma[\tau/a] \Leftrightarrow \forall\,a_2{:}\,\kappa'[\tau'/a'].\,\sigma'[\tau'/a']$$

and we are done by Lemma E.5.

Case TS_ALLC: Almost identical to the previous case.

Case TS_RED: Follows from Lemma E.8.

Case TS_VARRED: If the coercion in question isn't $c{:}\,a \sim a'$, then this case is trivial. Otherwise, we have by assumption that $(\Gamma', \Gamma'')[\tau/a][\tau'/a'] \models \tau \Leftrightarrow \tau'$, which is exactly what we need to prove.

Case TS_APP: Similar to the previous cases. By induction (suppose we are substituting σ'', σ''',

$$(\Gamma', \Gamma'')[\sigma''/a][\sigma'''/a'] \models \tau[\sigma''/a] \Leftrightarrow \tau'[\sigma'''/a']$$

and

$$(\Gamma', \Gamma'')[\sigma''/a][\sigma'''/a'] \models \sigma[\sigma''/a] \Leftrightarrow \sigma'[\sigma'''/a'].$$

So, by applying TS_APP, we have

$$(\Gamma', \Gamma'')[\sigma''/a][\sigma'''/a'] \models \tau[\sigma''/a]\,(\sigma[\sigma''/a]) \Leftrightarrow \tau'[\sigma'''/a']\,(\sigma[\sigma''/a])$$

and

$$(\Gamma', \Gamma'')[\sigma''/a][\sigma'''/a'] \models \tau'[\sigma'''/a']\,(\sigma[\sigma''/a]) \Leftrightarrow \tau'[\sigma'''/a']\,(\sigma'[\sigma'''/a'])$$

so we are done by Lemma E.5.

Case TS_CAPP: Immediate, by induction.

□

Lemma E.10 (Substitution)**.** *Suppose* $\mathbf{Good}\,\Gamma$ *and* $\Gamma \models \sigma \rightsquigarrow^* \sigma'$*, with* a, a' *free in* σ, σ'*, and* $\Gamma = \Gamma', c{:}\, a \sim a', \Gamma''$*. Suppose also that* $(\Gamma', \Gamma'')[\tau/a][\tau'/a'] \models \tau \Leftrightarrow \tau'$*. Then,* $(\Gamma', \Gamma'')[\tau/a][\tau'/a'] \models \sigma[\tau/a] \Leftrightarrow \sigma'[\tau'/a']$.

Proof. Induction on the length of reduction $\Gamma \models \sigma \rightsquigarrow^* \sigma'$. The base case is trivial. The inductive step follows by Appendix E and Lemma E.5. □

Corollary E.11 (Joinability substitution)**.** *Suppose* $\mathbf{Good}\,\Gamma$ *and* $\Gamma \models \sigma \Leftrightarrow \sigma'$*, with* a, a' *free in* σ, σ'*, and* $\Gamma = \Gamma', c{:}\, a \sim a', \Gamma''$*. Suppose also that* $(\Gamma', \Gamma'')[\tau/a][\tau'/a'] \models \tau \Leftrightarrow \tau'$*. Then,* $(\Gamma', \Gamma'')[\tau/a][\tau'/a'] \models \sigma[\tau/a] \Leftrightarrow \sigma'[\tau'/a']$.

Proof. By induction on the number of transitions in $\Gamma \models \sigma \Leftrightarrow \sigma'$. The base case is trivial. For the induction step, we can use the induction hypothesis, combined with Lemma E.10. □

Lemma E.12 (Joinability strengthening)**.** *If* $\mathbf{Good}\,(\Gamma,\ a{:}\,\kappa, \Gamma')$ *and* $\Gamma,\ a{:}\,\kappa, \Gamma' \models \tau_1 \Leftrightarrow \tau_2$*, then* $\mathbf{Good}\,(\Gamma, \Gamma')\ \Gamma, \Gamma' \models \tau_1 \Leftrightarrow \tau_2$.

