Dynamic Updating of Information-Flow Policies

Michael Hicks* Boniface Hicks† Stephen Tse‡ Steve Zdancewic‡
* University of Maryland † Pennsylvania State University ‡ University of Pennsylvania

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This report is automatically generated by the tool `lf2tex` from the semantics specification `update.elf` (Twelf source) and the syntactic specification `update.tex` (LaTeX source).

Some notes about the typesetting used in this report:

- A syntax class is displayed boxed like $\boxed{\mathtt{e} ::= \cdots}$ (*terms*).
- A syntax entity is displayed unboxed like $\mathtt{e} ::= \lambda \mathtt{x{:}t.\ e}$ (*functions*).
- A judgement form is displayed boxed like $\boxed{\Pi;\Gamma \vdash \mathtt{e} : \mathtt{t}}$ (*typings*).
- A judgement rule is displayed unboxed like $\dfrac{\Pi;\Gamma,\mathtt{x{:}t_1} \vdash \mathtt{e} : \mathtt{t_2}}{\Pi;\Gamma \vdash \lambda \mathtt{x{:}t.e} : \mathtt{t_1} \rightarrow \mathtt{t_2}}$ (*typing for functions*).
- A theorem is displayed boxed like $\boxed{\dfrac{\Pi_1;\Gamma \vdash \mathtt{e_1{:}t} \quad (\Pi_1;\mathtt{e_1}) \longrightarrow (\Pi_2;\mathtt{e_2})}{\Pi_2;\Gamma \vdash \mathtt{e_2} : \mathtt{t}}}$ (*preservations*).
- A proof is displayed as a numbered list with the last one of the list being the conclusion. (A rule name with downarrow ($\downarrow$ ty-fun) means by the inversion of such rule.)

Contents

1 Syntax

1.1 Constants

$$\boxed{\texttt{c} ::= \cdots} \qquad \boxed{\texttt{c}}$$

1.2 Principals

$$\boxed{\texttt{p} ::= \cdots} \qquad \boxed{\texttt{p}}$$

$$\texttt{p} ::= \texttt{c} \qquad \text{(pc)}$$

$$\texttt{p} ::= \texttt{p,p} \qquad \text{(pcat)}$$

1.3 Labels

$$\boxed{\ell ::= \cdots} \qquad \boxed{\text{l}}$$

$$\ell ::= \texttt{p:}\ \texttt{p} \qquad \text{(lp)}$$

$$\ell ::= \ell, \ell \qquad \text{(lcat)}$$

1.4 Hierarchies

$$\boxed{\Pi ::= \cdots} \qquad \boxed{\text{a}}$$

$$\Pi ::= \cdot \qquad \text{(az)}$$

$$\Pi ::= \Pi, \texttt{p} \leq \texttt{p} \qquad \text{(ap)}$$

1.5 Types

$$\boxed{\texttt{t} ::= \cdots} \qquad \boxed{\texttt{t}}$$

$$\texttt{t} ::= \texttt{bool}_{\ell} \qquad \text{(tbool)}$$

$$\texttt{t} ::= \texttt{t} \rightarrow \texttt{t} \qquad \text{(tfun)}$$

1.6 Variables

$$\boxed{\texttt{x} ::= \cdots} \qquad \boxed{\texttt{x}}$$

1.7 Terms

$$\boxed{\texttt{e} ::= \cdots} \qquad \boxed{\texttt{e}}$$

$$\texttt{e} ::= \texttt{true}_{\ell} \qquad \text{(true)}$$

$$\texttt{e} ::= \texttt{false}_{\ell} \qquad \text{(false)}$$

$$\texttt{e} ::= \texttt{x} \qquad \text{(var)}$$

$$\texttt{e} ::= \lambda[\Pi]\texttt{x:t.}\ \texttt{e} \qquad \text{(fun)}$$

$$\texttt{e} ::= \texttt{e e} \qquad \text{(app)}$$

$$\texttt{e} ::= \texttt{if e e e} \qquad \text{(ifb)}$$

$$\texttt{e} ::= \texttt{if (p} \leq \texttt{p) e e} \qquad \text{(ifp)}$$

$$\texttt{e} ::= [\ell \sqsubseteq \ell]\texttt{e} \qquad \text{(tag)}$$

1.8 Contexts

$\boxed{\Gamma ::= \cdots}$	$\boxed{\text{g}}$
$\Gamma ::= \cdot$	(gz)
$\Gamma ::= \Gamma, \mathtt{x}:\mathtt{t}$	(gx)

2 Static semantics

2.1 Substitutions

$$\boxed{\mathtt{e}\{\mathtt{e}/\mathtt{x}\} = \mathtt{e}} \qquad \boxed{\text{sub}}$$

$$(\mathtt{true}_\ell)\{\mathtt{e}/\mathtt{x}\} = (\mathtt{true}_\ell) \qquad \text{(sub-true)}$$

$$(\mathtt{false}_\ell)\{\mathtt{e}/\mathtt{x}\} = (\mathtt{false}_\ell) \qquad \text{(sub-false)}$$

$$(\mathtt{x})\{\mathtt{e}/\mathtt{x}\} = \mathtt{e} \qquad \text{(sub-var1)}$$

$$(\mathtt{x}_1)\{\mathtt{e}/\mathtt{x}_2\} = (\mathtt{x}_1) \qquad \text{(sub-var2)}$$

$$\frac{\mathtt{e}_1\{\mathtt{e}/\mathtt{x}\} = \mathtt{e}_2}{(\lambda[\Pi]\mathtt{x}_1 : \mathtt{t}.\ \mathtt{e}_1)\{\mathtt{e}/\mathtt{x}\} = (\lambda[\Pi]\mathtt{x}_1 : \mathtt{t}.\ \mathtt{e}_2)} \qquad \text{(sub-fun)}$$

$$\frac{\mathtt{e}_1\{\mathtt{e}/\mathtt{x}\} = \mathtt{e}_3 \quad \mathtt{e}_2\{\mathtt{e}/\mathtt{x}\} = \mathtt{e}_4}{(\mathtt{e}_1\ \mathtt{e}_2)\{\mathtt{e}/\mathtt{x}\} = (\mathtt{e}_3\ \mathtt{e}_4)} \qquad \text{(sub-app)}$$

$$\frac{\mathtt{e}_1\{\mathtt{e}/\mathtt{x}\} = \mathtt{e}_4 \quad \mathtt{e}_2\{\mathtt{e}/\mathtt{x}\} = \mathtt{e}_5 \quad \mathtt{e}_3\{\mathtt{e}/\mathtt{x}\} = \mathtt{e}_6}{(\mathtt{if}\ \mathtt{e}_1\ \mathtt{e}_2\ \mathtt{e}_3)\{\mathtt{e}/\mathtt{x}\} = (\mathtt{if}\ \mathtt{e}_4\ \mathtt{e}_5\ \mathtt{e}_6)} \qquad \text{(sub-ifb)}$$

$$\frac{\mathtt{e}_1\{\mathtt{e}/\mathtt{x}\} = \mathtt{e}_3 \quad \mathtt{e}_2\{\mathtt{e}/\mathtt{x}\} = \mathtt{e}_4}{(\mathtt{if}\ (\mathtt{p}_1 \leq \mathtt{p}_2)\ \mathtt{e}_1\ \mathtt{e}_2)\{\mathtt{e}/\mathtt{x}\} = (\mathtt{if}\ (\mathtt{p}_1 \leq \mathtt{p}_2)\ \mathtt{e}_3\ \mathtt{e}_4)} \qquad \text{(sub-ifp)}$$

$$\frac{\mathtt{e}_1\{\mathtt{e}/\mathtt{x}\} = \mathtt{e}_2}{([\ell_1 \sqsubseteq \ell_2]\mathtt{e}_1)\{\mathtt{e}/\mathtt{x}\} = ([\ell_1 \sqsubseteq \ell_2]\mathtt{e}_2)} \qquad \text{(sub-tag)}$$

2.2 Principal subtypings

$$\boxed{\mathtt{some\ p}} \qquad \boxed{\text{psome}}$$

$$\boxed{\Pi \vdash \mathtt{p} \leq \mathtt{p}} \qquad \boxed{\text{pst}}$$

$$\boxed{\Pi \nvdash \mathtt{p} \leq \mathtt{p}} \qquad \boxed{\text{pnst}}$$

$$\Pi \vdash \mathtt{p} \leq \mathtt{p} \qquad \text{(pst-z)}$$

$$\frac{\mathtt{some\ p}_2 \quad \Pi \vdash \mathtt{p}_1 \leq \mathtt{p}_2 \quad \Pi \vdash \mathtt{p}_2 \leq \mathtt{p}_3}{\Pi \vdash \mathtt{p}_1 \leq \mathtt{p}_3} \qquad \text{(pst-x)}$$

$$(\Pi, \mathtt{p}_1 \leq \mathtt{p}_2) \vdash \mathtt{p}_1 \leq \mathtt{p}_2 \qquad \text{(pst-a1)}$$

$$\frac{\Pi \vdash \mathtt{p}_3 \leq \mathtt{p}_4}{(\Pi, \mathtt{p}_1 \leq \mathtt{p}_2) \vdash \mathtt{p}_3 \leq \mathtt{p}_4} \qquad \text{(pst-a2)}$$

$$\Pi \vdash (\mathtt{p}_1, \mathtt{p}_2) \leq \mathtt{p}_1 \qquad \text{(pst-cat1)}$$

$$\Pi \vdash (\mathtt{p}_1, \mathtt{p}_2) \leq \mathtt{p}_2 \qquad \text{(pst-cat2)}$$

2.3 Label subtypings

$$\boxed{\Pi \vdash \ell \sqsubseteq \ell} \qquad \boxed{\text{lst}}$$

$$\boxed{\Pi \vdash \ell \not\preceq \ell} \qquad \boxed{\text{lnst}}$$

$$\boxed{\mathtt{some}\ \ell} \qquad \boxed{\text{lsome}}$$

$$\Pi \vdash \ell \sqsubseteq \ell \qquad \text{(lst-z)}$$

$$\frac{\texttt{some}\ \ell_2 \quad \Pi \vdash \ell_1 \sqsubseteq \ell_2 \quad \Pi \vdash \ell_2 \sqsubseteq \ell_3}{\Pi \vdash \ell_1 \sqsubseteq \ell_3} \qquad \text{(lst-x)}$$

$$\frac{\Pi \vdash p_1 \leq p_3 \quad \Pi \vdash p_2 \leq p_4}{\Pi \vdash (p_1 : p_2) \sqsubseteq (p_3 : p_4)} \qquad \text{(lst-p)}$$

$$\Pi \vdash \ell_1 \sqsubseteq (\ell_1, \ell_2) \qquad \text{(lst-cat1)}$$

$$\Pi \vdash \ell_2 \sqsubseteq (\ell_1, \ell_2) \qquad \text{(lst-cat2)}$$

2.4 Hierarchy subtypings

$$\boxed{\texttt{some}\ \Pi} \qquad \boxed{\text{asome}}$$

$$\boxed{\Pi \leq \Pi} \qquad \boxed{\text{ast}}$$

$$\Pi \leq \Pi \qquad \text{(ast-z)}$$

$$\frac{\texttt{some}\ \Pi_2 \quad \Pi_1 \leq \Pi_2 \quad \Pi_2 \leq \Pi_3}{\Pi_1 \leq \Pi_3} \qquad \text{(ast-x)}$$

$$\frac{\Pi_1 \leq \Pi_2}{(\Pi_1, p_1 \leq p_2) \leq \Pi_2} \qquad \text{(ast-p1)}$$

$$\frac{\Pi_1 \leq \Pi_2 \quad \Pi_1 \vdash p_3 \leq p_4}{\Pi_1 \leq (\Pi_2, p_3 \leq p_4)} \qquad \text{(ast-p2)}$$

2.5 Type subtypings

$$\boxed{\Pi \vdash t \preceq t} \qquad \boxed{\text{tst}}$$

$$\frac{\Pi \vdash \ell_1 \sqsubseteq \ell_2}{\Pi \vdash (\texttt{bool}_{\ell_1}) \preceq (\texttt{bool}_{\ell_2})} \qquad \text{(tst-bool)}$$

$$\frac{\Pi \vdash t_3 \preceq t_1 \quad \Pi \vdash t_2 \preceq t_4}{\Pi \vdash (t_1 \to t_2) \preceq (t_3 \to t_4)} \qquad \text{(tst-fun)}$$

2.6 Type labels

$$\boxed{\texttt{lab(t)} = \ell} \qquad \boxed{\text{lab}}$$

$$\texttt{lab}((\texttt{bool}_\ell)) = \ell \qquad \text{(lab-bool)}$$

$$\frac{\texttt{lab}(t_2) = \ell}{\texttt{lab}((t_1 \to t_2)) = \ell} \qquad \text{(lab-fun)}$$

2.7 Typings

$$\boxed{\Pi; \Gamma \vdash e : t} \qquad \boxed{\text{ty}}$$

$$\Pi; \Gamma \vdash (\texttt{true}_\ell) : (\texttt{bool}_\ell) \qquad \text{(ty-true)}$$

$$\Pi; \Gamma \vdash (\texttt{false}_\ell) : (\texttt{bool}_\ell) \qquad \text{(ty-false)}$$

$$\frac{\Pi; \Gamma \vdash e_1 : (\texttt{bool}_\ell) \quad \Pi; \Gamma \vdash e_2 : t \quad \Pi; \Gamma \vdash e_3 : t \quad \texttt{lab(t)} = \ell}{\Pi; \Gamma \vdash (\texttt{if}\ e_1\ e_2\ e_3) : t} \qquad \text{(ty-ifb)}$$

$$\Pi; (\Gamma, x{:}t) \vdash (x) : t \qquad \text{(ty-var1)}$$

$$\frac{\Pi;\Gamma \vdash (x_2) : t_2}{\Pi;(\Gamma, x_1 : t_1) \vdash (x_2) : t_2} \quad \text{(ty-var2)}$$

$$\frac{\Pi \leq \Pi_1 \quad \Pi_1;(\Gamma, x : t_1) \vdash e : t_2}{\Pi;\Gamma \vdash (\lambda[\Pi_1] x : t_1 . e) : (t_1 \to t_2)} \quad \text{(ty-fun)}$$

$$\frac{\Pi;\Gamma \vdash e_1 : (t_1 \to t_2) \quad \Pi;\Gamma \vdash e_2 : t_1}{\Pi;\Gamma \vdash (e_1\ e_2) : t_2} \quad \text{(ty-app)}$$

$$\frac{(\Pi, p_1 \leq p_2);\Gamma \vdash e_1 : t \quad \Pi;\Gamma \vdash e_2 : t}{\Pi;\Gamma \vdash (\texttt{if}\ (p_1 \leq p_2)\ e_1\ e_2) : t} \quad \text{(ty-ifp)}$$

$$\frac{\Pi \vdash \ell_1 \sqsubseteq \ell_2 \quad \Pi;\Gamma \vdash e : (\texttt{bool}_{\ell_1})}{\Pi;\Gamma \vdash ([\ell_1 \sqsubseteq \ell_2] e) : (\texttt{bool}_{\ell_2})} \quad \text{(ty-tag)}$$

3 Dynamic semantics

3.1 Values

$$\boxed{\texttt{val e}} \qquad \boxed{\text{val}}$$

$$\texttt{val}\ (\texttt{true}_\ell) \qquad \text{(val-true)}$$

$$\texttt{val}\ (\texttt{false}_\ell) \qquad \text{(val-false)}$$

$$\texttt{val}\ (\lambda[\Pi]\texttt{x:t.}\ \texttt{e}) \qquad \text{(val-fun)}$$

3.2 Holes

$$\boxed{\mathcal{E} ::= \cdots} \qquad \boxed{\text{q}}$$

$$\mathcal{E} ::= \mathcal{E}\ \texttt{e} \qquad \text{(qappa)}$$

$$\mathcal{E} ::= \texttt{e}\ \mathcal{E} \qquad \text{(qappb)}$$

$$\mathcal{E} ::= \texttt{if}\ \mathcal{E}\ \texttt{e}\ \texttt{e} \qquad \text{(qifb)}$$

$$\mathcal{E} ::= [\ell \sqsubseteq \ell]\mathcal{E} \qquad \text{(qtag)}$$

3.3 Splits

$$\boxed{\texttt{e} = \mathcal{E}[\texttt{e}]} \qquad \boxed{\text{split}}$$

$$(\texttt{if}\ \texttt{e}_1\ \texttt{e}_2\ \texttt{e}_3) = (\texttt{if}\ \mathcal{E}\ \texttt{e}_2\ \texttt{e}_3)[\texttt{e}_1] \qquad \text{(split-ifb)}$$

$$(\texttt{e}_1\ \texttt{e}_2) = (\mathcal{E}\ \texttt{e}_2)[\texttt{e}_1] \qquad \text{(split-app1)}$$

$$(\texttt{e}_1\ \texttt{e}_2) = (\texttt{e}_1\ \mathcal{E})[\texttt{e}_2] \qquad \text{(split-app2)}$$

$$([\ell_1 \sqsubseteq \ell_2]\texttt{e}) = ([\ell_1 \sqsubseteq \ell_2]\mathcal{E})[\texttt{e}] \qquad \text{(split-tag)}$$

3.4 Combines

$$\boxed{\mathcal{E}[\texttt{e}] = \texttt{e}} \qquad \boxed{\text{combine}}$$

$$(\mathcal{E}\ \texttt{e}_2)[\texttt{e}_1] = (\texttt{e}_1\ \texttt{e}_2) \qquad \text{(combine-app1)}$$

$$(\texttt{e}_1\ \mathcal{E})[\texttt{e}_2] = (\texttt{e}_1\ \texttt{e}_2) \qquad \text{(combine-app2)}$$

$$(\texttt{if}\ \mathcal{E}\ \texttt{e}_2\ \texttt{e}_3)[\texttt{e}_1] = (\texttt{if}\ \texttt{e}_1\ \texttt{e}_2\ \texttt{e}_3) \qquad \text{(combine-ifb)}$$

$$([\ell_1 \sqsubseteq \ell_2]\mathcal{E})[\texttt{e}] = ([\ell_1 \sqsubseteq \ell_2]\texttt{e}) \qquad \text{(combine-tag)}$$

