

Soundness Theorem for Simply-Typed Lambda Calculus in de Bruijn Index Representation

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Aug 1, 2004*

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*Last update: August 10, 2004

1 Syntax

1.1 Numbers

$n ::= \dots$	n
$n ::= 0$	(z)
$n ::= n + 1$	(s)

1.2 Types

$t ::= \dots$	t
$t ::= \langle \rangle$	(tunit)
$t ::= t \rightarrow t$	(tfun)

1.3 Terms

$e ::= \dots$	e
$e ::= \langle \rangle$	(unit)
$e ::= n$	(var)
$e ::= \lambda. t. e$	(fun)
$e ::= e e$	(app)

1.4 Contexts

$\Gamma ::= \dots$	Γ
$\Gamma ::= \cdot$	(gn)
$\Gamma ::= \Gamma, t$	(gt)

2 Operators

2.1 Arithmetic

$\boxed{n - 1 = n}$	$\boxed{\text{pred}}$
$n + 1 - 1 = n$	(pred-s)
$\boxed{n < n}$	$\boxed{\text{lt}}$
$0 < n + 1$	(lt-z)
$\frac{n_1 < n_2}{n_1 + 1 < n_2 + 1}$	(lt-s)
$\boxed{n \geq n}$	$\boxed{\text{ge}}$
$0 \geq 0$	(ge-zz)
$n + 1 \geq 0$	(ge-sz)
$\frac{n_1 \geq n_2}{n_1 + 1 \geq n_2 + 1}$	(ge-ss)
$\boxed{n \neq n}$	$\boxed{\text{ne}}$
$0 \neq n + 1$	(ne-zs)
$n + 1 \neq 0$	(ne-sz)
$\frac{n_1 \neq n_2}{n_1 + 1 \neq n_2 + 1}$	(ne-ss)

2.2 Mapping

$\boxed{\Gamma [n] = t}$	$\boxed{\text{map}}$
$\Gamma, t [0] = t$	(map-z)
$\frac{\Gamma [n] = t}{\Gamma, t_0 [n + 1] = t}$	(map-s)

2.3 Context up-shifting

$\boxed{\uparrow_n^t \Gamma = \Gamma}$	$\boxed{\text{gup}}$
$\uparrow_0^t \Gamma = \Gamma, t$	(gup-z)
$\frac{\uparrow_n^t \Gamma_1 = \Gamma_2}{\uparrow_{n+1}^t \Gamma_1, t_1 = \Gamma_2, t_1}$	(gup-s)

2.4 Term up-shifting

$\boxed{\uparrow_n e = e}$	$\boxed{\text{up}}$
$\uparrow_n \langle \rangle = \langle \rangle$	(up-unit)
$\frac{n_1 < n_2}{\uparrow_{n_2} n_1 = n_1}$	(up-var1)

$$\frac{n_1 \geq n_2}{\uparrow_{n_2} n_1 = n_1 + 1} \quad (\text{up-var2})$$

$$\frac{\uparrow_{n+1} e_1 = e_2}{\uparrow_n \lambda. :t. e_1 = \lambda. :t. e_2} \quad (\text{up-fun})$$

$$\frac{\uparrow_n e_1 = e_3 \quad \uparrow_n e_2 = e_4}{\uparrow_n e_1 e_2 = e_3 e_4} \quad (\text{up-app})$$

2.5 Down-shifting

$$\boxed{\downarrow_n e = e} \quad \boxed{\text{down}}$$

$$\downarrow_n \langle \rangle = \langle \rangle \quad (\text{down-unit})$$

$$\frac{n_1 < n_2}{\downarrow_{n_2} n_1 = n_1} \quad (\text{down-var1})$$

$$\frac{n_1 \geq n_2 \quad n_1 - 1 = n_3}{\downarrow_{n_2} n_1 = n_3} \quad (\text{down-var2})$$

$$\frac{\downarrow_{n+1} e_1 = e_2}{\downarrow_n \lambda. :t. e_1 = \lambda. :t. e_2} \quad (\text{down-fun})$$

$$\frac{\downarrow_n e_1 = e_3 \quad \downarrow_n e_2 = e_4}{\downarrow_n e_1 e_2 = e_3 e_4} \quad (\text{down-app})$$