Proof. By inspection on the rewrite relation. The rewrite relation does not depend on any type bindings in the context, only axioms. □

Lemma E.13 (Basic implicit substitution)**.**

1. *If* $\Gamma,\ a{:}\,\kappa_1, \Gamma' \models \tau : \kappa_2$ *and* $\Gamma \models \sigma : \kappa_1$*, then* $\Gamma, \Gamma'[\sigma/a] \models \tau[\sigma/a] : \kappa_2[\sigma/a]$.
2. *If* $\models \Gamma,\ a{:}\,\kappa_1, \Gamma'$ *and* $\Gamma \models \sigma : \kappa_1$*, then* $\models \Gamma, \Gamma'[\sigma/a]$.
3. *If* $\Gamma,\ a{:}\,\kappa_1, \Gamma' \models \phi$ *ok and* $\Gamma \models \sigma : \kappa_1$*, then* $\Gamma, \Gamma'[\sigma/a] \models \phi[\sigma/a]$ *ok.*

Proof. Straightforward mutual induction. □

Lemma E.14 (Implicit regularity/generation)**.** *If* $\Gamma \models \tau : \kappa$*, then* $\Gamma \models \kappa : \star$ *and the height of this derivation is at most the height of* $\Gamma \models \tau : \kappa$.

Proof. Straightforward induction, appealing to Lemma E.13 in the IT_TAPP case. The base cases appeal to rules IT_STARINSTAR and IT_ARROWK. □

Lemma E.15 (Weakening for implicit system)**.** *If* $\Gamma \models \tau : \kappa$*, then* $\Gamma, \Gamma' \models \tau : \kappa$ *for any* Γ' *such that* $\models \Gamma, \Gamma'$*, and there exists a derivation of* $\Gamma \models \tau : \kappa$ *with height at most the height of the derivation of* $\Gamma, \Gamma' \models \tau : \kappa$.

Proof. Straightforward induction. □

We need a lemma to deal with the $\mathbf{kind}\,\gamma$ construct. Essentially, this lemma states that we don't need the $\mathbf{kind}\,\gamma$ construct, as it is already internalized in our system.

Lemma E.16 (Admissibility of "kind")**.** *Suppose we have a derivation* $\Gamma \models \gamma : \tau_1 \sim \tau_2$*, such that* $\Gamma \models \tau_1 : \kappa_1$ *and* $\Gamma \models \tau_2 : \kappa_2$ *and* $fcv(\gamma) \subseteq dom\,\Gamma'$ *for some subcontext* Γ' *satisfying* $\mathbf{Good}\,\Gamma'$*. Then, there exists a derivation* $\Gamma \models \eta : \kappa_1 \sim \kappa_2$ *at strictly* lower *height, for some* η*, such that* $fcv(\eta) \subseteq dom\,\Gamma'$.

Proof Sketch. By induction on the derivation $\Gamma \models \gamma : \tau_1 \sim \tau_2$. Most cases are straightforward. We consider two here.

Case ICT_TRANS: Given rule:

$$\frac{\Gamma \models \gamma_1 : \tau_1 \sim \tau_2 \quad \Gamma \models \gamma_2 : \tau_2 \sim \tau_3}{\Gamma \models \gamma_1 \mathbin{⨾} \gamma_2 : \tau_1 \sim \tau_3}\ \text{ICT\_TRANS}$$

Note that the free coercion variables of $\gamma_1 \mathbin{⨾} \gamma_2$ lie in a good context, so the same is true of γ_1 and γ_2. Hence, by induction, we are able to find derivations of $\Gamma \models \eta_1 : \kappa_1 \sim \kappa_2$ and $\Gamma \models \eta_2 : \kappa_2 \sim \kappa_3$, both of height less than that of any premises of ICT_TRANS. Now, by ICT_TRANS, we are able to create a proof $\Gamma \models \eta_1 \mathbin{⨾} \eta_2 : \kappa_1 \sim \kappa_3$ at height strictly less than that of the conclusion, and we are done.