3.5 Small-step evaluations

$$\boxed{\Pi \vdash \texttt{e} \longrightarrow \texttt{e}} \qquad \boxed{\text{ev}}$$

$$\frac{\texttt{val}\ \texttt{e}_2 \quad \texttt{e}_1\{\texttt{e}_2/\texttt{x}\} = \texttt{e}_3}{\Pi \vdash ((\lambda[\Pi_1]\texttt{x:t.}\ \texttt{e}_1)\ \texttt{e}_2) \longrightarrow \texttt{e}_3} \qquad \text{(ev-app)}$$

$$\Pi \vdash (\texttt{if}\ (\texttt{true}_\ell)\ \texttt{e}_1\ \texttt{e}_2) \longrightarrow \texttt{e}_1 \qquad \text{(ev-ifb1)}$$

$$\Pi \vdash (\texttt{if}\ (\texttt{false}_\ell)\ \texttt{e}_1\ \texttt{e}_2) \longrightarrow \texttt{e}_2 \qquad \text{(ev-ifb2)}$$

$$\frac{\Pi \vdash \texttt{p}_1 \leq \texttt{p}_2}{\Pi \vdash (\texttt{if}\ (\texttt{p}_1 \leq \texttt{p}_2)\ \texttt{e}_1\ \texttt{e}_2) \longrightarrow \texttt{e}_1} \qquad \text{(ev-ifp1)}$$

$$\frac{\Pi \not\vdash p_1 \leq p_2}{\Pi \vdash (\texttt{if } (p_1 \leq p_2)\ e_1\ e_2) \longrightarrow e_2} \quad \text{(ev-ifp2)}$$

$$\Pi \vdash ([\ell_1 \sqsubseteq \ell_2](\texttt{true}_{\ell_1})) \longrightarrow (\texttt{true}_{\ell_2}) \quad \text{(ev-tag1)}$$

$$\Pi \vdash ([\ell_1 \sqsubseteq \ell_2](\texttt{false}_{\ell_1})) \longrightarrow (\texttt{false}_{\ell_2}) \quad \text{(ev-tag2)}$$

$$\frac{e_1 = \mathcal{E}[e_2] \quad \Pi \vdash e_2 \longrightarrow e_3 \quad \mathcal{E}[e_3] = e_4}{\Pi \vdash e_1 \longrightarrow e_4} \quad \text{(ev-hole)}$$

3.6 Tag checkings

$$\boxed{\Pi \vdash e} \qquad \boxed{\text{ae}}$$

$$\Pi \vdash (\texttt{true}_\ell) \quad \text{(ae-true)}$$

$$\Pi \vdash (\texttt{false}_\ell) \quad \text{(ae-false)}$$

$$\frac{\Pi \leq \Pi_1}{\Pi \vdash (\lambda[\Pi_1]x{:}t.\ e)} \quad \text{(ae-fun)}$$

$$\frac{\Pi \vdash e_1 \quad \Pi \vdash e_2}{\Pi \vdash (e_1\ e_2)} \quad \text{(ae-app)}$$

$$\frac{\Pi \vdash e_1 \quad \Pi \vdash e_2 \quad \Pi \vdash e_3}{\Pi \vdash (\texttt{if } e_1\ e_2\ e_3)} \quad \text{(ae-ifb)}$$

$$\frac{(\Pi, p_1 \leq p_2) \vdash e_1 \quad \Pi \vdash e_2}{\Pi \vdash (\texttt{if } (p_1 \leq p_2)\ e_1\ e_2)} \quad \text{(ae-ifp)}$$

$$\frac{\Pi \vdash \ell_1 \sqsubseteq \ell_2 \quad \Pi \vdash e}{\Pi \vdash ([\ell_1 \sqsubseteq \ell_2]e)} \quad \text{(ae-tag)}$$

3.7 Top-level evaluations

$$\boxed{(\Pi; e) | \Pi \longrightarrow (\Pi; e)} \qquad \boxed{\text{topev}}$$

$$\frac{\Pi \vdash e_1 \longrightarrow e_2}{(\Pi; e_1) | \Pi \longrightarrow (\Pi; e_2)} \quad \text{(topev-ev)}$$

$$\frac{\Pi_2 \vdash e}{(\Pi_1; e) | \Pi_2 \longrightarrow (\Pi_2; e)} \quad \text{(topev-upd)}$$

3.8 Totalities

$$\boxed{\overline{\texttt{some } \Pi}} \qquad \boxed{\text{asome-total}}$$

$$\boxed{\overline{e_1\{e/x\} = e_2}} \qquad \boxed{\text{sub-total}}$$

$$\boxed{\Pi \vdash p \diamond p} \qquad \boxed{\text{psty}}$$

$$\frac{\Pi \vdash p_1 \leq p_2}{\Pi \vdash p_1 \diamond p_2} \quad \text{(psty-pst)}$$

$$\frac{\Pi \not\vdash p_1 \leq p_2}{\Pi \vdash p_1 \diamond p_2} \quad \text{(psty-pnst)}$$

$$\boxed{\overline{\Pi \vdash p_1 \diamond p_2}} \qquad \boxed{\text{pst-total}}$$

3.9 Program evaluations

$$\boxed{(\Pi;\mathtt{e}) \Downarrow (\Pi;\mathtt{e})} \qquad \boxed{\text{bigev}}$$

$$\frac{\mathtt{val\ e}}{(\Pi;\mathtt{e}) \Downarrow (\Pi;\mathtt{e})} \qquad \text{(bigev-v)}$$

$$\frac{\mathtt{some}\ \Pi' \quad (\Pi_1;\mathtt{e}_1)|\Pi \longrightarrow (\Pi_2;\mathtt{e}_2) \quad (\Pi_2;\mathtt{e}_2) \Downarrow (\Pi_3;\mathtt{e}_3)}{(\Pi_1;\mathtt{e}_1) \Downarrow (\Pi_2;\mathtt{e}_3)} \qquad \text{(bigev-e)}$$

4 Lemmas

4.1 Principal subtypings

$$\boxed{\frac{\Pi \vdash p_1 \leq p_2}{(\Pi, p_3 \leq p_4) \vdash p_1 \leq p_2}}$$ appst

(appst-z)

1: $\Pi \vdash p_1 \leq p_1$ *given
2: $(\Pi, p_2 \leq p_3) \vdash p_1 \leq p_1$ pst-z

(appst-x)

1: $\Pi \vdash p_5 \leq p_2$ *given
2: some p_1 ↓pst-x: 1
3: $\Pi \vdash p_5 \leq p_1$ ↓pst-x: 1
4: $(\Pi, p_3 \leq p_4) \vdash p_5 \leq p_1$ appst: 3
5: $\Pi \vdash p_1 \leq p_2$ ↓pst-x: 1
6: $(\Pi, p_3 \leq p_4) \vdash p_1 \leq p_2$ appst: 5
7: $(\Pi, p_3 \leq p_4) \vdash p_5 \leq p_2$ pst-x: 2,4,6

(appst-a1)

1: $(\Pi, p_1 \leq p_2) \vdash p_1 \leq p_2$ *given
2: $((\Pi, p_1 \leq p_2), p_3 \leq p_4) \vdash p_1 \leq p_2$ pst-a2: 1

(appst-a2)

1: $(\Pi, p_3 \leq p_4) \vdash p_1 \leq p_2$ *given
2: $\Pi \vdash p_1 \leq p_2$ ↓pst-a2: 1
3: $((\Pi, p_3 \leq p_4), p_5 \leq p_6) \vdash p_1 \leq p_2$ pst-a2: 1

(appst-cat1)

1: $\Pi \vdash (p_1, p_2) \leq p_1$ *given
2: $(\Pi, p_3 \leq p_4) \vdash (p_1, p_2) \leq p_1$ pst-cat1

(appst-cat2)

1: $\Pi \vdash (p_1, p_2) \leq p_2$ *given
2: $(\Pi, p_3 \leq p_4) \vdash (p_1, p_2) \leq p_2$ pst-cat2

$$\boxed{\frac{(\Pi, p_1 \leq p_2) \vdash p_3 \leq p_4 \quad \Pi \vdash p_1 \leq p_2}{\Pi \vdash p_3 \leq p_4}}$$ pstpst

(pstpst-z)

1: $(\Pi, p_1 \leq p_2) \vdash p_3 \leq p_3$ *given
2: $\Pi \vdash p_1 \leq p_2$ *given
3: $\Pi \vdash p_3 \leq p_3$ pst-z

(pstpst-x)

1: $(\Pi, p_1 \leq p_2) \vdash p_5 \leq p_4$ *given
2: some p_3 ↓pst-x: 1
3: $(\Pi, p_1 \leq p_2) \vdash p_5 \leq p_3$ ↓pst-x: 1
4: $\Pi \vdash p_1 \leq p_2$ *given
5: $\Pi \vdash p_5 \leq p_3$ pstpst: 3,4
6: $(\Pi, p_1 \leq p_2) \vdash p_3 \leq p_4$ ↓pst-x: 1
7: $\Pi \vdash p_3 \leq p_4$ pstpst: 6,4
8: $\Pi \vdash p_5 \leq p_4$ pst-x: 2,5,7

(pstpst-a1)

1 : $(\Pi, p_1 \leq p_2) \vdash p_1 \leq p_2$ *given
2 : $\Pi \vdash p_1 \leq p_2$ *given

(pstpst-a2)

1 : $\Pi \vdash p_1 \leq p_2$ *given
2 : $(\Pi, p_1 \leq p_2) \vdash p_3 \leq p_4$ *given
3 : $\Pi \vdash p_3 \leq p_4$ ↓pst-a2: 2

(pstpst-cat1)

1 : $(\Pi, p_1 \leq p_2) \vdash (p_3, p_4) \leq p_3$ *given
2 : $\Pi \vdash p_1 \leq p_2$ *given
3 : $\Pi \vdash (p_3, p_4) \leq p_3$ pst-cat1

(pstpst-cat2)

1 : $(\Pi, p_1 \leq p_2) \vdash (p_3, p_4) \leq p_4$ *given
2 : $\Pi \vdash p_1 \leq p_2$ *given
3 : $\Pi \vdash (p_3, p_4) \leq p_4$ pst-cat2

$$\frac{\Pi_1 \vdash p_1 \leq p_2 \quad \Pi_2 \leq \Pi_1}{\Pi_2 \vdash p_1 \leq p_2}$$ astpst

(astpst-z)

1 : $\Pi_1 \vdash p \leq p$ *given
2 : $\Pi_2 \leq \Pi_1$ *given
3 : $\Pi_2 \vdash p \leq p$ pst-z

(astpst-x)

1 : $\Pi_1 \vdash p_3 \leq p_2$ *given
2 : some p_1 ↓pst-x: 1
3 : $\Pi_1 \vdash p_3 \leq p_1$ ↓pst-x: 1
4 : $\Pi_2 \leq \Pi_1$ *given
5 : $\Pi_2 \vdash p_3 \leq p_1$ astpst: 3,4
6 : $\Pi_1 \vdash p_1 \leq p_2$ ↓pst-x: 1
7 : $\Pi_2 \vdash p_1 \leq p_2$ astpst: 6,4
8 : $\Pi_2 \vdash p_3 \leq p_2$ pst-x: 2,5,7

(astpst-a1)

1 : $(\Pi_1, p_1 \leq p_2) \vdash p_1 \leq p_2$ *given
2 : $(\Pi_2, p_1 \leq p_2) \leq (\Pi_1, p_1 \leq p_2)$ *given
3 : $(\Pi_2, p_1 \leq p_2) \vdash p_1 \leq p_2$ pst-a1

(astpst-a2)

1 : $\Pi_2 \leq (\Pi_1, p_3 \leq p_4)$ *given
2 : $\Pi_2 \vdash p_3 \leq p_4$ ↓ast-p2: 1
3 : $(\Pi_1, p_3 \leq p_4) \vdash p_1 \leq p_2$ *given
4 : $\Pi_1 \vdash p_1 \leq p_2$ ↓pst-a2: 3
5 : $\Pi_2 \leq \Pi_1$ ↓ast-p2: 1
6 : $\Pi_2 \vdash p_1 \leq p_2$ astpst: 4,5

(astpst-cat1)

1 : $\Pi_1 \vdash (p_1, p_2) \leq p_1$ *given
2 : $\Pi_2 \leq \Pi_1$ *given
3 : $\Pi_2 \vdash (p_1, p_2) \leq p_1$ pst-cat1

(astpst-cat2)

1 : $\Pi_1 \vdash (p_1, p_2) \leq p_2$ *given
2 : $\Pi_2 \leq \Pi_1$ *given
3 : $\Pi_2 \vdash (p_1, p_2) \leq p_2$ pst-cat2

4.2 Label subtypings

$$\frac{(\Pi, p_1 \leq p_2) \vdash \ell_1 \sqsubseteq \ell_2 \quad \Pi \vdash p_1 \leq p_2}{\Pi \vdash \ell_1 \sqsubseteq \ell_2}$$ pstlst

(pstlst-z)

1 : $(\Pi, p_1 \leq p_2) \vdash \ell \sqsubseteq \ell$ *given
2 : $\Pi \vdash p_1 \leq p_2$ *given
3 : $\Pi \vdash \ell \sqsubseteq \ell$ lst-z

(pstlst-x)

1 : $(\Pi, p_1 \leq p_2) \vdash \ell_3 \sqsubseteq \ell_2$ *given
2 : some ℓ_1 ↓lst-x: 1
3 : $(\Pi, p_1 \leq p_2) \vdash \ell_3 \sqsubseteq \ell_1$ ↓lst-x: 1
4 : $\Pi \vdash p_1 \leq p_2$ *given
5 : $\Pi \vdash \ell_3 \sqsubseteq \ell_1$ pstlst: 3,4
6 : $(\Pi, p_1 \leq p_2) \vdash \ell_1 \sqsubseteq \ell_2$ ↓lst-x: 1
7 : $\Pi \vdash \ell_1 \sqsubseteq \ell_2$ pstlst: 6,4
8 : $\Pi \vdash \ell_3 \sqsubseteq \ell_2$ lst-x: 2,5,7

(pstlst-p)

1 : $(\Pi, p_1 \leq p_2) \vdash (p_5 : p_3) \sqsubseteq (p_6 : p_4)$ *given
2 : $(\Pi, p_1 \leq p_2) \vdash p_5 \leq p_6$ ↓lst-p: 1
3 : $\Pi \vdash p_1 \leq p_2$ *given
4 : $\Pi \vdash p_5 \leq p_6$ pstpst: 2,3
5 : $(\Pi, p_1 \leq p_2) \vdash p_3 \leq p_4$ ↓lst-p: 1
6 : $\Pi \vdash p_3 \leq p_4$ pstpst: 5,3
7 : $\Pi \vdash (p_5 : p_3) \sqsubseteq (p_6 : p_4)$ lst-p: 4,6

(pstlst-cat1)

1 : $(\Pi, p_1 \leq p_2) \vdash \ell_1 \sqsubseteq (\ell_1, \ell_2)$ *given
2 : $\Pi \vdash p_1 \leq p_2$ *given
3 : $\Pi \vdash \ell_1 \sqsubseteq (\ell_1, \ell_2)$ lst-cat1

(pstlst-cat2)

1 : $(\Pi, p_1 \leq p_2) \vdash \ell_1 \sqsubseteq (\ell_2, \ell_1)$ *given
2 : $\Pi \vdash p_1 \leq p_2$ *given
3 : $\Pi \vdash \ell_1 \sqsubseteq (\ell_2, \ell_1)$ lst-cat2

$$\frac{\Pi_1 \vdash \ell_1 \sqsubseteq \ell_2 \quad \Pi_2 \leq \Pi_1}{\Pi_2 \vdash \ell_1 \sqsubseteq \ell_2}$$ astlst

(astlst-z)

1 : $\Pi_1 \vdash \ell \sqsubseteq \ell$ *given
2 : $\Pi_2 \leq \Pi_1$ *given
3 : $\Pi_2 \vdash \ell \sqsubseteq \ell$ lst-z

(astlst-x)

1 : $\Pi_1 \vdash \ell_3 \sqsubseteq \ell_2$ *given
2 : some ℓ_1 ↓lst-x: 1
3 : $\Pi_1 \vdash \ell_3 \sqsubseteq \ell_1$ ↓lst-x: 1

4 : $\Pi_2 \leq \Pi_1$ *given
5 : $\Pi_2 \vdash \ell_3 \sqsubseteq \ell_1$ astlst: 3,4
6 : $\Pi_1 \vdash \ell_1 \sqsubseteq \ell_2$ ↓lst-x: 1
7 : $\Pi_2 \vdash \ell_1 \sqsubseteq \ell_2$ astlst: 6,4
8 : $\Pi_2 \vdash \ell_3 \sqsubseteq \ell_2$ lst-x: 2,5,7

(astlst-p)
1 : $\Pi_1 \vdash (p_3 : p_1) \sqsubseteq (p_4 : p_2)$ *given
2 : $\Pi_1 \vdash p_3 \leq p_4$ ↓lst-p: 1
3 : $\Pi_2 \leq \Pi_1$ *given
4 : $\Pi_2 \vdash p_3 \leq p_4$ astpst: 2,3
5 : $\Pi_1 \vdash p_1 \leq p_2$ ↓lst-p: 1
6 : $\Pi_2 \vdash p_1 \leq p_2$ astpst: 5,3
7 : $\Pi_2 \vdash (p_3 : p_1) \sqsubseteq (p_4 : p_2)$ lst-p: 4,6

(astlst-cat1)
1 : $\Pi_1 \vdash \ell_1 \sqsubseteq (\ell_1, \ell_2)$ *given
2 : $\Pi_2 \leq \Pi_1$ *given
3 : $\Pi_2 \vdash \ell_1 \sqsubseteq (\ell_1, \ell_2)$ lst-cat1

(astlst-cat2)
1 : $\Pi_1 \vdash \ell_1 \sqsubseteq (\ell_2, \ell_1)$ *given
2 : $\Pi_2 \leq \Pi_1$ *given
3 : $\Pi_2 \vdash \ell_1 \sqsubseteq (\ell_2, \ell_1)$ lst-cat2

4.3 Hierarchy subtypings

$$\frac{(\Pi_1, p_1 \leq p_2) \leq \Pi_2 \quad \Pi_1 \vdash p_1 \leq p_2}{\Pi_1 \leq \Pi_2}$$ pstast

(pstast-x)
1 : $(\Pi_1, p_1 \leq p_2) \leq \Pi_3$ *given
2 : some Π_2 ↓ast-x: 1
3 : $(\Pi_1, p_1 \leq p_2) \leq \Pi_2$ ↓ast-x: 1
4 : $\Pi_1 \vdash p_1 \leq p_2$ *given
5 : $\Pi_1 \leq \Pi_2$ pstast: 3,4
6 : $\Pi_2 \leq \Pi_3$ ↓ast-x: 1
7 : $\Pi_1 \leq \Pi_3$ ast-x: 2,5,6

(pstast-p1)
1 : $\Pi_1 \vdash p_1 \leq p_2$ *given
2 : $(\Pi_1, p_1 \leq p_2) \leq \Pi_2$ *given
3 : $\Pi_1 \leq \Pi_2$ ↓ast-p1: 2