2.6 Discharge

$$\boxed{\Gamma\{n\} = \Gamma} \quad \boxed{\text{dis}}$$

$$\Gamma, t\{0\} = \Gamma \quad (\text{dis-z})$$

$$\frac{\Gamma_1\{n\} = \Gamma_2}{\Gamma_1, t\{n+1\} = \Gamma_2, t} \quad (\text{dis-s})$$

2.7 Independence

$$\boxed{n \notin \text{fv}(e)} \quad \boxed{\text{indep}}$$

$$n \notin \text{fv}(\langle \rangle) \quad (\text{indep-unit})$$

$$\frac{n_1 \neq n_2}{n_2 \notin \text{fv}(n_1)} \quad (\text{indep-var})$$

$$\frac{n+1 \notin \text{fv}(e)}{n \notin \text{fv}(\lambda. :t. e)} \quad (\text{indep-fun})$$

$$\frac{n \notin \text{fv}(e_1) \quad n \notin \text{fv}(e_2)}{n \notin \text{fv}(e_1 e_2)} \quad (\text{indep-app})$$

2.8 Substitution

$$\boxed{e\{e/n\} = e} \quad \boxed{\text{sub}}$$

$$\langle \rangle\{e_2/n\} = \langle \rangle \quad (\text{sub-unit})$$

$$n\{e_2/n\} = e_2 \quad (\text{sub-var1})$$

$$\frac{n_1 \neq n_2}{n_1\{e_2/n_2\} = n_1} \quad (\text{sub-var2})$$

$$\frac{\uparrow_0 \mathbf{e}_2 = \mathbf{e}_3 \quad \mathbf{e}_1 \{ \mathbf{e}_3 / \mathbf{n} + 1 \} = \mathbf{e}_4}{(\lambda. : \mathbf{t}. \mathbf{e}_1) \{ \mathbf{e}_2 / \mathbf{n} \} = \lambda. : \mathbf{t}. \mathbf{e}_4} \quad (\text{sub-fun})$$

$$\frac{\mathbf{e}_4 \{ \mathbf{e}_2 / \mathbf{n} \} = \mathbf{e}_6 \quad \mathbf{e}_5 \{ \mathbf{e}_2 / \mathbf{n} \} = \mathbf{e}_7}{(\mathbf{e}_4 \ \mathbf{e}_5) \{ \mathbf{e}_2 / \mathbf{n} \} = \mathbf{e}_5 \ \mathbf{e}_7} \quad (\text{sub-app})$$

3 Semantics

3.1 Typing

$\boxed{\Gamma \vdash e : t}$	$\boxed{\text{ty}}$
$\Gamma \vdash \langle \rangle : \langle \rangle$	(ty-unit)
$\frac{\Gamma [n] = t}{\Gamma \vdash n : t}$	(ty-var)
$\frac{\Gamma, t_1 \vdash e : t_2}{\Gamma \vdash \lambda. : t_1. e : t_1 \rightarrow t_2}$	(ty-fun)
$\frac{\Gamma \vdash e_1 : t_1 \rightarrow t_2 \quad \Gamma \vdash e_2 : t_1}{\Gamma \vdash e_1 e_2 : t_2}$	(ty-app)

3.2 Value

$\boxed{\text{val } e}$	$\boxed{\text{val}}$
$\text{val } \langle \rangle$	(val-unit)
$\text{val } \lambda. : t. e$	(val-fun)

3.3 Evaluation

$\boxed{e \rightarrow e}$	$\boxed{\text{ev}}$
$\frac{e_1 \rightarrow e_3}{e_1 e_2 \rightarrow e_3 e_2}$	(ev-app1)
$\frac{\text{val } e_1 \quad e_2 \rightarrow e_3}{e_1 e_2 \rightarrow e_1 e_3}$	(ev-app2)
$\frac{\text{val } e_2 \quad \uparrow_0 e_2 = e_3 \quad e_1 \{e_3/0\} = e_4 \quad \downarrow_0 e_4 = e_5}{(\lambda. : t. e_1) e_2 \rightarrow e_5}$	(ev-app3)