Case ICT_AXIOM: Given rule:

$$\frac{C{:}\, \forall\,\Delta.\,(\tau_1 \sim \tau_2) \in \Gamma \quad \Gamma \models \overline{\rho} : \Delta}{\Gamma \models C\,\overline{\rho} : \tau_1[\overline{\rho}/\Delta] \sim \tau_2[\overline{\rho}/\Delta]}\ \text{ICT\_AXIOM}$$

Note that the free coercion variables of $C\,\overline{\rho}$ lie in a good context, so the same is true of C and $\overline{\rho}$. Thus, the axiom lies in a good subcontext. By definition of $\mathbf{Good}\,\Gamma'$, we have that both sides are kind κ for a closed kind. Hence, simply choosing $\eta = \langle\kappa\rangle$ suffices. This η will have no free coercion variables (appealing to Lemma E.1), so the restriction on free coercion variables is vacuously satisfied.

But, what is the height of the derivation of $\Gamma \models \langle\kappa\rangle : \kappa \sim \kappa$? It is one more than the height of $\Gamma \models \kappa : \sigma$, the premise of $\Gamma \models \langle\kappa\rangle : \kappa \sim \kappa$. This derivation $\Gamma \models \kappa : \sigma$ must be a part of the derivation concluding in ICT_AXIOM: One of the premises to ICT_AXIOM is $\Gamma \models \overline{\rho} : \Delta$. That judgement, in turn, must eventually appeal to IT2_EMPTY, which depends on $\models \Gamma$. Because $C \in dom\,\Gamma$, the proof of $\models \Gamma$ must appeal to IV_AX, which in turn depends on $\Gamma, \Delta \models \tau_1 \sim \tau_2$ ok. This depends on $\Gamma, \Delta \models \tau_1 : \kappa$. By Lemma E.14, the proof that $\Gamma, \Delta \models \kappa : \sigma$ (for $\sigma = \star$) is strictly smaller than that of $\Gamma, \Delta \models \tau_1 : \kappa$. From the fact that C lies in a good context, we know that κ must not contain variables introduced in Δ, so there exists a derivation $\Gamma \models \kappa : \sigma$ and the height of this derivation is no larger than the height of $\Gamma, \Delta \models \kappa : \sigma$, invoking Lemma E.15. Thus, the height of $\Gamma \models \langle\kappa\rangle : \kappa \sim \kappa$ is strictly smaller than the height of the derivation concluding in ICT_AXIOM, as desired.

□

Lemma E.17 (Nth joinability)**.** *Suppose that* $\Gamma \models H\,\overline{\rho} \Leftrightarrow H\,\overline{\rho}'$*, and* $\mathbf{Good}\,\Gamma$*. Then,* $\Gamma \models \rho_i \Leftrightarrow \rho_i'$.

Proof. By induction on the length of the telescopes (by inversion, both have the same length). The base case is trivial. For induction, note that $H\,\overline{\rho}, H\,\overline{\rho}'$ must both step by TS_APP. Hence, by the form of that rewrite rule, say that $\overline{\rho} = \overline{\rho}_0, \rho_0$ and $\overline{\rho}' = \overline{\rho}_0', \rho_0'$, and the length of the telescopes are preserved. So, $\Gamma \models \rho_0 \Leftrightarrow \rho_0'$. If we want the last element in the telescope, we are done. Otherwise, $\Gamma \models H\,\overline{\rho}_0 \Leftrightarrow H\,\overline{\rho}_0'$, and we are done by induction. □

From these lemmas we see that joinability is complete.

Proof of Lemma 6.7 (Completeness) Suppose that $\Gamma \models \gamma : \sigma_1 \sim \sigma_2$, and $fcv(\gamma) \subseteq dom\,\Gamma'$ for some subcontext Γ' satisfying $\mathbf{Good}\,\Gamma'$. Then $\Gamma \models \sigma_1 \Leftrightarrow \sigma_2$.

Proof. By induction on the structure of $\Gamma \models \gamma : \sigma_1 \sim \sigma_2$.

Case ICT_CApp: We have rule:

$$\frac{\begin{array}{l}\Gamma \models \gamma : \tau \sim \tau' \\ \Gamma \models \tau \bullet_{\mathsf{co}} : \kappa \quad \Gamma \models \tau' \bullet_{\mathsf{co}} : \kappa'\end{array}}{\Gamma \models \gamma(\bullet_{\mathsf{co}}, \bullet_{\mathsf{co}}) : \tau \bullet_{\mathsf{co}} \sim \tau' \bullet_{\mathsf{co}}} \quad \text{ICT\_CAPP}$$

Note that the free coercion variables of $\gamma(\bullet_{\mathsf{co}}, \bullet_{\mathsf{co}})$ lie in a good context, so the same is true of γ. Hence, by induction, $\Gamma \models \tau \Leftrightarrow \tau'$. Then, by Lemma E.2, we are done.