(pstast-p2)
1 : $(\Pi_1, p_1 \leq p_2) \leq (\Pi_2, p_3 \leq p_4)$ *given
2 : $(\Pi_1, p_1 \leq p_2) \leq \Pi_2$ ↓ast-p2: 1
3 : $\Pi_1 \vdash p_1 \leq p_2$ *given
4 : $\Pi_1 \leq \Pi_2$ pstast: 2,3
5 : $(\Pi_1, p_1 \leq p_2) \vdash p_3 \leq p_4$ ↓ast-p2: 1
6 : $\Pi_1 \vdash p_3 \leq p_4$ pstpst: 5,3
7 : $\Pi_1 \leq (\Pi_2, p_3 \leq p_4)$ ast-p2: 4,6

$$\frac{\Pi_1 \leq \Pi_2}{(\Pi_1, p_1 \leq p_2) \leq \Pi_2}$$ apast

(apast-z)

1 : $\Pi \leq \Pi$ *given
2 : $(\Pi, p_1 \leq p_2) \leq \Pi$ ast-p1: 1

(apast-x)

1 : $\Pi_1 \leq \Pi_3$ *given
2 : some Π_2 ↓ast-x: 1
3 : $\Pi_1 \leq \Pi_2$ ↓ast-x: 1
4 : $(\Pi_1, p_1 \leq p_2) \leq \Pi_2$ apast: 3
5 : $\Pi_2 \leq \Pi_3$ ↓ast-x: 1
6 : $(\Pi_1, p_1 \leq p_2) \leq \Pi_3$ ast-x: 2,4,5

(apast-p1)

1 : $(\Pi_1, p_1 \leq p_2) \leq \Pi_2$ *given
2 : $\Pi_1 \leq \Pi_2$ ↓ast-p1: 1
3 : $((\Pi_1, p_1 \leq p_2), p_3 \leq p_4) \leq \Pi_2$ ast-p1: 1

(apast-p2)

1 : $\Pi_1 \leq (\Pi_2, p_1 \leq p_2)$ *given
2 : $\Pi_1 \leq \Pi_2$ ↓ast-p2: 1
3 : $(\Pi_1, p_3 \leq p_4) \leq \Pi_2$ apast: 2
4 : $\Pi_1 \vdash p_1 \leq p_2$ ↓ast-p2: 1
5 : $(\Pi_1, p_3 \leq p_4) \vdash p_1 \leq p_2$ appst: 4
6 : $(\Pi_1, p_3 \leq p_4) \leq (\Pi_2, p_1 \leq p_2)$ ast-p2: 3,5

$$\boxed{\frac{\Pi_1 \leq \Pi_2}{(\Pi_1, p_1 \leq p_2) \leq (\Pi_2, \ell_1 \leq \ell_2)}}$$ apapast

(apapast-l)

1 : $\Pi_1 \leq \Pi_2$ *given
2 : $(\Pi_1, p_1 \leq p_2) \leq \Pi_2$ apast: 1
3 : $(\Pi_1, p_1 \leq p_2) \vdash p_1 \leq p_2$ pst-a1
4 : $(\Pi_1, p_1 \leq p_2) \leq (\Pi_2, p_1 \leq p_2)$ ast-p2: 2,3

4.4 Type subtypings

$$\boxed{\frac{(\Pi, p_1 \leq p_2); \Gamma \vdash e : t \quad \Pi \vdash p_1 \leq p_2}{\Pi; \Gamma \vdash e : t}}$$ pstty

(pstty-true)

1 : $(\Pi, p_1 \leq p_2); \Gamma \vdash (\texttt{true}_\ell) : (\texttt{bool}_\ell)$ *given
2 : $\Pi \vdash p_1 \leq p_2$ *given
3 : $\Pi; \Gamma \vdash (\texttt{true}_\ell) : (\texttt{bool}_\ell)$ ty-true

(pstty-false)

1 : $(\Pi, p_1 \leq p_2); \Gamma \vdash (\texttt{false}_\ell) : (\texttt{bool}_\ell)$ *given
2 : $\Pi \vdash p_1 \leq p_2$ *given
3 : $\Pi; \Gamma \vdash (\texttt{false}_\ell) : (\texttt{bool}_\ell)$ ty-false

(pstty-ifb)

1 : $(\Pi, p_1 \leq p_2); \Gamma \vdash (\texttt{if } e_1\ e_2\ e_1) : (\texttt{bool}_\ell)$ *given
2 : $(\Pi, p_1 \leq p_2); \Gamma \vdash e_1 : (\texttt{bool}_\ell)$ ↓ty-ifb: 1
3 : $\Pi \vdash p_1 \leq p_2$ *given
4 : $\Pi; \Gamma \vdash e_1 : (\texttt{bool}_\ell)$ pstty: 2,3
5 : $(\Pi, p_1 \leq p_2); \Gamma \vdash e_2 : (\texttt{bool}_\ell)$ ↓ty-ifb: 1
6 : $\Pi; \Gamma \vdash e_2 : (\texttt{bool}_\ell)$ pstty: 5,3

$7:\ (\Pi, \mathtt{p}_1 \leq \mathtt{p}_2); \Gamma \vdash \mathtt{e}_1 : (\mathtt{bool}_\ell)$ ↓ty-ifb: 1
$8:\ \Pi; \Gamma \vdash \mathtt{e}_1 : (\mathtt{bool}_\ell)$ pstty: 7,3
$9:\ \mathtt{lab}((\mathtt{bool}_\ell)) = \ell$ ↓ty-ifb: 1
$10:\ \Pi; \Gamma \vdash (\mathtt{if}\ \mathtt{e}_1\ \mathtt{e}_2\ \mathtt{e}_1) : (\mathtt{bool}_\ell)$ ty-ifb: 4,6,8,9

(pstty-var1)
$1:\ (\Pi, \mathtt{p}_1 \leq \mathtt{p}_2); (\Gamma, \mathtt{x}{:}\mathtt{t}) \vdash (\mathtt{x}) : \mathtt{t}$ *given
$2:\ \Pi \vdash \mathtt{p}_1 \leq \mathtt{p}_2$ *given
$3:\ \Pi; (\Gamma, \mathtt{x}{:}\mathtt{t}) \vdash (\mathtt{x}) : \mathtt{t}$ ty-var1

(pstty-var2)
$1:\ (\Pi, \mathtt{p}_1 \leq \mathtt{p}_2); (\Gamma, \mathtt{x}_2{:}\mathtt{t}_2) \vdash (\mathtt{x}_1) : \mathtt{t}_1$ *given
$2:\ (\Pi, \mathtt{p}_1 \leq \mathtt{p}_2); \Gamma \vdash (\mathtt{x}_1) : \mathtt{t}_1$ ↓ty-var2: 1
$3:\ \Pi \vdash \mathtt{p}_1 \leq \mathtt{p}_2$ *given
$4:\ \Pi; \Gamma \vdash (\mathtt{x}_1) : \mathtt{t}_1$ pstty: 2,3
$5:\ \Pi; (\Gamma, \mathtt{x}_2{:}\mathtt{t}_2) \vdash (\mathtt{x}_1) : \mathtt{t}_1$ ty-var2: 4

(pstty-fun)
$1:\ (\Pi_1, \mathtt{p}_1 \leq \mathtt{p}_2); \Gamma \vdash (\lambda[\Pi_2]\mathtt{x}{:}\mathtt{t}_1\,.\,\mathtt{e}) : (\mathtt{t}_1 \to \mathtt{t}_2)$ *given
$2:\ (\Pi_1, \mathtt{p}_1 \leq \mathtt{p}_2) \leq \Pi_2$ ↓ty-fun: 1
$3:\ \Pi_1 \vdash \mathtt{p}_1 \leq \mathtt{p}_2$ *given
$4:\ \Pi_1 \leq \Pi_2$ pstast: 2,3
$5:\ \Pi_2; (\Gamma, \mathtt{x}{:}\mathtt{t}_1) \vdash \mathtt{e} : \mathtt{t}_2$ ↓ty-fun: 1
$6:\ \Pi_1; \Gamma \vdash (\lambda[\Pi_2]\mathtt{x}{:}\mathtt{t}_1\,.\,\mathtt{e}) : (\mathtt{t}_1 \to \mathtt{t}_2)$ ty-fun: 4,5

(pstty-app)
$1:\ (\Pi, \mathtt{p}_1 \leq \mathtt{p}_2); \Gamma \vdash (\mathtt{e}_2\ \mathtt{e}_1) : \mathtt{t}_2$ *given
$2:\ (\Pi, \mathtt{p}_1 \leq \mathtt{p}_2); \Gamma \vdash \mathtt{e}_2 : (\mathtt{t}_1 \to \mathtt{t}_2)$ ↓ty-app: 1
$3:\ \Pi \vdash \mathtt{p}_1 \leq \mathtt{p}_2$ *given
$4:\ \Pi; \Gamma \vdash \mathtt{e}_2 : (\mathtt{t}_1 \to \mathtt{t}_2)$ pstty: 2,3
$5:\ (\Pi, \mathtt{p}_1 \leq \mathtt{p}_2); \Gamma \vdash \mathtt{e}_1 : \mathtt{t}_1$ ↓ty-app: 1
$6:\ \Pi; \Gamma \vdash \mathtt{e}_1 : \mathtt{t}_1$ pstty: 5,3
$7:\ \Pi; \Gamma \vdash (\mathtt{e}_2\ \mathtt{e}_1) : \mathtt{t}_2$ ty-app: 4,6

(pstty-ifp)
$1:\ (\Pi, \mathtt{p}_1 \leq \mathtt{p}_2); \Gamma \vdash (\mathtt{if}\ (\mathtt{p}_1 \leq \mathtt{p}_2)\ \mathtt{e}_2\ \mathtt{e}_1) : \mathtt{t}$ *given
$2:\ ((\Pi, \mathtt{p}_1 \leq \mathtt{p}_2), \mathtt{p}_1 \leq \mathtt{p}_2); \Gamma \vdash \mathtt{e}_2 : \mathtt{t}$ ↓ty-ifp: 1
$3:\ \Pi \vdash \mathtt{p}_1 \leq \mathtt{p}_2$ *given
$4:\ (\Pi, \mathtt{p}_1 \leq \mathtt{p}_2) \vdash \mathtt{p}_1 \leq \mathtt{p}_2$ appst: 3
$5:\ (\Pi, \mathtt{p}_1 \leq \mathtt{p}_2); \Gamma \vdash \mathtt{e}_2 : \mathtt{t}$ pstty: 2,4
$6:\ (\Pi, \mathtt{p}_1 \leq \mathtt{p}_2); \Gamma \vdash \mathtt{e}_1 : \mathtt{t}$ ↓ty-ifp: 1
$7:\ \Pi; \Gamma \vdash \mathtt{e}_1 : \mathtt{t}$ pstty: 6,3
$8:\ \Pi; \Gamma \vdash (\mathtt{if}\ (\mathtt{p}_1 \leq \mathtt{p}_2)\ \mathtt{e}_2\ \mathtt{e}_1) : \mathtt{t}$ ty-ifp: 5,7

(pstty-tag)
$1:\ (\Pi, \mathtt{p}_1 \leq \mathtt{p}_2); \Gamma \vdash ([\ell_1 \sqsubseteq \ell_2]\mathtt{e}) : (\mathtt{bool}_{\ell_2})$ *given
$2:\ (\Pi, \mathtt{p}_1 \leq \mathtt{p}_2) \vdash \ell_1 \sqsubseteq \ell_2$ ↓ty-tag: 1
$3:\ \Pi \vdash \mathtt{p}_1 \leq \mathtt{p}_2$ *given
$4:\ \Pi \vdash \ell_1 \sqsubseteq \ell_2$ pstlst: 2,3
$5:\ (\Pi, \mathtt{p}_1 \leq \mathtt{p}_2); \Gamma \vdash \mathtt{e} : (\mathtt{bool}_{\ell_1})$ ↓ty-tag: 1
$6:\ \Pi; \Gamma \vdash \mathtt{e} : (\mathtt{bool}_{\ell_1})$ pstty: 5,3
$7:\ \Pi; \Gamma \vdash ([\ell_1 \sqsubseteq \ell_2]\mathtt{e}) : (\mathtt{bool}_{\ell_2})$ ty-tag: 4,6

$$\frac{\Pi_1; \Gamma \vdash \mathtt{e} : \mathtt{t} \quad \Pi_2 \leq \Pi_1}{\Pi_2; \Gamma \vdash \mathtt{e} : \mathtt{t}}$$ astty

(astty-true)

1 : $\Pi_1;\Gamma \vdash (\texttt{true}_\ell) : (\texttt{bool}_\ell)$ *given
2 : $\Pi_2 \leq \Pi_1$ *given
3 : $\Pi_2;\Gamma \vdash (\texttt{true}_\ell) : (\texttt{bool}_\ell)$ ty-true

(astty-false)

1 : $\Pi_1;\Gamma \vdash (\texttt{false}_\ell) : (\texttt{bool}_\ell)$ *given
2 : $\Pi_2 \leq \Pi_1$ *given
3 : $\Pi_2;\Gamma \vdash (\texttt{false}_\ell) : (\texttt{bool}_\ell)$ ty-false

(astty-ifb)

1 : $\Pi_1;\Gamma \vdash (\texttt{if e}_1\ \texttt{e}_2\ \texttt{e}_1) : (\texttt{bool}_\ell)$ *given
2 : $\Pi_1;\Gamma \vdash \texttt{e}_1 : (\texttt{bool}_\ell)$ ↓ty-ifb: 1
3 : $\Pi_2 \leq \Pi_1$ *given
4 : $\Pi_2;\Gamma \vdash \texttt{e}_1 : (\texttt{bool}_\ell)$ astty: 2,3
5 : $\Pi_1;\Gamma \vdash \texttt{e}_2 : (\texttt{bool}_\ell)$ ↓ty-ifb: 1
6 : $\Pi_2;\Gamma \vdash \texttt{e}_2 : (\texttt{bool}_\ell)$ astty: 5,3
7 : $\Pi_1;\Gamma \vdash \texttt{e}_1 : (\texttt{bool}_\ell)$ ↓ty-ifb: 1
8 : $\Pi_2;\Gamma \vdash \texttt{e}_1 : (\texttt{bool}_\ell)$ astty: 7,3
9 : $\texttt{lab}((\texttt{bool}_\ell)) = \ell$ ↓ty-ifb: 1
10 : $\Pi_2;\Gamma \vdash (\texttt{if e}_1\ \texttt{e}_2\ \texttt{e}_1) : (\texttt{bool}_\ell)$ ty-ifb: 4,6,8,9

(astty-var1)

1 : $\Pi_1;(\Gamma,\texttt{x:t}) \vdash (\texttt{x}) : \texttt{t}$ *given
2 : $\Pi_2 \leq \Pi_1$ *given
3 : $\Pi_2;(\Gamma,\texttt{x:t}) \vdash (\texttt{x}) : \texttt{t}$ ty-var1

(astty-var2)

1 : $\Pi_1;(\Gamma,\texttt{x}_2\texttt{:t}_2) \vdash (\texttt{x}_1) : \texttt{t}_1$ *given
2 : $\Pi_1;\Gamma \vdash (\texttt{x}_1) : \texttt{t}_1$ ↓ty-var2: 1
3 : $\Pi_2 \leq \Pi_1$ *given
4 : $\Pi_2;\Gamma \vdash (\texttt{x}_1) : \texttt{t}_1$ astty: 2,3
5 : $\Pi_2;(\Gamma,\texttt{x}_2\texttt{:t}_2) \vdash (\texttt{x}_1) : \texttt{t}_1$ ty-var2: 4

(astty-fun)

1 : $\texttt{some}\ \Pi_1$ asome-total
2 : $\Pi_3 \leq \Pi_1$ *given
3 : $\Pi_1;\Gamma \vdash (\lambda[\Pi_2]\texttt{x:t}_1\texttt{.}\ \texttt{e}) : (\texttt{t}_1 \to \texttt{t}_2)$ *given
4 : $\Pi_1 \leq \Pi_2$ ↓ty-fun: 3
5 : $\Pi_3 \leq \Pi_2$ ast-x: 1,2,4
6 : $\Pi_2;(\Gamma,\texttt{x:t}_1) \vdash \texttt{e} : \texttt{t}_2$ ↓ty-fun: 3
7 : $\Pi_3;\Gamma \vdash (\lambda[\Pi_2]\texttt{x:t}_1\texttt{.}\ \texttt{e}) : (\texttt{t}_1 \to \texttt{t}_2)$ ty-fun: 5,6

(astty-app)

1 : $\Pi_1;\Gamma \vdash (\texttt{e}_2\ \texttt{e}_1) : \texttt{t}_2$ *given
2 : $\Pi_1;\Gamma \vdash \texttt{e}_2 : (\texttt{t}_1 \to \texttt{t}_2)$ ↓ty-app: 1
3 : $\Pi_2 \leq \Pi_1$ *given
4 : $\Pi_2;\Gamma \vdash \texttt{e}_2 : (\texttt{t}_1 \to \texttt{t}_2)$ astty: 2,3
5 : $\Pi_1;\Gamma \vdash \texttt{e}_1 : \texttt{t}_1$ ↓ty-app: 1
6 : $\Pi_2;\Gamma \vdash \texttt{e}_1 : \texttt{t}_1$ astty: 5,3
7 : $\Pi_2;\Gamma \vdash (\texttt{e}_2\ \texttt{e}_1) : \texttt{t}_2$ ty-app: 4,6

(astty-ifp)

1 : $\Pi_1;\Gamma \vdash (\texttt{if}\ (\texttt{p}_1 \leq \texttt{p}_2)\ \texttt{e}_2\ \texttt{e}_1) : \texttt{t}$ *given
2 : $(\Pi_1,\texttt{p}_1 \leq \texttt{p}_2);\Gamma \vdash \texttt{e}_2 : \texttt{t}$ ↓ty-ifp: 1
3 : $\Pi_2 \leq \Pi_1$ *given
4 : $(\Pi_2,\texttt{p}_1 \leq \texttt{p}_2) \leq (\Pi_1,\texttt{p}_1 \leq \texttt{p}_2)$ apapast: 3
5 : $(\Pi_2,\texttt{p}_1 \leq \texttt{p}_2);\Gamma \vdash \texttt{e}_2 : \texttt{t}$ astty: 2,4

6 : $\Pi_1;\Gamma \vdash \mathtt{e}_1 : \mathtt{t}$ ↓ty-ifp: 1
7 : $\Pi_2;\Gamma \vdash \mathtt{e}_1 : \mathtt{t}$ astty: 6,3
8 : $\Pi_2;\Gamma \vdash (\mathtt{if}\ (\mathtt{p}_1 \leq \mathtt{p}_2)\ \mathtt{e}_2\ \mathtt{e}_1) : \mathtt{t}$ ty-ifp: 5,7

(astty-tag)
1 : $\Pi_1;\Gamma \vdash ([\ell_1 \sqsubseteq \ell_2]\mathtt{e}) : (\mathtt{bool}_{\ell_2})$ *given
2 : $\Pi_1 \vdash \ell_1 \sqsubseteq \ell_2$ ↓ty-tag: 1
3 : $\Pi_2 \leq \Pi_1$ *given
4 : $\Pi_2 \vdash \ell_1 \sqsubseteq \ell_2$ astlst: 2,3
5 : $\Pi_1;\Gamma \vdash \mathtt{e} : (\mathtt{bool}_{\ell_1})$ ↓ty-tag: 1
6 : $\Pi_2;\Gamma \vdash \mathtt{e} : (\mathtt{bool}_{\ell_1})$ astty: 5,3
7 : $\Pi_2;\Gamma \vdash ([\ell_1 \sqsubseteq \ell_2]\mathtt{e}) : (\mathtt{bool}_{\ell_2})$ ty-tag: 4,6

4.5 Tag checkings

$$\boxed{\frac{\Pi;\cdot \vdash \mathtt{e} : \mathtt{t}}{\Pi \vdash \mathtt{e}}}$$ tyae

(tyae-true)
1 : $\Pi;\cdot \vdash (\mathtt{true}_\ell) : (\mathtt{bool}_\ell)$ *given
2 : $\Pi \vdash (\mathtt{true}_\ell)$ ae-true

(tyae-false)
1 : $\Pi;\cdot \vdash (\mathtt{false}_\ell) : (\mathtt{bool}_\ell)$ *given
2 : $\Pi \vdash (\mathtt{false}_\ell)$ ae-false

(tyae-ifb)
1 : $\Pi;\cdot \vdash (\mathtt{if}\ \mathtt{e}_3\ \mathtt{e}_2\ \mathtt{e}_1) : \mathtt{t}$ *given
2 : $\mathtt{lab(t)} = \ell$ ↓ty-ifb: 1
3 : $\Pi;\cdot \vdash \mathtt{e}_3 : (\mathtt{bool}_\ell)$ ↓ty-ifb: 1
4 : $\Pi \vdash \mathtt{e}_3$ tyae: 3
5 : $\Pi;\cdot \vdash \mathtt{e}_2 : \mathtt{t}$ ↓ty-ifb: 1
6 : $\Pi \vdash \mathtt{e}_2$ tyae: 5
7 : $\Pi;\cdot \vdash \mathtt{e}_1 : \mathtt{t}$ ↓ty-ifb: 1
8 : $\Pi \vdash \mathtt{e}_1$ tyae: 7
9 : $\Pi \vdash (\mathtt{if}\ \mathtt{e}_3\ \mathtt{e}_2\ \mathtt{e}_1)$ ae-ifb: 4,6,8

(tyae-fun)
1 : $\Pi_1;\cdot \vdash (\lambda[\Pi_2]\mathtt{x}{:}\mathtt{t}_1.\ \mathtt{e}) : (\mathtt{t}_1 \to \mathtt{t}_2)$ *given
2 : $\Pi_2;(\cdot, \mathtt{x}{:}\mathtt{t}_1) \vdash \mathtt{e} : \mathtt{t}_2$ ↓ty-fun: 1
3 : $\Pi_1 \leq \Pi_2$ ↓ty-fun: 1
4 : $\Pi_1 \vdash (\lambda[\Pi_2]\mathtt{x}{:}\mathtt{t}_1.\ \mathtt{e})$ ae-fun: 3

(tyae-app)
1 : $\Pi;\cdot \vdash (\mathtt{e}_2\ \mathtt{e}_1) : \mathtt{t}_2$ *given
2 : $\Pi;\cdot \vdash \mathtt{e}_2 : (\mathtt{t}_1 \to \mathtt{t}_2)$ ↓ty-app: 1
3 : $\Pi \vdash \mathtt{e}_2$ tyae: 2
4 : $\Pi;\cdot \vdash \mathtt{e}_1 : \mathtt{t}_1$ ↓ty-app: 1
5 : $\Pi \vdash \mathtt{e}_1$ tyae: 4
6 : $\Pi \vdash (\mathtt{e}_2\ \mathtt{e}_1)$ ae-app: 3,5

(tyae-ifp)
1 : $\Pi;\cdot \vdash (\mathtt{if}\ (\mathtt{p}_1 \leq \mathtt{p}_2)\ \mathtt{e}_2\ \mathtt{e}_1) : \mathtt{t}$ *given
2 : $(\Pi, \mathtt{p}_1 \leq \mathtt{p}_2);\cdot \vdash \mathtt{e}_2 : \mathtt{t}$ ↓ty-ifp: 1
3 : $(\Pi, \mathtt{p}_1 \leq \mathtt{p}_2) \vdash \mathtt{e}_2$ tyae: 2
4 : $\Pi;\cdot \vdash \mathtt{e}_1 : \mathtt{t}$ ↓ty-ifp: 1
5 : $\Pi \vdash \mathtt{e}_1$ tyae: 4

6 : $\Pi \vdash (\texttt{if}\ (\texttt{p}_1 \leq \texttt{p}_2)\ \texttt{e}_2\ \texttt{e}_1)$ — ae-ifp: 3,5

(tyae-tag)
1 : $\Pi;\cdot \vdash ([\ell_1 \sqsubseteq \ell_2]\texttt{e}) : (\texttt{bool}_{\ell_2})$ — *given
2 : $\Pi \vdash \ell_1 \sqsubseteq \ell_2$ — ↓ty-tag: 1
3 : $\Pi;\cdot \vdash \texttt{e} : (\texttt{bool}_{\ell_1})$ — ↓ty-tag: 1
4 : $\Pi \vdash \texttt{e}$ — tyae: 3
5 : $\Pi \vdash ([\ell_1 \sqsubseteq \ell_2]\texttt{e})$ — ae-tag: 2,4

$$\frac{\Pi_1;\cdot \vdash \texttt{e} : \texttt{t} \quad \Pi_2 \vdash \texttt{e}}{\Pi_2;\cdot \vdash \texttt{e} : \texttt{t}}$$ aety

(aety-true)
1 : $\Pi_1;\cdot \vdash (\texttt{true}_\ell) : (\texttt{bool}_\ell)$ — *given
2 : $\Pi_2 \vdash (\texttt{true}_\ell)$ — *given
3 : $\Pi_2;\cdot \vdash (\texttt{true}_\ell) : (\texttt{bool}_\ell)$ — ty-true

(aety-false)
1 : $\Pi_1;\cdot \vdash (\texttt{false}_\ell) : (\texttt{bool}_\ell)$ — *given
2 : $\Pi_2 \vdash (\texttt{false}_\ell)$ — *given
3 : $\Pi_2;\cdot \vdash (\texttt{false}_\ell) : (\texttt{bool}_\ell)$ — ty-false

(aety-ifb)
1 : $\Pi_1;\cdot \vdash (\texttt{if}\ \texttt{e}_3\ \texttt{e}_2\ \texttt{e}_1) : \texttt{t}$ — *given
2 : $\Pi_1;\cdot \vdash \texttt{e}_3 : (\texttt{bool}_\ell)$ — ↓ty-ifb: 1
3 : $\Pi_2 \vdash (\texttt{if}\ \texttt{e}_3\ \texttt{e}_2\ \texttt{e}_1)$ — *given
4 : $\Pi_2 \vdash \texttt{e}_3$ — ↓ae-ifb: 3
5 : $\Pi_2;\cdot \vdash \texttt{e}_3 : (\texttt{bool}_\ell)$ — aety: 2,4
6 : $\Pi_1;\cdot \vdash \texttt{e}_2 : \texttt{t}$ — ↓ty-ifb: 1
7 : $\Pi_2 \vdash \texttt{e}_2$ — ↓ae-ifb: 3
8 : $\Pi_2;\cdot \vdash \texttt{e}_2 : \texttt{t}$ — aety: 6,7
9 : $\Pi_1;\cdot \vdash \texttt{e}_1 : \texttt{t}$ — ↓ty-ifb: 1
10 : $\Pi_2 \vdash \texttt{e}_1$ — ↓ae-ifb: 3
11 : $\Pi_2;\cdot \vdash \texttt{e}_1 : \texttt{t}$ — aety: 9,10
12 : $\texttt{lab(t)} = \ell$ — ↓ty-ifb: 1
13 : $\Pi_2;\cdot \vdash (\texttt{if}\ \texttt{e}_3\ \texttt{e}_2\ \texttt{e}_1) : \texttt{t}$ — ty-ifb: 5,8,11,12

(aety-fun)
1 : $\Pi_1;\cdot \vdash (\lambda[\Pi_2]\texttt{x}{:}\texttt{t}_1.\ \texttt{e}) : (\texttt{t}_1 \to \texttt{t}_2)$ — *given
2 : $\Pi_1 \leq \Pi_2$ — ↓ty-fun: 1
3 : $\Pi_3 \vdash (\lambda[\Pi_2]\texttt{x}{:}\texttt{t}_1.\ \texttt{e})$ — *given
4 : $\Pi_3 \leq \Pi_2$ — ↓ae-fun: 3
5 : $\Pi_2;(\cdot, \texttt{x}{:}\texttt{t}_1) \vdash \texttt{e} : \texttt{t}_2$ — ↓ty-fun: 1
6 : $\Pi_3;\cdot \vdash (\lambda[\Pi_2]\texttt{x}{:}\texttt{t}_1.\ \texttt{e}) : (\texttt{t}_1 \to \texttt{t}_2)$ — ty-fun: 4,5

(aety-app)
1 : $\Pi_1;\cdot \vdash (\texttt{e}_2\ \texttt{e}_1) : \texttt{t}_2$ — *given
2 : $\Pi_1;\cdot \vdash \texttt{e}_2 : (\texttt{t}_1 \to \texttt{t}_2)$ — ↓ty-app: 1
3 : $\Pi_2 \vdash (\texttt{e}_2\ \texttt{e}_1)$ — *given
4 : $\Pi_2 \vdash \texttt{e}_2$ — ↓ae-app: 3
5 : $\Pi_2;\cdot \vdash \texttt{e}_2 : (\texttt{t}_1 \to \texttt{t}_2)$ — aety: 2,4
6 : $\Pi_1;\cdot \vdash \texttt{e}_1 : \texttt{t}_1$ — ↓ty-app: 1
7 : $\Pi_2 \vdash \texttt{e}_1$ — ↓ae-app: 3
8 : $\Pi_2;\cdot \vdash \texttt{e}_1 : \texttt{t}_1$ — aety: 6,7
9 : $\Pi_2;\cdot \vdash (\texttt{e}_2\ \texttt{e}_1) : \texttt{t}_2$ — ty-app: 5,8

(aety-ifp)

1 : $\Pi_1;\cdot \vdash (\texttt{if } (\texttt{p}_1 \leq \texttt{p}_2)\ \texttt{e}_2\ \texttt{e}_1) : \texttt{t}$ *given
2 : $(\Pi_1, \texttt{p}_1 \leq \texttt{p}_2);\cdot \vdash \texttt{e}_2 : \texttt{t}$ ↓ty-ifp: 1
3 : $\Pi_2 \vdash (\texttt{if } (\texttt{p}_1 \leq \texttt{p}_2)\ \texttt{e}_2\ \texttt{e}_1)$ *given
4 : $(\Pi_2, \texttt{p}_1 \leq \texttt{p}_2) \vdash \texttt{e}_2$ ↓ae-ifp: 3
5 : $(\Pi_2, \texttt{p}_1 \leq \texttt{p}_2);\cdot \vdash \texttt{e}_2 : \texttt{t}$ aety: 2,4
6 : $\Pi_1;\cdot \vdash \texttt{e}_1 : \texttt{t}$ ↓ty-ifp: 1
7 : $\Pi_2 \vdash \texttt{e}_1$ ↓ae-ifp: 3
8 : $\Pi_2;\cdot \vdash \texttt{e}_1 : \texttt{t}$ aety: 6,7
9 : $\Pi_2;\cdot \vdash (\texttt{if } (\texttt{p}_1 \leq \texttt{p}_2)\ \texttt{e}_2\ \texttt{e}_1) : \texttt{t}$ ty-ifp: 5,8

(aety-tag)

1 : $\Pi_1;\cdot \vdash ([\ell_1 \sqsubseteq \ell_2]\texttt{e}) : (\texttt{bool}_{\ell_2})$ *given
2 : $\Pi_1 \vdash \ell_1 \sqsubseteq \ell_2$ ↓ty-tag: 1
3 : $\Pi_2 \vdash ([\ell_1 \sqsubseteq \ell_2]\texttt{e})$ *given
4 : $\Pi_2 \vdash \ell_1 \sqsubseteq \ell_2$ ↓ae-tag: 3
5 : $\Pi_1;\cdot \vdash \texttt{e} : (\texttt{bool}_{\ell_1})$ ↓ty-tag: 1
6 : $\Pi_2 \vdash \texttt{e}$ ↓ae-tag: 3
7 : $\Pi_2;\cdot \vdash \texttt{e} : (\texttt{bool}_{\ell_1})$ aety: 5,6
8 : $\Pi_2;\cdot \vdash ([\ell_1 \sqsubseteq \ell_2]\texttt{e}) : (\texttt{bool}_{\ell_2})$ ty-tag: 4,7

4.6 Permutations

$$\boxed{\Gamma \equiv \Gamma}$$ perm

$$((\Gamma, \texttt{x}_1 : \texttt{t}_1), \texttt{x}_2 : \texttt{t}_2) \equiv ((\Gamma, \texttt{x}_2 : \texttt{t}_2), \texttt{x}_1 : \texttt{t}_1)$$ (perm-x1)

$$\frac{\Gamma_1 \equiv \Gamma_2}{(\Gamma_1, \texttt{x}:\texttt{t}) \equiv (\Gamma_2, \texttt{x}:\texttt{t})}$$ (perm-x2)

$$\boxed{\frac{\Pi;\Gamma_1 \vdash \texttt{e} : \texttt{t} \quad \Gamma_1 \equiv \Gamma_2}{\Pi;\Gamma_2 \vdash \texttt{e} : \texttt{t}}}$$ permty

(permty-true)

1 : $\Pi;\Gamma_1 \vdash (\texttt{true}_\ell) : (\texttt{bool}_\ell)$ *given
2 : $\Gamma_1 \equiv \Gamma_2$ *given
3 : $\Pi;\Gamma_2 \vdash (\texttt{true}_\ell) : (\texttt{bool}_\ell)$ ty-true

(permty-false)

1 : $\Pi;\Gamma_1 \vdash (\texttt{false}_\ell) : (\texttt{bool}_\ell)$ *given
2 : $\Gamma_1 \equiv \Gamma_2$ *given
3 : $\Pi;\Gamma_2 \vdash (\texttt{false}_\ell) : (\texttt{bool}_\ell)$ ty-false

(permty-ifb)

1 : $\Pi;\Gamma_1 \vdash (\texttt{if } \texttt{e}_1\ \texttt{e}_2\ \texttt{e}_1) : (\texttt{bool}_\ell)$ *given
2 : $\Pi;\Gamma_1 \vdash \texttt{e}_1 : (\texttt{bool}_\ell)$ ↓ty-ifb: 1
3 : $\Gamma_1 \equiv \Gamma_2$ *given
4 : $\Pi;\Gamma_2 \vdash \texttt{e}_1 : (\texttt{bool}_\ell)$ permty: 2,3
5 : $\Pi;\Gamma_1 \vdash \texttt{e}_2 : (\texttt{bool}_\ell)$ ↓ty-ifb: 1
6 : $\Pi;\Gamma_2 \vdash \texttt{e}_2 : (\texttt{bool}_\ell)$ permty: 5,3
7 : $\Pi;\Gamma_1 \vdash \texttt{e}_1 : (\texttt{bool}_\ell)$ ↓ty-ifb: 1
8 : $\Pi;\Gamma_2 \vdash \texttt{e}_1 : (\texttt{bool}_\ell)$ permty: 7,3
9 : $\texttt{lab}((\texttt{bool}_\ell)) = \ell$ ↓ty-ifb: 1
10 : $\Pi;\Gamma_2 \vdash (\texttt{if } \texttt{e}_1\ \texttt{e}_2\ \texttt{e}_1) : (\texttt{bool}_\ell)$ ty-ifb: 4,6,8,9

(permty-var1)

1 : $\Pi;(\Gamma_1, \texttt{x}:\texttt{t}) \vdash (\texttt{x}) : \texttt{t}$ *given
2 : $(\Gamma_1, \texttt{x}:\texttt{t}) \equiv (\Gamma_2, \texttt{x}:\texttt{t})$ *given

3 : $\Pi;(\Gamma_2, \mathtt{x}:\mathtt{t}) \vdash (\mathtt{x}) : \mathtt{t}$ — ty-var1

(permty-var2)

1 : $\Pi;((\Gamma, \mathtt{x}_1:\mathtt{t}_1), \mathtt{x}_2:\mathtt{t}_2) \vdash (\mathtt{x}_1) : \mathtt{t}_1$ — *given
2 : $\Pi;(\Gamma, \mathtt{x}_1:\mathtt{t}_1) \vdash (\mathtt{x}_1) : \mathtt{t}_1$ — ↓ty-var2: 1
3 : $((\Gamma, \mathtt{x}_1:\mathtt{t}_1), \mathtt{x}_2:\mathtt{t}_2) \equiv ((\Gamma, \mathtt{x}_2:\mathtt{t}_2), \mathtt{x}_1:\mathtt{t}_1)$ — *given
4 : $\Pi;((\Gamma, \mathtt{x}_2:\mathtt{t}_2), \mathtt{x}_1:\mathtt{t}_1) \vdash (\mathtt{x}_1) : \mathtt{t}_1$ — ty-var1

(permty-var3)

1 : $((\Gamma, \mathtt{x}_1:\mathtt{t}_1), \mathtt{x}_2:\mathtt{t}_2) \equiv ((\Gamma, \mathtt{x}_2:\mathtt{t}_2), \mathtt{x}_1:\mathtt{t}_1)$ — *given
2 : $\Pi;((\Gamma, \mathtt{x}_1:\mathtt{t}_1), \mathtt{x}_2:\mathtt{t}_2) \vdash (\mathtt{x}_3) : \mathtt{t}_3$ — *given
3 : $\Pi;(\Gamma, \mathtt{x}_1:\mathtt{t}_1) \vdash (\mathtt{x}_3) : \mathtt{t}_3$ — ↓ty-var2: 2
4 : $\Pi;\Gamma \vdash (\mathtt{x}_3) : \mathtt{t}_3$ — ↓ty-var2: 3
5 : $\Pi;(\Gamma, \mathtt{x}_2:\mathtt{t}_2) \vdash (\mathtt{x}_3) : \mathtt{t}_3$ — ty-var2: 4
6 : $\Pi;((\Gamma, \mathtt{x}_2:\mathtt{t}_2), \mathtt{x}_1:\mathtt{t}_1) \vdash (\mathtt{x}_3) : \mathtt{t}_3$ — ty-var2: 5