Case ICT_AllC:

$$\frac{\begin{array}{l}\Gamma \models \eta_1 : \sigma_1 \sim \sigma_1' \quad \phi_1 = \sigma_1 \sim \sigma_2 \\ \Gamma \models \eta_2 : \sigma_2 \sim \sigma_2' \quad \phi_2 = \sigma_1' \sim \sigma_2' \\ c_1 \# \gamma \quad c_2 \# \gamma \\ \Gamma,\, c_1{:}\,\phi_1,\, c_2{:}\,\phi_2 \models \gamma : \tau_1 \sim \tau_2 \\ \Gamma \models \forall\, c_1{:}\,\phi_1.\, \tau_1 : \star \quad \Gamma \models \forall\, c_2{:}\,\phi_2.\, \tau_2 : \star\end{array}}{\Gamma \models \forall_{(\eta_1,\eta_2)}(c_1, c_2).\gamma : (\forall\, c_1{:}\,\phi_1.\, \tau_1) \sim (\forall\, c_2{:}\,\phi_2.\, \tau_2)}$$

Note that the free coercion variables of $\forall_{(\eta_1,\eta_2)}(c_1, c_2).\gamma$ lie in a good context, so the same is true of γ, η_1, and η_2. Hence, by induction, there is a join point σ_1'' for σ_1 and σ_1', and there is a join point σ_2'' for σ_2 and σ_2'. Let $\phi = \sigma_1'' \sim \sigma_2''$. Also by induction, there is a join point τ for τ_1, τ_2. By rule TS_AllC, we have that

$$\Gamma \models \forall\, c_1{:}\,\phi_1.\, \tau_1 \rightsquigarrow^{*} \forall\, c_1{:}\,\phi.\, \tau$$

and

$$\Gamma \models \forall\, c_2{:}\,\phi_2.\, \tau_2 \rightsquigarrow^{*} \forall\, c_2{:}\,\phi.\, \tau$$

and hence they are joinable.

Case ICT_Inst:

$$\frac{\begin{array}{l}\Gamma \models \gamma_1 : (\forall\, a_1{:}\,\kappa_1.\, \tau_1) \sim (\forall\, a_2{:}\,\kappa_2.\, \tau_2) \\ \Gamma \models \gamma_2 : \sigma_1 \sim \sigma_2 \\ \Gamma \models \sigma_1 : \kappa_1 \quad \Gamma \models \sigma_2 : \kappa_2\end{array}}{\Gamma \models \gamma_1 @ \gamma_2 : \tau_1[\sigma_1/a_1] \sim \tau_2[\sigma_2/a_2]} \quad \text{ICT\_INST}$$

Note that the free coercion variables of $\gamma @ \gamma'$ lie in a good context, so the same is true of γ and γ. Hence, by induction, $\Gamma \models \sigma_1 \Leftrightarrow \sigma_2$, and $\Gamma \models (\forall\, a_1{:}\,\kappa_1.\, \tau_1) \Leftrightarrow (\forall\, a_2{:}\,\kappa_2.\, \tau_2)$. Now, by inversion on the step relation for quantified types, we find that $\Gamma,\; c{:}\, a_1 \sim a_2 \models \tau_1 \Leftrightarrow \tau_2$. By substitution (Lemma E.10), we have $\Gamma \models \tau_1[\sigma_1/a_1] \Leftrightarrow \tau_2[\sigma_2/a_2]$, as desired.