(permty-var4)

1 : $\Pi;(\Gamma_1, \mathtt{x}_2:\mathtt{t}_2) \vdash (\mathtt{x}_1) : \mathtt{t}_1$ — *given
2 : $\Pi;\Gamma_1 \vdash (\mathtt{x}_1) : \mathtt{t}_1$ — ↓ty-var2: 1
3 : $(\Gamma_1, \mathtt{x}_2:\mathtt{t}_2) \equiv (\Gamma_2, \mathtt{x}_2:\mathtt{t}_2)$ — *given
4 : $\Gamma_1 \equiv \Gamma_2$ — ↓perm-x2: 3
5 : $\Pi;\Gamma_2 \vdash (\mathtt{x}_1) : \mathtt{t}_1$ — permty: 2,4
6 : $\Pi;(\Gamma_2, \mathtt{x}_2:\mathtt{t}_2) \vdash (\mathtt{x}_1) : \mathtt{t}_1$ — ty-var2: 5

(permty-fun)

1 : $\Pi_2;\Gamma_1 \vdash (\lambda[\Pi_1]\mathtt{x}:\mathtt{t}_1 . \mathtt{e}) : (\mathtt{t}_1 \to \mathtt{t}_2)$ — *given
2 : $\Pi_2 \leq \Pi_1$ — ↓ty-fun: 1
3 : $\Pi_1;(\Gamma_1, \mathtt{x}:\mathtt{t}_1) \vdash \mathtt{e} : \mathtt{t}_2$ — ↓ty-fun: 1
4 : $\Gamma_1 \equiv \Gamma_2$ — *given
5 : $(\Gamma_1, \mathtt{x}:\mathtt{t}_1) \equiv (\Gamma_2, \mathtt{x}:\mathtt{t}_1)$ — perm-x2: 4
6 : $\Pi_1;(\Gamma_2, \mathtt{x}:\mathtt{t}_1) \vdash \mathtt{e} : \mathtt{t}_2$ — permty: 3,5
7 : $\Pi_2;\Gamma_2 \vdash (\lambda[\Pi_1]\mathtt{x}:\mathtt{t}_1 . \mathtt{e}) : (\mathtt{t}_1 \to \mathtt{t}_2)$ — ty-fun: 2,6

(permty-app)

1 : $\Pi;\Gamma_1 \vdash (\mathtt{e}_2\ \mathtt{e}_1) : \mathtt{t}_2$ — *given
2 : $\Pi;\Gamma_1 \vdash \mathtt{e}_2 : (\mathtt{t}_1 \to \mathtt{t}_2)$ — ↓ty-app: 1
3 : $\Gamma_1 \equiv \Gamma_2$ — *given
4 : $\Pi;\Gamma_2 \vdash \mathtt{e}_2 : (\mathtt{t}_1 \to \mathtt{t}_2)$ — permty: 2,3
5 : $\Pi;\Gamma_1 \vdash \mathtt{e}_1 : \mathtt{t}_1$ — ↓ty-app: 1
6 : $\Pi;\Gamma_2 \vdash \mathtt{e}_1 : \mathtt{t}_1$ — permty: 5,3
7 : $\Pi;\Gamma_2 \vdash (\mathtt{e}_2\ \mathtt{e}_1) : \mathtt{t}_2$ — ty-app: 4,6

(permty-ifp)

1 : $\Pi;\Gamma_1 \vdash (\mathtt{if}\ (\mathtt{p}_1 \leq \mathtt{p}_2)\ \mathtt{e}_2\ \mathtt{e}_1) : \mathtt{t}$ — *given
2 : $(\Pi, \mathtt{p}_1 \leq \mathtt{p}_2);\Gamma_1 \vdash \mathtt{e}_2 : \mathtt{t}$ — ↓ty-ifp: 1
3 : $\Gamma_1 \equiv \Gamma_2$ — *given
4 : $(\Pi, \mathtt{p}_1 \leq \mathtt{p}_2);\Gamma_2 \vdash \mathtt{e}_2 : \mathtt{t}$ — permty: 2,3
5 : $\Pi;\Gamma_1 \vdash \mathtt{e}_1 : \mathtt{t}$ — ↓ty-ifp: 1
6 : $\Pi;\Gamma_2 \vdash \mathtt{e}_1 : \mathtt{t}$ — permty: 5,3
7 : $\Pi;\Gamma_2 \vdash (\mathtt{if}\ (\mathtt{p}_1 \leq \mathtt{p}_2)\ \mathtt{e}_2\ \mathtt{e}_1) : \mathtt{t}$ — ty-ifp: 4,6

(permty-tag)

1 : $\Pi;\Gamma_1 \vdash ([\ell_1 \sqsubseteq \ell_2]\mathtt{e}) : (\mathtt{bool}_{\ell_2})$ — *given
2 : $\Pi \vdash \ell_1 \sqsubseteq \ell_2$ — ↓ty-tag: 1
3 : $\Pi;\Gamma_1 \vdash \mathtt{e} : (\mathtt{bool}_{\ell_1})$ — ↓ty-tag: 1
4 : $\Gamma_1 \equiv \Gamma_2$ — *given
5 : $\Pi;\Gamma_2 \vdash \mathtt{e} : (\mathtt{bool}_{\ell_1})$ — permty: 3,4
6 : $\Pi;\Gamma_2 \vdash ([\ell_1 \sqsubseteq \ell_2]\mathtt{e}) : (\mathtt{bool}_{\ell_2})$ — ty-tag: 2,5

4.7 Substitutions

$$\frac{\Pi;(\Gamma,\mathtt{x}{:}\mathtt{t}_1)\vdash \mathtt{e}_1 : \mathtt{t}_2 \quad \cdot;\cdot\vdash \mathtt{e}_2 : \mathtt{t}_1 \quad \mathtt{e}_1\{\mathtt{e}_2/\mathtt{x}\}=\mathtt{e}_3}{\Pi;\Gamma\vdash \mathtt{e}_3 : \mathtt{t}_2}$$

subty

(subty-true)

1 : $\Pi;(\Gamma,\mathtt{x}{:}\mathtt{t})\vdash(\mathtt{true}_\ell):(\mathtt{bool}_\ell)$ *given
2 : $\cdot;\cdot\vdash \mathtt{e}:\mathtt{t}$ *given
3 : $(\mathtt{true}_\ell)\{\mathtt{e}/\mathtt{x}\}=(\mathtt{true}_\ell)$ *given
4 : $\Pi;\Gamma\vdash(\mathtt{true}_\ell):(\mathtt{bool}_\ell)$ ty-true

(subty-false)

1 : $\Pi;(\Gamma,\mathtt{x}{:}\mathtt{t})\vdash(\mathtt{false}_\ell):(\mathtt{bool}_\ell)$ *given
2 : $\cdot;\cdot\vdash \mathtt{e}:\mathtt{t}$ *given
3 : $(\mathtt{false}_\ell)\{\mathtt{e}/\mathtt{x}\}=(\mathtt{false}_\ell)$ *given
4 : $\Pi;\Gamma\vdash(\mathtt{false}_\ell):(\mathtt{bool}_\ell)$ ty-false

(subty-if)

1 : $\Pi;(\Gamma,\mathtt{x}{:}\mathtt{t}_1)\vdash(\mathtt{if}\ \mathtt{e}_6\ \mathtt{e}_4\ \mathtt{e}_1):\mathtt{t}_2$ *given
2 : $\Pi;(\Gamma,\mathtt{x}{:}\mathtt{t}_1)\vdash \mathtt{e}_6:(\mathtt{bool}_\ell)$ ↓ty-ifb: 1
3 : $\cdot;\cdot\vdash \mathtt{e}_2:\mathtt{t}_1$ *given
4 : $(\mathtt{if}\ \mathtt{e}_6\ \mathtt{e}_4\ \mathtt{e}_1)\{\mathtt{e}_2/\mathtt{x}\}=(\mathtt{if}\ \mathtt{e}_7\ \mathtt{e}_5\ \mathtt{e}_3)$ *given
5 : $\mathtt{e}_6\{\mathtt{e}_2/\mathtt{x}\}=\mathtt{e}_7$ ↓sub-ifb: 4
6 : $\Pi;\Gamma\vdash \mathtt{e}_7:(\mathtt{bool}_\ell)$ subty: 2,3,5
7 : $\Pi;(\Gamma,\mathtt{x}{:}\mathtt{t}_1)\vdash \mathtt{e}_4:\mathtt{t}_2$ ↓ty-ifb: 1
8 : $\mathtt{e}_4\{\mathtt{e}_2/\mathtt{x}\}=\mathtt{e}_5$ ↓sub-ifb: 4
9 : $\Pi;\Gamma\vdash \mathtt{e}_5:\mathtt{t}_2$ subty: 7,3,8
10 : $\Pi;(\Gamma,\mathtt{x}{:}\mathtt{t}_1)\vdash \mathtt{e}_1:\mathtt{t}_2$ ↓ty-ifb: 1
11 : $\mathtt{e}_1\{\mathtt{e}_2/\mathtt{x}\}=\mathtt{e}_3$ ↓sub-ifb: 4
12 : $\Pi;\Gamma\vdash \mathtt{e}_3:\mathtt{t}_2$ subty: 10,3,11
13 : $\mathtt{lab}(\mathtt{t}_2)=\ell$ ↓ty-ifb: 1
14 : $\Pi;\Gamma\vdash(\mathtt{if}\ \mathtt{e}_7\ \mathtt{e}_5\ \mathtt{e}_3):\mathtt{t}_2$ ty-ifb: 6,9,12,13

(subty-var1)

1 : $\cdot;(\cdot,\mathtt{x}{:}\mathtt{t})\vdash(\mathtt{x}):\mathtt{t}$ *given
2 : $(\mathtt{x})\{\mathtt{e}/\mathtt{x}\}=\mathtt{e}$ *given
3 : $\cdot;\cdot\vdash \mathtt{e}:\mathtt{t}$ *given

(subty-var2)

1 : $(\mathtt{x}_2)\{(\mathtt{x}_2)/\mathtt{x}_1\}=(\mathtt{x}_2)$ *given
2 : $\cdot;(\cdot,\mathtt{x}_1{:}\mathtt{t})\vdash(\mathtt{x}_2):\mathtt{t}$ *given
3 : $\cdot;\cdot\vdash(\mathtt{x}_2):\mathtt{t}$ ↓ty-var2: 2

(subty-fun)

1 : $\Pi_2;(\Gamma,\mathtt{x}_2{:}\mathtt{t}_2)\vdash(\lambda[\Pi_1]\mathtt{x}_1{:}\mathtt{t}_1.\ \mathtt{e}_1):(\mathtt{t}_1\to\mathtt{t}_3)$ *given
2 : $\Pi_2\le\Pi_1$ ↓ty-fun: 1
3 : $\Pi_1;((\Gamma,\mathtt{x}_2{:}\mathtt{t}_2),\mathtt{x}_1{:}\mathtt{t}_1)\vdash \mathtt{e}_1:\mathtt{t}_3$ ↓ty-fun: 1
4 : $((\Gamma,\mathtt{x}_2{:}\mathtt{t}_2),\mathtt{x}_1{:}\mathtt{t}_1)\equiv((\Gamma,\mathtt{x}_1{:}\mathtt{t}_1),\mathtt{x}_2{:}\mathtt{t}_2)$ perm-x1
5 : $\Pi_1;((\Gamma,\mathtt{x}_1{:}\mathtt{t}_1),\mathtt{x}_2{:}\mathtt{t}_2)\vdash \mathtt{e}_1:\mathtt{t}_3$ permty: 3,4
6 : $\cdot;\cdot\vdash \mathtt{e}_2:\mathtt{t}_2$ *given
7 : $(\lambda[\Pi_1]\mathtt{x}_1{:}\mathtt{t}_1.\ \mathtt{e}_1)\{\mathtt{e}_2/\mathtt{x}_2\}=(\lambda[\Pi_1]\mathtt{x}_1{:}\mathtt{t}_1.\ \mathtt{e}_3)$ *given
8 : $\mathtt{e}_1\{\mathtt{e}_2/\mathtt{x}_2\}=\mathtt{e}_3$ ↓sub-fun: 7
9 : $\Pi_1;(\Gamma,\mathtt{x}_1{:}\mathtt{t}_1)\vdash \mathtt{e}_3:\mathtt{t}_3$ subty: 5,6,8
10 : $\Pi_2;\Gamma\vdash(\lambda[\Pi_1]\mathtt{x}_1{:}\mathtt{t}_1.\ \mathtt{e}_3):(\mathtt{t}_1\to\mathtt{t}_3)$ ty-fun: 2,9

(subty-app)

1 : $\Pi;(\Gamma,\mathtt{x}{:}\mathtt{t}_1)\vdash(\mathtt{e}_4\ \mathtt{e}_1):\mathtt{t}_3$ *given
2 : $\Pi;(\Gamma,\mathtt{x}{:}\mathtt{t}_1)\vdash\mathtt{e}_4:(\mathtt{t}_2\to\mathtt{t}_3)$ ↓ty-app: 1
3 : $\cdot;\cdot\vdash\mathtt{e}_2:\mathtt{t}_1$ *given
4 : $(\mathtt{e}_4\ \mathtt{e}_1)\{\mathtt{e}_2/\mathtt{x}\}=(\mathtt{e}_5\ \mathtt{e}_3)$ *given
5 : $\mathtt{e}_4\{\mathtt{e}_2/\mathtt{x}\}=\mathtt{e}_5$ ↓sub-app: 4
6 : $\Pi;\Gamma\vdash\mathtt{e}_5:(\mathtt{t}_2\to\mathtt{t}_3)$ subty: 2,3,5
7 : $\Pi;(\Gamma,\mathtt{x}{:}\mathtt{t}_1)\vdash\mathtt{e}_1:\mathtt{t}_2$ ↓ty-app: 1
8 : $\mathtt{e}_1\{\mathtt{e}_2/\mathtt{x}\}=\mathtt{e}_3$ ↓sub-app: 4
9 : $\Pi;\Gamma\vdash\mathtt{e}_3:\mathtt{t}_2$ subty: 7,3,8
10 : $\Pi;\Gamma\vdash(\mathtt{e}_5\ \mathtt{e}_3):\mathtt{t}_3$ ty-app: 6,9

$$\frac{\Pi;\Gamma\vdash\mathtt{e}_1:\mathtt{t}_1\quad \mathtt{e}_1=\mathcal{E}[\mathtt{e}_2]}{\Pi;\Gamma\vdash\mathtt{e}_2:\mathtt{t}_2}$$ splitty

(splitty-ifb)

1 : $\Pi;\Gamma\vdash(\mathtt{if}\ \mathtt{e}_1\ \mathtt{e}_2\ \mathtt{e}_3):\mathtt{t}$ *given
2 : $\Pi;\Gamma\vdash\mathtt{e}_2:\mathtt{t}$ ↓ty-ifb: 1
3 : $\Pi;\Gamma\vdash\mathtt{e}_3:\mathtt{t}$ ↓ty-ifb: 1
4 : $\mathtt{lab(t)}=\ell$ ↓ty-ifb: 1
5 : $(\mathtt{if}\ \mathtt{e}_1\ \mathtt{e}_2\ \mathtt{e}_3)=(\mathtt{if}\ \mathcal{E}\ \mathtt{e}_2\ \mathtt{e}_3)[\mathtt{e}_1]$ *given
6 : $\Pi;\Gamma\vdash\mathtt{e}_1:(\mathtt{bool}_\ell)$ ↓ty-ifb: 1

(splitty-app1)

1 : $\Pi;\Gamma\vdash(\mathtt{e}_1\ \mathtt{e}_2):\mathtt{t}_1$ *given
2 : $\Pi;\Gamma\vdash\mathtt{e}_2:\mathtt{t}_2$ ↓ty-app: 1
3 : $(\mathtt{e}_1\ \mathtt{e}_2)=(\mathcal{E}\ \mathtt{e}_2)[\mathtt{e}_1]$ *given
4 : $\Pi;\Gamma\vdash\mathtt{e}_1:(\mathtt{t}_2\to\mathtt{t}_1)$ ↓ty-app: 1

(splitty-app2)

1 : $\Pi;\Gamma\vdash(\mathtt{e}_1\ \mathtt{e}_2):\mathtt{t}_1$ *given
2 : $\Pi;\Gamma\vdash\mathtt{e}_1:(\mathtt{t}_2\to\mathtt{t}_1)$ ↓ty-app: 1
3 : $(\mathtt{e}_1\ \mathtt{e}_2)=(\mathtt{e}_1\ \mathcal{E})[\mathtt{e}_2]$ *given
4 : $\Pi;\Gamma\vdash\mathtt{e}_2:\mathtt{t}_2$ ↓ty-app: 1

(splitty-tag)

1 : $\Pi;\Gamma\vdash([\ell_1\sqsubseteq\ell_2]\mathtt{e}):(\mathtt{bool}_{\ell_2})$ *given
2 : $\Pi\vdash\ell_1\sqsubseteq\ell_2$ ↓ty-tag: 1
3 : $([\ell_1\sqsubseteq\ell_2]\mathtt{e})=([\ell_1\sqsubseteq\ell_2]\mathcal{E})[\mathtt{e}]$ *given
4 : $\Pi;\Gamma\vdash\mathtt{e}:(\mathtt{bool}_{\ell_1})$ ↓ty-tag: 1

$$\frac{\Pi;\Gamma\vdash\mathtt{e}_1:\mathtt{t}_1\quad \mathtt{e}_1=\mathcal{E}[\mathtt{e}_2]\quad \Pi;\Gamma\vdash\mathtt{e}_2:\mathtt{t}_2\quad \Pi;\Gamma\vdash\mathtt{e}_3:\mathtt{t}_2\quad \mathcal{E}[\mathtt{e}_3]=\mathtt{e}_4}{\Pi;\Gamma\vdash\mathtt{e}_4:\mathtt{t}_1}$$ combinety

(combinety-ifb)

1 : $\Pi;\Gamma\vdash(\mathtt{if}\ \mathtt{e}_1\ \mathtt{e}_2\ \mathtt{e}_3):\mathtt{t}$ *given
2 : $\Pi;\Gamma\vdash\mathtt{e}_1:(\mathtt{bool}_\ell)$ ↓ty-ifb: 1
3 : $(\mathtt{if}\ \mathtt{e}_1\ \mathtt{e}_2\ \mathtt{e}_3)=(\mathtt{if}\ \mathcal{E}\ \mathtt{e}_2\ \mathtt{e}_3)[\mathtt{e}_1]$ *given
4 : $(\mathtt{if}\ \mathcal{E}\ \mathtt{e}_2\ \mathtt{e}_3)[\mathtt{e}_4]=(\mathtt{if}\ \mathtt{e}_4\ \mathtt{e}_2\ \mathtt{e}_3)$ *given
5 : $\Pi;\Gamma\vdash\mathtt{e}_4:(\mathtt{bool}_\ell)$ *given
6 : $\Pi;\Gamma\vdash\mathtt{e}_2:\mathtt{t}$ ↓ty-ifb: 1
7 : $\Pi;\Gamma\vdash\mathtt{e}_3:\mathtt{t}$ ↓ty-ifb: 1
8 : $\mathtt{lab(t)}=\ell$ ↓ty-ifb: 1
9 : $\Pi;\Gamma\vdash(\mathtt{if}\ \mathtt{e}_4\ \mathtt{e}_2\ \mathtt{e}_3):\mathtt{t}$ ty-ifb: 5,6,7,8

(combinety-app1)

1 : $\Pi;\Gamma\vdash(\mathtt{e}_1\ \mathtt{e}_2):\mathtt{t}_1$ *given

2 : $\Pi;\Gamma \vdash \mathtt{e}_1 : (\mathtt{t}_2 \to \mathtt{t}_1)$ ↓ty-app: 1
3 : $(\mathtt{e}_1\ \mathtt{e}_2) = (\mathcal{E}\ \mathtt{e}_2)[\mathtt{e}_1]$ *given
4 : $(\mathcal{E}\ \mathtt{e}_2)[\mathtt{e}_3] = (\mathtt{e}_3\ \mathtt{e}_2)$ *given
5 : $\Pi;\Gamma \vdash \mathtt{e}_3 : (\mathtt{t}_2 \to \mathtt{t}_1)$ *given
6 : $\Pi;\Gamma \vdash \mathtt{e}_2 : \mathtt{t}_2$ ↓ty-app: 1
7 : $\Pi;\Gamma \vdash (\mathtt{e}_3\ \mathtt{e}_2) : \mathtt{t}_1$ ty-app: 5,6

(combinety-app2)

1 : $\Pi;\Gamma \vdash (\mathtt{e}_1\ \mathtt{e}_2) : \mathtt{t}_1$ *given
2 : $\Pi;\Gamma \vdash \mathtt{e}_2 : \mathtt{t}_2$ ↓ty-app: 1
3 : $(\mathtt{e}_1\ \mathtt{e}_2) = (\mathtt{e}_1\ \mathcal{E})[\mathtt{e}_2]$ *given
4 : $(\mathtt{e}_1\ \mathcal{E})[\mathtt{e}_3] = (\mathtt{e}_1\ \mathtt{e}_3)$ *given
5 : $\Pi;\Gamma \vdash \mathtt{e}_1 : (\mathtt{t}_2 \to \mathtt{t}_1)$ ↓ty-app: 1
6 : $\Pi;\Gamma \vdash \mathtt{e}_3 : \mathtt{t}_2$ *given
7 : $\Pi;\Gamma \vdash (\mathtt{e}_1\ \mathtt{e}_3) : \mathtt{t}_1$ ty-app: 5,6

(combinety-tag)

1 : $\Pi;\Gamma \vdash ([\ell_1 \sqsubseteq \ell_2]\mathtt{e}_1) : (\mathtt{bool}_{\ell_2})$ *given
2 : $\Pi;\Gamma \vdash \mathtt{e}_1 : (\mathtt{bool}_{\ell_1})$ ↓ty-tag: 1
3 : $([\ell_1 \sqsubseteq \ell_2]\mathtt{e}_1) = ([\ell_1 \sqsubseteq \ell_2]\mathcal{E})[\mathtt{e}_1]$ *given
4 : $([\ell_1 \sqsubseteq \ell_2]\mathcal{E})[\mathtt{e}_2] = ([\ell_1 \sqsubseteq \ell_2]\mathtt{e}_2)$ *given
5 : $\Pi \vdash \ell_1 \sqsubseteq \ell_2$ ↓ty-tag: 1
6 : $\Pi;\Gamma \vdash \mathtt{e}_2 : (\mathtt{bool}_{\ell_1})$ *given
7 : $\Pi;\Gamma \vdash ([\ell_1 \sqsubseteq \ell_2]\mathtt{e}_2) : (\mathtt{bool}_{\ell_2})$ ty-tag: 5,6

5 Theorems

5.1 Preservation

$$\boxed{\frac{\Pi;\Gamma \vdash \mathtt{e} : \mathtt{t} \quad \mathtt{val}\ \mathtt{e}}{\cdot;\cdot \vdash \mathtt{e} : \mathtt{t}}}$$

valty

(valty-true)

1 : $\Pi;\Gamma \vdash (\mathtt{true}_\ell) : (\mathtt{bool}_\ell)$ *given
2 : $\mathtt{val}\ (\mathtt{true}_\ell)$ *given
3 : $\cdot;\cdot \vdash (\mathtt{true}_\ell) : (\mathtt{bool}_\ell)$ ty-true

(valty-false)

1 : $\Pi;\Gamma \vdash (\mathtt{false}_\ell) : (\mathtt{bool}_\ell)$ *given
2 : $\mathtt{val}\ (\mathtt{false}_\ell)$ *given
3 : $\cdot;\cdot \vdash (\mathtt{false}_\ell) : (\mathtt{bool}_\ell)$ ty-false

(valty-fun)

1 : $\cdot;\cdot \vdash (\lambda[\Pi]\mathtt{x}{:}\mathtt{t}_1\,.\,\mathtt{e}) : (\mathtt{t}_1 \to \mathtt{t}_2)$ *given
2 : $\cdot \leq \Pi$ ↓ty-fun: 1
3 : $\Pi;(\cdot,\mathtt{x}{:}\mathtt{t}_1) \vdash \mathtt{e} : \mathtt{t}_2$ ↓ty-fun: 1
4 : $\mathtt{val}\ (\lambda[\Pi]\mathtt{x}{:}\mathtt{t}_1\,.\,\mathtt{e})$ *given

$$\boxed{\frac{\Pi;\cdot \vdash \mathtt{e}_1 : \mathtt{t} \quad \Pi \vdash \mathtt{e}_1 \longrightarrow \mathtt{e}_2}{\Pi;\cdot \vdash \mathtt{e}_2 : \mathtt{t}}}$$

ps

(ps-app)

1 : $\Pi;\cdot \vdash ((\lambda[\Pi]\mathtt{x}{:}\mathtt{t}_1\,.\,\mathtt{e}_1)\ \mathtt{e}_2) : \mathtt{t}_2$ *given
2 : $\Pi;\cdot \vdash (\lambda[\Pi]\mathtt{x}{:}\mathtt{t}_1\,.\,\mathtt{e}_1) : (\mathtt{t}_1 \to \mathtt{t}_2)$ ↓ty-app: 1
3 : $\Pi \leq \Pi$ ↓ty-fun: 2
4 : $\Pi;(\cdot,\mathtt{x}{:}\mathtt{t}_1) \vdash \mathtt{e}_1 : \mathtt{t}_2$ ↓ty-fun: 2
5 : $\Pi;\cdot \vdash \mathtt{e}_2 : \mathtt{t}_1$ ↓ty-app: 1
6 : $\Pi \vdash ((\lambda[\Pi]\mathtt{x}{:}\mathtt{t}_1\,.\,\mathtt{e}_1)\ \mathtt{e}_2) \longrightarrow \mathtt{e}_3$ *given
7 : $\mathtt{val}\ \mathtt{e}_2$ ↓ev-app: 6
8 : $\cdot;\cdot \vdash \mathtt{e}_2 : \mathtt{t}_1$ valty: 5,7
9 : $\mathtt{e}_1\{\mathtt{e}_2/\mathtt{x}\} = \mathtt{e}_3$ ↓ev-app: 6
10 : $\Pi;\cdot \vdash \mathtt{e}_3 : \mathtt{t}_2$ subty: 4,8,9

(ps-ifb1)

1 : $\Pi;\cdot \vdash (\mathtt{if}\ (\mathtt{true}_{\ell_1})\ \mathtt{e}_1\ \mathtt{e}_2) : \mathtt{t}$ *given
2 : $\Pi;\cdot \vdash (\mathtt{true}_{\ell_1}) : (\mathtt{bool}_{\ell_2})$ ↓ty-ifb: 1
3 : $\Pi;\cdot \vdash \mathtt{e}_2 : \mathtt{t}$ ↓ty-ifb: 1
4 : $\mathtt{lab}(\mathtt{t}) = \ell_2$ ↓ty-ifb: 1
5 : $\Pi \vdash (\mathtt{if}\ (\mathtt{true}_{\ell_1})\ \mathtt{e}_1\ \mathtt{e}_2) \longrightarrow \mathtt{e}_1$ *given
6 : $\Pi;\cdot \vdash \mathtt{e}_1 : \mathtt{t}$ ↓ty-ifb: 1

(ps-ifb2)

1 : $\Pi;\cdot \vdash (\mathtt{if}\ (\mathtt{false}_{\ell_1})\ \mathtt{e}_1\ \mathtt{e}_2) : \mathtt{t}$ *given
2 : $\Pi;\cdot \vdash (\mathtt{false}_{\ell_1}) : (\mathtt{bool}_{\ell_2})$ ↓ty-ifb: 1
3 : $\Pi;\cdot \vdash \mathtt{e}_1 : \mathtt{t}$ ↓ty-ifb: 1
4 : $\mathtt{lab}(\mathtt{t}) = \ell_2$ ↓ty-ifb: 1
5 : $\Pi \vdash (\mathtt{if}\ (\mathtt{false}_{\ell_1})\ \mathtt{e}_1\ \mathtt{e}_2) \longrightarrow \mathtt{e}_2$ *given
6 : $\Pi;\cdot \vdash \mathtt{e}_2 : \mathtt{t}$ ↓ty-ifb: 1

(ps-ifp1)

1 : $\Pi;\cdot \vdash (\mathtt{if}\ (\mathtt{p}_1 \leq \mathtt{p}_2)\ \mathtt{e}_1\ \mathtt{e}_2) : \mathtt{t}$ *given
2 : $\Pi;\cdot \vdash \mathtt{e}_2 : \mathtt{t}$ ↓ty-ifp: 1

3 : $(\Pi, \mathtt{p}_1 \leq \mathtt{p}_2);\cdot \vdash \mathtt{e}_1 : \mathtt{t}$ ↓ty-ifp: 1
4 : $\Pi \vdash (\mathtt{if}\ (\mathtt{p}_1 \leq \mathtt{p}_2)\ \mathtt{e}_1\ \mathtt{e}_2) \longrightarrow \mathtt{e}_1$ *given
5 : $\Pi \vdash \mathtt{p}_1 \leq \mathtt{p}_2$ ↓ev-ifp1: 4
6 : $\Pi;\cdot \vdash \mathtt{e}_1 : \mathtt{t}$ pstty: 3,5

(ps-ifp2)

1 : $\Pi;\cdot \vdash (\mathtt{if}\ (\mathtt{p}_1 \leq \mathtt{p}_2)\ \mathtt{e}_1\ \mathtt{e}_2) : \mathtt{t}$ *given
2 : $(\Pi, \mathtt{p}_1 \leq \mathtt{p}_2);\cdot \vdash \mathtt{e}_1 : \mathtt{t}$ ↓ty-ifp: 1
3 : $\Pi \vdash (\mathtt{if}\ (\mathtt{p}_1 \leq \mathtt{p}_2)\ \mathtt{e}_1\ \mathtt{e}_2) \longrightarrow \mathtt{e}_2$ *given
4 : $\Pi \not\vdash \mathtt{p}_1 \leq \mathtt{p}_2$ ↓ev-ifp2: 3
5 : $\Pi;\cdot \vdash \mathtt{e}_2 : \mathtt{t}$ ↓ty-ifp: 1

(ps-tag1)

1 : $\Pi;\cdot \vdash ([\ell_1 \sqsubseteq \ell_2](\mathtt{true}_{\ell_1})) : (\mathtt{bool}_{\ell_2})$ *given
2 : $\Pi \vdash \ell_1 \sqsubseteq \ell_2$ ↓ty-tag: 1
3 : $\Pi;\cdot \vdash (\mathtt{true}_{\ell_1}) : (\mathtt{bool}_{\ell_1})$ ↓ty-tag: 1
4 : $\Pi \vdash ([\ell_1 \sqsubseteq \ell_2](\mathtt{true}_{\ell_1})) \longrightarrow (\mathtt{true}_{\ell_2})$ *given
5 : $\Pi;\cdot \vdash (\mathtt{true}_{\ell_2}) : (\mathtt{bool}_{\ell_2})$ ty-true

(ps-tag2)

1 : $\Pi;\cdot \vdash ([\ell_1 \sqsubseteq \ell_2](\mathtt{false}_{\ell_1})) : (\mathtt{bool}_{\ell_2})$ *given
2 : $\Pi \vdash \ell_1 \sqsubseteq \ell_2$ ↓ty-tag: 1
3 : $\Pi;\cdot \vdash (\mathtt{false}_{\ell_1}) : (\mathtt{bool}_{\ell_1})$ ↓ty-tag: 1
4 : $\Pi \vdash ([\ell_1 \sqsubseteq \ell_2](\mathtt{false}_{\ell_1})) \longrightarrow (\mathtt{false}_{\ell_2})$ *given
5 : $\Pi;\cdot \vdash (\mathtt{false}_{\ell_2}) : (\mathtt{bool}_{\ell_2})$ ty-false

(ps-hole)

1 : $\Pi;\cdot \vdash \mathtt{e}_1 : \mathtt{t}_1$ *given
2 : $\Pi \vdash \mathtt{e}_1 \longrightarrow \mathtt{e}_4$ *given
3 : $\mathtt{e}_1 = \mathcal{E}[\mathtt{e}_2]$ ↓ev-hole: 2
4 : $\Pi;\cdot \vdash \mathtt{e}_2 : \mathtt{t}_2$ splitty: 1,3
5 : $\Pi \vdash \mathtt{e}_2 \longrightarrow \mathtt{e}_3$ ↓ev-hole: 2
6 : $\Pi;\cdot \vdash \mathtt{e}_3 : \mathtt{t}_2$ ps: 4,5
7 : $\mathcal{E}[\mathtt{e}_3] = \mathtt{e}_4$ ↓ev-hole: 2
8 : $\Pi;\cdot \vdash \mathtt{e}_4 : \mathtt{t}_1$ combinety: 1,3,4,6,7

$$\boxed{\frac{\Pi_1;\cdot \vdash \mathtt{e}_1 : \mathtt{t} \quad (\Pi_1;\mathtt{e}_1)|\Pi^\bullet \longrightarrow (\Pi_2;\mathtt{e}_2)}{\Pi_2;\cdot \vdash \mathtt{e}_2 : \mathtt{t}}}$$ topps

(topps-ev)

1 : $\Pi;\cdot \vdash \mathtt{e}_1 : \mathtt{t}$ *given
2 : $(\Pi;\mathtt{e}_1)|\Pi \longrightarrow (\Pi;\mathtt{e}_2)$ *given
3 : $\Pi \vdash \mathtt{e}_1 \longrightarrow \mathtt{e}_2$ ↓topev-ev: 2
4 : $\Pi;\cdot \vdash \mathtt{e}_2 : \mathtt{t}$ ps: 1,3

(topps-upd)

1 : $\Pi_1;\cdot \vdash \mathtt{e} : \mathtt{t}$ *given
2 : $(\Pi_1;\mathtt{e})|\Pi_2 \longrightarrow (\Pi_2;\mathtt{e})$ *given
3 : $\Pi_2 \vdash \mathtt{e}$ ↓topev-upd: 2
4 : $\Pi_2;\cdot \vdash \mathtt{e} : \mathtt{t}$ aety: 1,3

5.2 Progress

$$\boxed{\Pi \vdash \mathtt{ve}\ \mathtt{e}}$$ ve

$$\frac{\mathtt{val}\ \mathtt{e}}{\Pi \vdash \mathtt{ve}\ \mathtt{e}}$$ (ve-v)

$$\frac{\Pi \vdash \mathtt{e}_1 \longrightarrow \mathtt{e}_2}{\Pi \vdash \mathtt{ve}\ \mathtt{e}_1} \qquad \text{(ve-e)}$$

$$\boxed{\frac{\Pi;\cdot \vdash \mathtt{e} : \mathtt{t}}{\Pi \vdash \mathtt{ve}\ \mathtt{e}}} \qquad \boxed{\text{pg}}$$

(pg-true)

1 :	$\Pi;\cdot \vdash (\mathtt{true}_\ell) : (\mathtt{bool}_\ell)$	*given
2 :	$\mathtt{val}\ (\mathtt{true}_\ell)$	val-true
3 :	$\Pi \vdash \mathtt{ve}\ (\mathtt{true}_\ell)$	ve-v: 2

(pg-false)

1 :	$\Pi;\cdot \vdash (\mathtt{false}_\ell) : (\mathtt{bool}_\ell)$	*given
2 :	$\mathtt{val}\ (\mathtt{false}_\ell)$	val-false
3 :	$\Pi \vdash \mathtt{ve}\ (\mathtt{false}_\ell)$	ve-v: 2

(pg-ifb1)

1 :	$\Pi;\cdot \vdash (\mathtt{if}\ \mathtt{e}_1\ \mathtt{e}_3\ \mathtt{e}_4) : \mathtt{t}$	*given
2 :	$\Pi;\cdot \vdash \mathtt{e}_3 : \mathtt{t}$	↓ty-ifb: 1
3 :	$\Pi;\cdot \vdash \mathtt{e}_4 : \mathtt{t}$	↓ty-ifb: 1
4 :	$\mathtt{lab(t)} = \ell$	↓ty-ifb: 1
5 :	$(\mathtt{if}\ \mathtt{e}_1\ \mathtt{e}_3\ \mathtt{e}_4) = (\mathtt{if}\ \mathcal{E}\ \mathtt{e}_3\ \mathtt{e}_4)[\mathtt{e}_1]$	split-ifb
6 :	$\Pi;\cdot \vdash \mathtt{e}_1 : (\mathtt{bool}_\ell)$	↓ty-ifb: 1
7 :	$\Pi \vdash \mathtt{ve}\ \mathtt{e}_1$	pg: 6
8 :	$\Pi \vdash \mathtt{e}_1 \longrightarrow \mathtt{e}_2$	↓ve-e: 7
9 :	$(\mathtt{if}\ \mathcal{E}\ \mathtt{e}_3\ \mathtt{e}_4)[\mathtt{e}_2] = (\mathtt{if}\ \mathtt{e}_2\ \mathtt{e}_3\ \mathtt{e}_4)$	combine-ifb
10 :	$\Pi \vdash (\mathtt{if}\ \mathtt{e}_1\ \mathtt{e}_3\ \mathtt{e}_4) \longrightarrow (\mathtt{if}\ \mathtt{e}_2\ \mathtt{e}_3\ \mathtt{e}_4)$	ev-hole: 5,8,9
11 :	$\Pi \vdash \mathtt{ve}\ (\mathtt{if}\ \mathtt{e}_1\ \mathtt{e}_3\ \mathtt{e}_4)$	ve-e: 10