Case ICT_InstC:

$$\frac{\begin{array}{l}\Gamma \models \gamma : (\forall\, c{:}\,\sigma_1 \sim \sigma_2.\, \tau) \sim (\forall\, c'{:}\,\sigma_1' \sim \sigma_2'.\, \tau') \\ \Gamma \models \gamma_1 : \sigma_1 \sim \sigma_2 \quad \Gamma \models \gamma_2 : \sigma_1' \sim \sigma_2'\end{array}}{\Gamma \models \gamma @ (\bullet_{\mathsf{co}}, \bullet_{\mathsf{co}}) : \tau \sim \tau'}$$

Note that the free coercion variables of $\gamma @ (\bullet_{\mathsf{co}}, \bullet_{\mathsf{co}})$ lie in a good context, so the same is true of γ. Hence, by induction, $\Gamma \models (\forall\, c{:}\,\sigma_1 \sim \sigma_2.\, \tau) \Leftrightarrow (\forall\, c{:}\,\sigma_1' \sim \sigma_2'.\, \tau')$. Now, by inversion on the step relation for quantified types, we find that $\Gamma \models \tau \Leftrightarrow \tau'$, and we are done.

Case ICT_Refl: Trivial.

Case ICT_Sym: Trivial.

Case ICT_Trans: Follows from Lemma E.5.

Case ICT_App: Follows from Lemma E.2.

Case ICT_AllT:

$$\frac{\begin{array}{l}\Gamma \models \eta : \kappa_1 \sim \kappa_2 \\ \Gamma,\, a_1{:}\,\kappa_1,\, a_2{:}\,\kappa_2,\, c{:}\, a_1 \sim a_2 \models \gamma : \tau_1 \sim \tau_2 \\ \Gamma \models \forall\, a_1{:}\,\kappa_1.\, \tau_1 : \star \quad \Gamma \models \forall\, a_2{:}\,\kappa_2.\, \tau_2 : \star\end{array}}{\Gamma \models \forall_{\eta}(a_1, a_2, c).\gamma : (\forall\, a_1{:}\,\kappa_1.\, \tau_1) \sim (\forall\, a_2{:}\,\kappa_2.\, \tau_2)}$$

Note that the free coercion variables of $\forall_{\eta}(a_1, a_2, c).\gamma$ lie in a good context, so the same is true of γ since $c{:}\, a_1 \sim a_2$ is a good assumption that doesn't overlap with the previous axioms, as the variables a_1, a_2 are fresh. Hence, by induction, we have $\Gamma,\, a_1{:}\,\kappa_1,\, a_2{:}\,\kappa_2,\, c{:}\, a_1 \sim a_2 \models \tau_1 \Leftrightarrow \tau_2$, which we can strengthen to $\Gamma,\, c{:}\, a_1 \sim a_2 \models \tau_1 \Leftrightarrow \tau_2$, by Lemma E.12. Also by induction, we have $\Gamma \models \kappa_1 \Leftrightarrow \kappa_2$, which allows us to finish the rule by TS_AllT.

Case ICT_Var: Trivial, all assumptions are rewrite rules in good contexts. Note that c must be a good assumption in the context.

Case ICT_Axiom: We have the rule:

$$\frac{C{:}\, \forall\, \Delta.\, (\tau_1 \sim \tau_2) \in \Gamma \quad \Gamma \models \overline{\rho} : \Delta}{\Gamma \models C\, \overline{\rho} : \tau_1[\overline{\rho}/\Delta] \sim \tau_2[\overline{\rho}/\Delta]} \quad \text{ICT\_AXIOM}$$

We know that C lies in a good context, so we know that τ_1 has the form $F\, \overline{\tau_1'}$. Thus, rule TS_Red shows that $\Gamma \models \tau_1[\overline{\rho}/\Delta] \rightsquigarrow \tau_2[\overline{\rho}/\Delta]$, so we are done, noting that TS_Refl shows that $\Gamma \models \tau_2[\overline{\rho}/\Delta] \rightsquigarrow \tau_2[\overline{\rho}/\Delta]$ as well.

Case ICT_Nth: We have the rule:

$$\frac{\Gamma \models \gamma : H\, \overline{\tau} \sim H\, \overline{\tau'}}{\Gamma \models \mathbf{nth}^{i}\, \gamma : \tau_i \sim \tau_i'} \quad \text{ICT\_NTH}$$

Note that the free coercion variables of $\mathbf{nth}^{i}\, \gamma$ lie in a good context, so the same is true of γ. Hence, by induction, then Lemma E.17, we are done.