(pg-ifb2)

1 :	$\Pi;\cdot \vdash (\mathtt{if}\ (\mathtt{true}_{\ell_1})\ \mathtt{e}_1\ \mathtt{e}_2) : \mathtt{t}$	*given
2 :	$\Pi;\cdot \vdash (\mathtt{true}_{\ell_1}) : (\mathtt{bool}_{\ell_2})$	↓ty-ifb: 1
3 :	$\Pi;\cdot \vdash \mathtt{e}_1 : \mathtt{t}$	↓ty-ifb: 1
4 :	$\Pi;\cdot \vdash \mathtt{e}_2 : \mathtt{t}$	↓ty-ifb: 1
5 :	$\mathtt{lab(t)} = \ell_2$	↓ty-ifb: 1
6 :	$\Pi \vdash \mathtt{ve}\ (\mathtt{true}_{\ell_1})$	pg: 2
7 :	$\mathtt{val}\ (\mathtt{true}_{\ell_1})$	↓ve-v: 6
8 :	$\Pi \vdash (\mathtt{if}\ (\mathtt{true}_{\ell_1})\ \mathtt{e}_1\ \mathtt{e}_2) \longrightarrow \mathtt{e}_1$	ev-ifb1
9 :	$\Pi \vdash \mathtt{ve}\ (\mathtt{if}\ (\mathtt{true}_{\ell_1})\ \mathtt{e}_1\ \mathtt{e}_2)$	ve-e: 8

(pg-ifb3)

1 :	$\Pi;\cdot \vdash (\mathtt{if}\ (\mathtt{false}_{\ell_1})\ \mathtt{e}_1\ \mathtt{e}_2) : \mathtt{t}$	*given
2 :	$\Pi;\cdot \vdash (\mathtt{false}_{\ell_1}) : (\mathtt{bool}_{\ell_2})$	↓ty-ifb: 1
3 :	$\Pi;\cdot \vdash \mathtt{e}_1 : \mathtt{t}$	↓ty-ifb: 1
4 :	$\Pi;\cdot \vdash \mathtt{e}_2 : \mathtt{t}$	↓ty-ifb: 1
5 :	$\mathtt{lab(t)} = \ell_2$	↓ty-ifb: 1
6 :	$\Pi \vdash \mathtt{ve}\ (\mathtt{false}_{\ell_1})$	pg: 2
7 :	$\mathtt{val}\ (\mathtt{false}_{\ell_1})$	↓ve-v: 6
8 :	$\Pi \vdash (\mathtt{if}\ (\mathtt{false}_{\ell_1})\ \mathtt{e}_1\ \mathtt{e}_2) \longrightarrow \mathtt{e}_2$	ev-ifb2
9 :	$\Pi \vdash \mathtt{ve}\ (\mathtt{if}\ (\mathtt{false}_{\ell_1})\ \mathtt{e}_1\ \mathtt{e}_2)$	ve-e: 8

(pg-fun)

1 :	$\Pi_1;\cdot \vdash (\lambda[\Pi_2]\mathtt{x}{:}\mathtt{t}_1.\ \mathtt{e}) : (\mathtt{t}_1 \to \mathtt{t}_2)$	*given
2 :	$\Pi_1 \leq \Pi_2$	↓ty-fun: 1
3 :	$\Pi_2;(\cdot, \mathtt{x}{:}\mathtt{t}_1) \vdash \mathtt{e} : \mathtt{t}_2$	↓ty-fun: 1
4 :	$\mathtt{val}\ (\lambda[\Pi_2]\mathtt{x}{:}\mathtt{t}_1.\ \mathtt{e})$	val-fun
5 :	$\Pi_1 \vdash \mathtt{ve}\ (\lambda[\Pi_2]\mathtt{x}{:}\mathtt{t}_1.\ \mathtt{e})$	ve-v: 4

(pg-app1)

1 :	$\Pi;\cdot \vdash (\mathtt{e}_1\ \mathtt{e}_3) : \mathtt{t}_2$	*given
2 :	$\Pi;\cdot \vdash \mathtt{e}_3 : \mathtt{t}_1$	↓ty-app: 1
3 :	$(\mathtt{e}_1\ \mathtt{e}_3) = (\mathcal{E}\ \mathtt{e}_3)[\mathtt{e}_1]$	split-app1
4 :	$\Pi;\cdot \vdash \mathtt{e}_1 : (\mathtt{t}_1 \to \mathtt{t}_2)$	↓ty-app: 1
5 :	$\Pi \vdash \mathtt{ve}\ \mathtt{e}_1$	pg: 4
6 :	$\Pi \vdash \mathtt{e}_1 \longrightarrow \mathtt{e}_2$	↓ve-e: 5
7 :	$(\mathcal{E}\ \mathtt{e}_3)[\mathtt{e}_2] = (\mathtt{e}_2\ \mathtt{e}_3)$	combine-app1
8 :	$\Pi \vdash (\mathtt{e}_1\ \mathtt{e}_3) \longrightarrow (\mathtt{e}_2\ \mathtt{e}_3)$	ev-hole: 3,6,7
9 :	$\Pi \vdash \mathtt{ve}\ (\mathtt{e}_1\ \mathtt{e}_3)$	ve-e: 8

(pg-app2)

1 :	$\Pi;\cdot \vdash (\mathtt{e}_3\ \mathtt{e}_1) : \mathtt{t}_2$	*given
2 :	$\Pi;\cdot \vdash \mathtt{e}_3 : (\mathtt{t}_1 \to \mathtt{t}_2)$	↓ty-app: 1
3 :	$\Pi \vdash \mathtt{ve}\ \mathtt{e}_3$	pg: 2
4 :	$\mathtt{val}\ \mathtt{e}_3$	↓ve-v: 3
5 :	$(\mathtt{e}_3\ \mathtt{e}_1) = (\mathtt{e}_3\ \mathcal{E})[\mathtt{e}_1]$	split-app2
6 :	$\Pi;\cdot \vdash \mathtt{e}_1 : \mathtt{t}_1$	↓ty-app: 1
7 :	$\Pi \vdash \mathtt{ve}\ \mathtt{e}_1$	pg: 6
8 :	$\Pi \vdash \mathtt{e}_1 \longrightarrow \mathtt{e}_2$	↓ve-e: 7
9 :	$(\mathtt{e}_3\ \mathcal{E})[\mathtt{e}_2] = (\mathtt{e}_3\ \mathtt{e}_2)$	combine-app2
10 :	$\Pi \vdash (\mathtt{e}_3\ \mathtt{e}_1) \longrightarrow (\mathtt{e}_3\ \mathtt{e}_2)$	ev-hole: 5,8,9
11 :	$\Pi \vdash \mathtt{ve}\ (\mathtt{e}_3\ \mathtt{e}_1)$	ve-e: 10

(pg-app3)

1 :	$\Pi_1;\cdot \vdash ((\lambda[\Pi_2]\mathtt{x}{:}\mathtt{t}_2.\ \mathtt{e}_1)\ \mathtt{e}_2) : \mathtt{t}_3$	*given
2 :	$\Pi_1;\cdot \vdash (\lambda[\Pi_2]\mathtt{x}{:}\mathtt{t}_2.\ \mathtt{e}_1) : (\mathtt{t}_1 \to \mathtt{t}_3)$	↓ty-app: 1
3 :	$\Pi_1 \vdash \mathtt{ve}\ (\lambda[\Pi_2]\mathtt{x}{:}\mathtt{t}_2.\ \mathtt{e}_1)$	pg: 2
4 :	$\mathtt{val}\ (\lambda[\Pi_2]\mathtt{x}{:}\mathtt{t}_2.\ \mathtt{e}_1)$	↓ve-v: 3
5 :	$\Pi_1;\cdot \vdash \mathtt{e}_2 : \mathtt{t}_1$	↓ty-app: 1
6 :	$\Pi_1 \vdash \mathtt{ve}\ \mathtt{e}_2$	pg: 5
7 :	$\mathtt{val}\ \mathtt{e}_2$	↓ve-v: 6
8 :	$\mathtt{e}_1\{\mathtt{e}_2/\mathtt{x}\} = \mathtt{e}_3$	sub-total
9 :	$\Pi_1 \vdash ((\lambda[\Pi_2]\mathtt{x}{:}\mathtt{t}_2.\ \mathtt{e}_1)\ \mathtt{e}_2) \longrightarrow \mathtt{e}_3$	ev-app: 7,8
10 :	$\Pi_1 \vdash \mathtt{ve}\ ((\lambda[\Pi_2]\mathtt{x}{:}\mathtt{t}_2.\ \mathtt{e}_1)\ \mathtt{e}_2)$	ve-e: 9

(pg-ifp1)

1 :	$\Pi;\cdot \vdash (\mathtt{if}\ (\mathtt{p}_1 \le \mathtt{p}_2)\ \mathtt{e}_1\ \mathtt{e}_2) : \mathtt{t}$	*given
2 :	$(\Pi, \mathtt{p}_1 \le \mathtt{p}_2);\cdot \vdash \mathtt{e}_1 : \mathtt{t}$	↓ty-ifp: 1
3 :	$\Pi;\cdot \vdash \mathtt{e}_2 : \mathtt{t}$	↓ty-ifp: 1
4 :	$\Pi \vdash \mathtt{p}_1 \diamond \mathtt{p}_2$	pst-total
5 :	$\Pi \vdash \mathtt{p}_1 \le \mathtt{p}_2$	↓psty-pst: 4
6 :	$\Pi \vdash (\mathtt{if}\ (\mathtt{p}_1 \le \mathtt{p}_2)\ \mathtt{e}_1\ \mathtt{e}_2) \longrightarrow \mathtt{e}_1$	ev-ifp1: 5
7 :	$\Pi \vdash \mathtt{ve}\ (\mathtt{if}\ (\mathtt{p}_1 \le \mathtt{p}_2)\ \mathtt{e}_1\ \mathtt{e}_2)$	ve-e: 6

(pg-ifp2)

1 :	$\Pi;\cdot \vdash (\mathtt{if}\ (\mathtt{p}_1 \le \mathtt{p}_2)\ \mathtt{e}_1\ \mathtt{e}_2) : \mathtt{t}$	*given
2 :	$(\Pi, \mathtt{p}_1 \le \mathtt{p}_2);\cdot \vdash \mathtt{e}_1 : \mathtt{t}$	↓ty-ifp: 1
3 :	$\Pi;\cdot \vdash \mathtt{e}_2 : \mathtt{t}$	↓ty-ifp: 1
4 :	$\Pi \vdash \mathtt{p}_1 \diamond \mathtt{p}_2$	pst-total
5 :	$\Pi \not\vdash \mathtt{p}_1 \le \mathtt{p}_2$	↓psty-pnst: 4
6 :	$\Pi \vdash (\mathtt{if}\ (\mathtt{p}_1 \le \mathtt{p}_2)\ \mathtt{e}_1\ \mathtt{e}_2) \longrightarrow \mathtt{e}_2$	ev-ifp2: 5
7 :	$\Pi \vdash \mathtt{ve}\ (\mathtt{if}\ (\mathtt{p}_1 \le \mathtt{p}_2)\ \mathtt{e}_1\ \mathtt{e}_2)$	ve-e: 6

(pg-tag1)

1 :	$\Pi;\cdot \vdash ([\ell_1 \sqsubseteq \ell_2](\mathtt{true}_{\ell_1})) : (\mathtt{bool}_{\ell_2})$	*given
2 :	$\Pi \vdash \ell_1 \sqsubseteq \ell_2$	↓ty-tag: 1
3 :	$\Pi;\cdot \vdash (\mathtt{true}_{\ell_1}) : (\mathtt{bool}_{\ell_1})$	↓ty-tag: 1
4 :	$\Pi \vdash \mathtt{ve}\ (\mathtt{true}_{\ell_1})$	pg: 3

5 : $\texttt{val}\ (\texttt{true}_{\ell_1})$ ↓ve-v: 4
6 : $\Pi \vdash ([\ell_1 \sqsubseteq \ell_2](\texttt{true}_{\ell_1})) \longrightarrow (\texttt{true}_{\ell_2})$ ev-tag1
7 : $\Pi \vdash \texttt{ve}\ ([\ell_1 \sqsubseteq \ell_2](\texttt{true}_{\ell_1}))$ ve-e: 6

(pg-tag2)
1 : $\Pi;\cdot \vdash ([\ell_1 \sqsubseteq \ell_2](\texttt{false}_{\ell_1})) : (\texttt{bool}_{\ell_2})$ *given
2 : $\Pi \vdash \ell_1 \sqsubseteq \ell_2$ ↓ty-tag: 1
3 : $\Pi;\cdot \vdash (\texttt{false}_{\ell_1}) : (\texttt{bool}_{\ell_1})$ ↓ty-tag: 1
4 : $\Pi \vdash \texttt{ve}\ (\texttt{false}_{\ell_1})$ pg: 3
5 : $\texttt{val}\ (\texttt{false}_{\ell_1})$ ↓ve-v: 4
6 : $\Pi \vdash ([\ell_1 \sqsubseteq \ell_2](\texttt{false}_{\ell_1})) \longrightarrow (\texttt{false}_{\ell_2})$ ev-tag2
7 : $\Pi \vdash \texttt{ve}\ ([\ell_1 \sqsubseteq \ell_2](\texttt{false}_{\ell_1}))$ ve-e: 6

$$\boxed{\Pi \vdash \texttt{topve e}}$$ topve

$$\frac{\texttt{val e}}{\Pi \vdash \texttt{topve e}}$$ (topve-v)

$$\frac{\texttt{some}\ \Pi' \quad (\Pi_1;\texttt{e}_1)|\Pi \longrightarrow (\Pi_2;\texttt{e}_2)}{\Pi_1 \vdash \texttt{topve e}_1}$$ (topve-e)

$$\boxed{\frac{\Pi;\cdot \vdash \texttt{e} : \texttt{t}}{\Pi \vdash \texttt{topve e}}}$$ toppg

(toppg-v)
1 : $\Pi;\cdot \vdash \texttt{e} : \texttt{t}$ *given
2 : $\Pi \vdash \texttt{ve e}$ pg: 1
3 : $\texttt{val e}$ ↓ve-v: 2
4 : $\Pi \vdash \texttt{topve e}$ topve-v: 3

(toppg-e)
1 : $\texttt{some}\ \Pi'$ asome-total
2 : $\Pi;\cdot \vdash \texttt{e}_1 : \texttt{t}$ *given
3 : $\Pi \vdash \texttt{ve e}_1$ pg: 2
4 : $\Pi \vdash \texttt{e}_1 \longrightarrow \texttt{e}_2$ ↓ve-e: 3
5 : $(\Pi;\texttt{e}_1)|\Pi \longrightarrow (\Pi';\texttt{e}_2)$ topev-ev: 4
6 : $\Pi \vdash \texttt{topve e}_1$ topve-e: 1,5

5.3 Soundness

$$\boxed{\frac{\Pi_1;\cdot \vdash \texttt{e}_1 : \texttt{t}}{(\Pi_1;\texttt{e}_1) \Downarrow (\Pi_2;\texttt{e}_2)}}$$ snd

(snd-v)
1 : $\Pi;\cdot \vdash \texttt{e} : \texttt{t}$ *given
2 : $\Pi \vdash \texttt{topve e}$ toppg: 1
3 : $\texttt{val e}$ ↓topve-v: 2
4 : $(\Pi;\texttt{e}) \Downarrow (\Pi;\texttt{e})$ bigev-v: 3

(snd-e)
1 : $\Pi_3;\cdot \vdash \texttt{e}_3 : \texttt{t}$ *given
2 : $\Pi_3 \vdash \texttt{topve e}_3$ toppg: 1
3 : $\texttt{some}\ \Pi_4$ ↓topve-e: 2
4 : $(\Pi_3;\texttt{e}_3)|\Pi_4 \longrightarrow (\Pi_1;\texttt{e}_1)$ ↓topve-e: 2
5 : $\Pi_1;\cdot \vdash \texttt{e}_1 : \texttt{t}$ topps: 1,4
6 : $(\Pi_1;\texttt{e}_1) \Downarrow (\Pi_2;\texttt{e}_2)$ snd: 5
7 : $(\Pi_3;\texttt{e}_3) \Downarrow (\Pi_1;\texttt{e}_2)$ bigev-e: 3,4,6

6 Translations

6.1 Types

$$\boxed{\mathtt{u} ::= \cdots} \qquad \boxed{\mathrm{u}}$$

$$\mathtt{u} ::= \mathtt{bool}_\ell \qquad \text{(ubool)}$$

$$\mathtt{u} ::= \mathtt{u} \to \mathtt{u} \qquad \text{(ufun)}$$

6.2 Terms

$$\boxed{\mathtt{m} ::= \cdots} \qquad \boxed{\mathrm{m}}$$

$$\mathtt{m} ::= \mathtt{true}_\ell \qquad \text{(mtrue)}$$

$$\mathtt{m} ::= \mathtt{false}_\ell \qquad \text{(mfalse)}$$

$$\mathtt{m} ::= \mathtt{x} \qquad \text{(mvar)}$$

$$\mathtt{m} ::= \lambda \mathtt{x}{:}\mathtt{u}.\ \mathtt{m} \qquad \text{(mfun)}$$

$$\mathtt{m} ::= \mathtt{m}\ \mathtt{m} \qquad \text{(mapp)}$$

$$\mathtt{m} ::= \mathtt{if}\ \mathtt{m}\ \mathtt{m}\ \mathtt{m} \qquad \text{(mifb)}$$

$$\mathtt{m} ::= \mathtt{if}\ (\mathtt{p} \preceq \mathtt{p})\ \mathtt{m}\ \mathtt{m} \qquad \text{(mifl)}$$

6.3 Contexts

$$\boxed{\Gamma ::= \cdots} \qquad \boxed{\mathrm{h}}$$

$$\Gamma ::= \cdot \qquad \text{(hz)}$$

$$\Gamma ::= \Gamma, \mathtt{x}{:}\mathtt{u} \qquad \text{(hx)}$$

6.4 Type labels

$$\boxed{\mathtt{lab(u)} = \ell} \qquad \boxed{\mathrm{ulab}}$$

$$\mathtt{lab}((\mathtt{bool}_\ell)) = \ell \qquad \text{(ulab-bool)}$$

$$\frac{\mathtt{lab}(\mathtt{u}_2) = \ell}{\mathtt{lab}((\mathtt{u}_1 \to \mathtt{u}_2)) = \ell} \qquad \text{(ulab-fun)}$$

6.5 Subtypings

$$\boxed{\Pi \vdash \mathtt{u} \preceq \mathtt{u}} \qquad \boxed{\mathrm{ust}}$$

$$\frac{\Pi \vdash \ell_1 \sqsubseteq \ell_2}{\Pi \vdash (\mathtt{bool}_{\ell_1}) \preceq (\mathtt{bool}_{\ell_2})} \qquad \text{(ust-bool)}$$

$$\frac{\Pi \vdash \mathtt{u}_3 \preceq \mathtt{u}_1 \quad \Pi \vdash \mathtt{u}_2 \preceq \mathtt{u}_4}{\Pi \vdash (\mathtt{u}_1 \to \mathtt{u}_2) \preceq (\mathtt{u}_3 \to \mathtt{u}_4)} \qquad \text{(ust-fun)}$$

6.6 Typings

$$\boxed{\mathtt{some\ u}} \qquad \boxed{\mathrm{usome}}$$

$$\boxed{\Pi; \Gamma \vdash \mathtt{m} : \mathtt{u}} \qquad \boxed{\mathrm{mty}}$$

$$\Pi;\Gamma \vdash (\texttt{true}_\ell) : (\texttt{bool}_\ell) \qquad \text{(mty-true)}$$

$$\Pi;\Gamma \vdash (\texttt{false}_\ell) : (\texttt{bool}_\ell) \qquad \text{(mty-false)}$$

$$\frac{\Pi;\Gamma \vdash \texttt{m}_1 : (\texttt{bool}_\ell) \quad \Pi;\Gamma \vdash \texttt{m}_2 : \texttt{u} \quad \Pi;\Gamma \vdash \texttt{m}_3 : \texttt{u} \quad \texttt{lab(u)} = \ell}{\Pi;\Gamma \vdash (\texttt{if m}_1\ \texttt{m}_2\ \texttt{m}_3) : \texttt{u}} \qquad \text{(mty-ifb)}$$

$$\Pi;(\Gamma,\texttt{x:u}) \vdash (\texttt{x}) : \texttt{u} \qquad \text{(mty-var1)}$$

$$\frac{\Pi;\Gamma \vdash (\texttt{x}_2) : \texttt{u}_2}{\Pi;(\Gamma,\texttt{x}_1\texttt{:u}_1) \vdash (\texttt{x}_2) : \texttt{u}_2} \qquad \text{(mty-var2)}$$

$$\frac{\Pi;(\Gamma,\texttt{x:u}_1) \vdash \texttt{m} : \texttt{u}_2}{\Pi;\Gamma \vdash (\lambda\texttt{x:u}_1\,.\,\texttt{m}) : (\texttt{u}_1 \to \texttt{u}_2)} \qquad \text{(mty-fun)}$$

$$\frac{\Pi;\Gamma \vdash \texttt{m}_1 : (\texttt{u}_1 \to \texttt{u}_2) \quad \Pi;\Gamma \vdash \texttt{m}_2 : \texttt{u}_1}{\Pi;\Gamma \vdash (\texttt{m}_1\ \texttt{m}_2) : \texttt{u}_2} \qquad \text{(mty-app)}$$

$$\frac{(\Pi,\texttt{p}_1 \le \texttt{p}_2);\Gamma \vdash \texttt{m}_1 : \texttt{u} \quad \Pi;\Gamma \vdash \texttt{m}_2 : \texttt{u}}{\Pi;\Gamma \vdash (\texttt{if}\ (\texttt{p}_1 \preceq \texttt{p}_2)\ \texttt{m}_1\ \texttt{m}_2) : \texttt{u}} \qquad \text{(mty-ifp)}$$

$$\frac{\Pi;\Gamma \vdash \texttt{m} : \texttt{u}_1 \quad \texttt{some u}_2 \quad \Pi \vdash \texttt{u}_1 \preceq \texttt{u}_2}{\Pi;\Gamma \vdash \texttt{m} : \texttt{u}_2} \qquad \text{(mty-sub)}$$

6.7 Theorems

$$\boxed{[\![\texttt{u}]\!] = \texttt{t}} \qquad \boxed{\text{ut}}$$

$$[\![(\texttt{bool}_\ell)]\!] = (\texttt{bool}_\ell) \qquad \text{(ut-bool)}$$

$$\frac{[\![\texttt{u}_1]\!] = \texttt{t}_1 \quad [\![\texttt{u}_2]\!] = \texttt{t}_2}{[\![(\texttt{u}_1 \to \texttt{u}_2)]\!] = (\texttt{t}_1 \to \texttt{t}_2)} \qquad \text{(ut-fun)}$$

$$\boxed{\frac{\texttt{lab(u)} = \ell \quad \Pi;\Gamma \vdash \texttt{e} : \texttt{t}}{\texttt{lab(t)} = \ell}} \qquad \boxed{\text{ulablab}}$$

(ulablab-bool)

1:	$\texttt{lab}((\texttt{bool}_\ell)) = \ell$	*given
2:	$\Pi;\Gamma \vdash \texttt{e} : (\texttt{bool}_\ell)$	*given
3:	$\texttt{lab}((\texttt{bool}_\ell)) = \ell$	lab-bool

(ulablab-fun)

1:	$\Pi_2;\Gamma \vdash (\lambda[\Pi_1]\texttt{x:t}_1\,.\,\texttt{e}) : (\texttt{t}_1 \to \texttt{t}_2)$	*given
2:	$\Pi_2 \le \Pi_1$	↓ty-fun: 1
3:	$\texttt{lab}((\texttt{u}_2 \to \texttt{u}_1)) = \ell$	*given
4:	$\texttt{lab}(\texttt{u}_1) = \ell$	↓ulab-fun: 3
5:	$\Pi_1;(\Gamma,\texttt{x:t}_1) \vdash \texttt{e} : \texttt{t}_2$	↓ty-fun: 1
6:	$\texttt{lab}(\texttt{t}_2) = \ell$	ulablab: 4,5
7:	$\texttt{lab}((\texttt{t}_1 \to \texttt{t}_2)) = \ell$	lab-fun: 6

$$\boxed{\texttt{fresh x}} \qquad \boxed{\text{fresh}}$$

$$\boxed{\frac{\Pi \vdash \texttt{u}_1 \preceq \texttt{u}_2}{\Pi;\Gamma \vdash \texttt{e} : (\texttt{t}_1 \to \texttt{t}_2)}} \qquad \boxed{\text{ustty}}$$

(ustty-bool)

1:	$\texttt{x}$	fresh
2:	$\Pi \le \Pi$	ast-z
3:	$\Pi \vdash (\texttt{bool}_{\ell_1}) \preceq (\texttt{bool}_{\ell_2})$	*given

4 :	$\Pi \vdash \ell_1 \sqsubseteq \ell_2$	↓ust-bool: 3
5 :	$\Pi; (\Gamma, \mathtt{x}:(\mathtt{bool}_{\ell_1})) \vdash (\mathtt{x}) : (\mathtt{bool}_{\ell_1})$	ty-var1
6 :	$\Pi; (\Gamma, \mathtt{x}:(\mathtt{bool}_{\ell_1})) \vdash ([\ell_1 \sqsubseteq \ell_2](\mathtt{x})) : (\mathtt{bool}_{\ell_2})$	ty-tag: 4,5
7 :	$\Pi; \Gamma \vdash (\lambda[\Pi]\mathtt{x}:(\mathtt{bool}_{\ell_1}).\ ([\ell_1 \sqsubseteq \ell_2](\mathtt{x}))) : ((\mathtt{bool}_{\ell_1}) \to (\mathtt{bool}_{\ell_2}))$	ty-fun: 2,6

(ustty-fun)

1 :	$\mathtt{x}_2$	fresh
2 :	$\mathtt{x}_1$	fresh
3 :	$\Pi \leq \Pi$	ast-z
4 :	$\Pi \vdash (\mathtt{u}_4 \to \mathtt{u}_1) \preceq (\mathtt{u}_3 \to \mathtt{u}_2)$	*given
5 :	$\Pi \vdash \mathtt{u}_1 \preceq \mathtt{u}_2$	↓ust-fun: 4
6 :	$\Pi; ((\Gamma, \mathtt{x}_1:(\mathtt{t}_1 \to \mathtt{t}_2)), \mathtt{x}_2:\mathtt{t}_3) \vdash \mathtt{e}_1 : (\mathtt{t}_2 \to \mathtt{t}_4)$	ustty: 5
7 :	$\Pi; (\Gamma, \mathtt{x}_1:(\mathtt{t}_1 \to \mathtt{t}_2)) \vdash (\mathtt{x}_1) : (\mathtt{t}_1 \to \mathtt{t}_2)$	ty-var1
8 :	$\Pi; ((\Gamma, \mathtt{x}_1:(\mathtt{t}_1 \to \mathtt{t}_2)), \mathtt{x}_2:\mathtt{t}_3) \vdash (\mathtt{x}_1) : (\mathtt{t}_1 \to \mathtt{t}_2)$	ty-var2: 7
9 :	$\Pi \vdash \mathtt{u}_3 \preceq \mathtt{u}_4$	↓ust-fun: 4
10 :	$\Pi; ((\Gamma, \mathtt{x}_1:(\mathtt{t}_1 \to \mathtt{t}_2)), \mathtt{x}_2:\mathtt{t}_3) \vdash \mathtt{e}_2 : (\mathtt{t}_3 \to \mathtt{t}_1)$	ustty: 9
11 :	$\Pi; ((\Gamma, \mathtt{x}_1:(\mathtt{t}_1 \to \mathtt{t}_2)), \mathtt{x}_2:\mathtt{t}_3) \vdash (\mathtt{x}_2) : \mathtt{t}_3$	ty-var1
12 :	$\Pi; ((\Gamma, \mathtt{x}_1:(\mathtt{t}_1 \to \mathtt{t}_2)), \mathtt{x}_2:\mathtt{t}_3) \vdash (\mathtt{e}_2\ (\mathtt{x}_2)) : \mathtt{t}_1$	ty-app: 10,11
13 :	$\Pi; ((\Gamma, \mathtt{x}_1:(\mathtt{t}_1 \to \mathtt{t}_2)), \mathtt{x}_2:\mathtt{t}_3) \vdash ((\mathtt{x}_1)\ (\mathtt{e}_2\ (\mathtt{x}_2))) : \mathtt{t}_2$	ty-app: 8,12
14 :	$\Pi; ((\Gamma, \mathtt{x}_1:(\mathtt{t}_1 \to \mathtt{t}_2)), \mathtt{x}_2:\mathtt{t}_3) \vdash (\mathtt{e}_1\ ((\mathtt{x}_1)\ (\mathtt{e}_2\ (\mathtt{x}_2)))) : \mathtt{t}_4$	ty-app: 6,13
15 :	$\Pi; (\Gamma, \mathtt{x}_1:(\mathtt{t}_1 \to \mathtt{t}_2)) \vdash (\lambda[\Pi]\mathtt{x}_2:\mathtt{t}_3.\ (\mathtt{e}_1\ ((\mathtt{x}_1)\ (\mathtt{e}_2\ (\mathtt{x}_2))))) : (\mathtt{t}_3 \to \mathtt{t}_4)$	ty-fun: 3,14
16 :	$\Pi; \Gamma \vdash (\lambda[\Pi]\mathtt{x}_1:(\mathtt{t}_1 \to \mathtt{t}_2).\ (\lambda[\Pi]\mathtt{x}_2:\mathtt{t}_3.\ (\mathtt{e}_1\ ((\mathtt{x}_1)\ (\mathtt{e}_2\ (\mathtt{x}_2)))))) : ((\mathtt{t}_1 \to \mathtt{t}_2) \to (\mathtt{t}_3 \to \mathtt{t}_4))$	ty-fun: 3,15

$$\boxed{\frac{\Pi; \Gamma \vdash \mathtt{m} : \mathtt{u}}{\Pi; \Gamma \vdash \mathtt{e} : \mathtt{t}}}$$ mtyty

(mtyty-true)

1 :	$\Pi; \Gamma \vdash (\mathtt{true}_\ell) : (\mathtt{bool}_\ell)$	*given
2 :	$\Pi; \Gamma \vdash (\mathtt{true}_\ell) : (\mathtt{bool}_\ell)$	ty-true

(mtyty-false)

1 :	$\Pi; \Gamma \vdash (\mathtt{false}_\ell) : (\mathtt{bool}_\ell)$	*given
2 :	$\Pi; \Gamma \vdash (\mathtt{false}_\ell) : (\mathtt{bool}_\ell)$	ty-false

(mtyty-if)

1 :	$\Pi; \Gamma \vdash (\mathtt{if}\ \mathtt{m}_3\ \mathtt{m}_2\ \mathtt{m}_1) : \mathtt{u}$	*given
2 :	$\Pi; \Gamma \vdash \mathtt{m}_3 : (\mathtt{bool}_\ell)$	↓mty-ifb: 1
3 :	$\Pi; \Gamma \vdash \mathtt{e}_1 : (\mathtt{bool}_\ell)$	mtyty: 2
4 :	$\Pi; \Gamma \vdash \mathtt{m}_2 : \mathtt{u}$	↓mty-ifb: 1
5 :	$\Pi; \Gamma \vdash \mathtt{e}_3 : (\mathtt{bool}_\ell)$	mtyty: 4
6 :	$\Pi; \Gamma \vdash \mathtt{m}_1 : \mathtt{u}$	↓mty-ifb: 1
7 :	$\Pi; \Gamma \vdash \mathtt{e}_2 : (\mathtt{bool}_\ell)$	mtyty: 6
8 :	$\mathtt{lab(u)} = \ell$	↓mty-ifb: 1
9 :	$\mathtt{lab}((\mathtt{bool}_\ell)) = \ell$	ulablab: 8,3
10 :	$\Pi; \Gamma \vdash (\mathtt{if}\ \mathtt{e}_1\ \mathtt{e}_3\ \mathtt{e}_2) : (\mathtt{bool}_\ell)$	ty-ifb: 3,5,7,9

(mtyty-var1)

1 :	$\Pi; (\Gamma, \mathtt{x}:\mathtt{u}) \vdash (\mathtt{x}) : \mathtt{u}$	*given
2 :	$\Pi; (\Gamma, \mathtt{x}:\mathtt{t}) \vdash (\mathtt{x}) : \mathtt{t}$	ty-var1

(mtyty-var2)

1 :	$\Pi; (\Gamma, \mathtt{x}_1:\mathtt{u}_1) \vdash (\mathtt{x}_2) : \mathtt{u}_2$	*given
2 :	$\Pi; \Gamma \vdash (\mathtt{x}_2) : \mathtt{u}_2$	↓mty-var2: 1
3 :	$\Pi; \Gamma \vdash (\mathtt{x}_2) : \mathtt{t}_2$	mtyty: 2
4 :	$\Pi; (\Gamma, \mathtt{x}_1:\mathtt{t}_1) \vdash (\mathtt{x}_2) : \mathtt{t}_2$	ty-var2: 3

(mtyty-fun)

1 :	$[\![\mathtt{u}_1]\!] = \mathtt{t}_1$	ut
2 :	$\Pi \leq \Pi$	ast-z
3 :	$\Pi;\Gamma \vdash (\lambda \mathtt{x}{:}\mathtt{u}_1\,.\,\mathtt{m}) : (\mathtt{u}_1 \to \mathtt{u}_2)$	*given
4 :	$\Pi;(\Gamma,\mathtt{x}{:}\mathtt{u}_1) \vdash \mathtt{m} : \mathtt{u}_2$	↓mty-fun: 3
5 :	$\Pi;(\Gamma,\mathtt{x}{:}\mathtt{t}_1) \vdash \mathtt{e} : \mathtt{t}_2$	mtyty: 4
6 :	$\Pi;\Gamma \vdash (\lambda[\Pi]\mathtt{x}{:}\mathtt{t}_1\,.\,\mathtt{e}) : (\mathtt{t}_1 \to \mathtt{t}_2)$	ty-fun: 2,5

(mtyty-app)

1 :	$\Pi;\Gamma \vdash (\mathtt{m}_2\ \mathtt{m}_1) : \mathtt{u}_2$	*given
2 :	$\Pi;\Gamma \vdash \mathtt{m}_2 : (\mathtt{u}_1 \to \mathtt{u}_2)$	↓mty-app: 1
3 :	$\Pi;\Gamma \vdash \mathtt{e}_2 : (\mathtt{t}_1 \to \mathtt{t}_2)$	mtyty: 2
4 :	$\Pi;\Gamma \vdash \mathtt{m}_1 : \mathtt{u}_1$	↓mty-app: 1
5 :	$\Pi;\Gamma \vdash \mathtt{e}_1 : \mathtt{t}_1$	mtyty: 4
6 :	$\Pi;\Gamma \vdash (\mathtt{e}_2\ \mathtt{e}_1) : \mathtt{t}_2$	ty-app: 3,5

(mtyty-ifp)

1 :	$\Pi;\Gamma \vdash (\mathtt{if}\ (\mathtt{p}_1 \preceq \mathtt{p}_2)\ \mathtt{m}_2\ \mathtt{m}_1) : \mathtt{u}$	*given
2 :	$(\Pi,\mathtt{p}_1 \leq \mathtt{p}_2);\Gamma \vdash \mathtt{m}_2 : \mathtt{u}$	↓mty-ifp: 1
3 :	$(\Pi,\mathtt{p}_1 \leq \mathtt{p}_2);\Gamma \vdash \mathtt{e}_2 : \mathtt{t}$	mtyty: 2
4 :	$\Pi;\Gamma \vdash \mathtt{m}_1 : \mathtt{u}$	↓mty-ifp: 1
5 :	$\Pi;\Gamma \vdash \mathtt{e}_1 : \mathtt{t}$	mtyty: 4
6 :	$\Pi;\Gamma \vdash (\mathtt{if}\ (\mathtt{p}_1 \leq \mathtt{p}_2)\ \mathtt{e}_2\ \mathtt{e}_1) : \mathtt{t}$	ty-ifp: 3,5

(mtyty-sub)

1 :	$\Pi;\Gamma \vdash \mathtt{m} : \mathtt{u}_2$	*given
2 :	$\mathtt{some}\ \mathtt{u}_2$	↓mty-sub: 1
3 :	$[\![\mathtt{u}_2]\!] = \mathtt{t}_2$	ut
4 :	$[\![\mathtt{u}_1]\!] = \mathtt{t}_1$	ut
5 :	$\Pi \vdash \mathtt{u}_1 \preceq \mathtt{u}_2$	↓mty-sub: 1
6 :	$\Pi;\Gamma \vdash \mathtt{e}_2 : (\mathtt{t}_1 \to \mathtt{t}_2)$	ustty: 5
7 :	$\Pi;\Gamma \vdash \mathtt{m} : \mathtt{u}_1$	↓mty-sub: 1
8 :	$\Pi;\Gamma \vdash \mathtt{e}_1 : \mathtt{t}_1$	mtyty: 7
9 :	$\Pi;\Gamma \vdash (\mathtt{e}_2\ \mathtt{e}_1) : \mathtt{t}_2$	ty-app: 6,8