Case ICT_Nth1TA: We have rule:

$$\frac{\Gamma \models \gamma_1 : (\forall\, a_1{:}\,\kappa_1.\, \tau_1) \sim (\forall\, a_2{:}\,\kappa_2.\, \tau_2)}{\Gamma \models \mathbf{nth}^{1}\, \gamma_1 : \kappa_1 \sim \kappa_2} \quad \text{ICT\_NTH1TA}$$

Note that the free coercion variables of $\mathbf{nth}^{1}\, \gamma_1$ lie in a good context, so the same is true of γ_1. Hence, by induction on γ_1, the two quantified types have a join point. By inversion on the rewrite relation, both sides must step via TS_AllT. Hence, we can find a join point for the kinds, and $\Gamma \models \kappa_1 \Leftrightarrow \kappa_2$ as desired.

Cases ICT_Nth1CA, ICT_Nth2CA: We have rules:

$$\frac{\Gamma \models \gamma : (\forall\, c{:}\,\kappa_1 \sim \kappa_2.\, \tau) \sim (\forall\, c'{:}\,\kappa_1' \sim \kappa_2'.\, \tau')}{\Gamma \models \mathbf{nth}^{1}\, \gamma : \kappa_1 \sim \kappa_1'} \quad \text{ICT\_NTH1CA}$$

$$\frac{\Gamma \models \gamma : (\forall\, c{:}\,\kappa_1 \sim \kappa_2.\, \tau) \sim (\forall\, c'{:}\,\kappa_1' \sim \kappa_2'.\, \tau')}{\Gamma \models \mathbf{nth}^{2}\, \gamma : \kappa_2 \sim \kappa_2'} \quad \text{ICT\_NTH2CA}$$

Virtually identical to the previous case.

Case ICT_Ext: We have rule:

$$\frac{\Gamma \models \gamma : \tau_1 \sim \tau_2 \quad \Gamma \models \tau_1 : \kappa_2 \quad \Gamma \models \tau_2 : \kappa_2}{\Gamma \models \mathbf{kind}\, \gamma : \kappa_1 \sim \kappa_2} \quad \text{ICT\_EXT}$$

By the admissibility of $\mathbf{kind}\, \gamma$ (Lemma E.16) we can construct a derivation of $\Gamma \models \eta : \kappa_1 \sim \kappa_2$ at strictly smaller height that proves the same equality, such that η has free variables in a good context. Then, we are done by induction.

□

Proof of the consistency lemma, Lemma 6.8:

Proof. Suppose $\Gamma \vdash_{\mathsf{co}} \gamma : \xi_1 \sim \xi_2$. Then, we have that $|\Gamma| \models |\gamma| : |\xi_1| \sim |\xi_2|$. By completeness, we have that those two types are joinable. There is some σ such that $|\Gamma| \models |\xi_1| \rightsquigarrow^{*} \sigma$ and $\Gamma \models |\xi_2| \rightsquigarrow^{*} \sigma$. However, by inversion on the rewriting relation, we see that it preserves the head forms of value types (since there exist no axioms for those by the first condition of $\mathbf{Good}\,|\Gamma|$). Also, we know that erasure preserves head forms. Thus, ξ_1 and ξ_2 (and σ) have the same head form. □

F. Metatheory for Progress

Using the consistency lemma, it is straightforward to prove progress. We refer the reader to previous work [Weirich et al. 2010] for this proof, which requires only one more case for the current system:

Lemma F.1 (Progress for T_CONTRA)**.** *Assume* Σ *is a closed, consistent context. If* $\Sigma \vdash_{\mathsf{tm}} \mathbf{contra}\,\gamma\,\tau \;:\; \tau$*, then there exists some* e *such that* $\mathbf{contra}\,\gamma\,\tau \longrightarrow e$*.*

Proof. By inversion on $\Sigma \vdash_{\mathsf{tm}} \mathbf{contra}\,\gamma\,\tau \;:\; \tau$, we get that $\Sigma \vdash_{\mathsf{co}} \gamma \;:\; H_1\,\overline{\rho}_1 \sim H_2\,\overline{\rho}_2$ and that $H_1 \neq H_2$. However, these facts exactly contradict the fact that Σ is consistent. Thus, our premises are absurd and we are done. □