4 Lemmas

4.1 Arithmetic

$\boxed{n \diamond n}$	$\boxed{\text{ltge}}$
$\frac{n_1 < n_2}{n_1 \diamond n_2}$	(ltge-lt)
$\frac{n_1 \geq n_2}{n_1 \diamond n_2}$	(ltge-ge)
$\boxed{\frac{n_1 < n_2}{n_1 \diamond n_2}}$	$\boxed{\text{ltgex}}$
$\boxed{\frac{n_1 < n_2}{n_1 < n_2 + 1}}$	$\boxed{\text{ltz}}$
$0 < n + 1$ \vdots $\frac{0 < n + 1 + 1}{0 < n + 1 + 1}$	lt-z
$\frac{0 < n + 1 + 1}{0 < n + 1 + 1}$ lt-z	(lts-z)
$\frac{\frac{n_1 + 1 < n_2 + 1}{n_1 < n_2} \downarrow \text{lt-s}}{n_1 < n_2 + 1} \text{ lts}$ $\frac{n_1 + 1 < n_2 + 1}{n_1 + 1 < n_2 + 1 + 1} \text{ lt-s}$	(lts-s)
$\boxed{\frac{n_1 \geq n_2}{n_1 + 1 \geq n_2}}$	$\boxed{\text{ges}}$
$\frac{0 \geq 0}{0 \geq 0} \text{ ge-zz}$ \vdots $\frac{0 + 1 \geq 0}{0 + 1 \geq 0} \text{ ge-sz}$	(ges-zz)
$n + 1 \geq 0$ \vdots $\frac{n + 1 + 1 \geq 0}{n + 1 + 1 \geq 0} \text{ ge-sz}$	(ges-sz)
$\frac{\frac{n_1 + 1 \geq n_2 + 1}{n_1 \geq n_2} \downarrow \text{ge-ss}}{n_1 + 1 \geq n_2} \text{ ges}$ $\frac{n_1 + 1 \geq n_2}{n_1 + 1 + 1 \geq n_2 + 1} \text{ ge-ss}$	(ges-ss)
$\boxed{\frac{n_1 \geq n_2 \quad n_3 < n_2}{n_1 \geq n_3}}$	$\boxed{\text{geltge}}$
$n_1 + 1 \geq n_2 + 1$ $\frac{n_1 + 1 \geq n_2 + 1}{n_1 \geq n_2} \downarrow \text{ge-ss}$ $0 < n_2 + 1$ \vdots $\frac{n_1 + 1 \geq 0}{n_1 + 1 \geq 0} \text{ ge-sz}$	(geltge-z)

$$\frac{\frac{n_1 + 1 \geq n_2 + 1}{n_1 \geq n_2} \downarrow \text{ge-ss} \quad \frac{n_3 + 1 < n_2 + 1}{n_3 < n_2} \downarrow \text{lt-s}}{\frac{n_1 \geq n_3}{n_1 + 1 \geq n_3 + 1} \text{ge-ss}} \text{geltge} \quad (\text{geltge-s})$$

$$\boxed{\frac{n_1 < n_2 \quad n_3 \geq n_2}{n_1 < n_3}} \quad (\text{ltgelt})$$

$$\frac{0 < n_1 + 1 \quad \begin{array}{c} n_2 + 1 \geq n_1 + 1 \\ \vdots \end{array}}{0 < n_2 + 1} \text{lt-z} \quad (\text{ltgelt-z})$$

$$\frac{\frac{n_1 + 1 < n_2 + 1}{n_1 < n_2} \downarrow \text{lt-s} \quad \frac{n_3 + 1 \geq n_2 + 1}{n_3 \geq n_2} \downarrow \text{ge-ss}}{\frac{n_1 < n_3}{n_1 + 1 < n_3 + 1} \text{lt-s}} \text{ltgelt} \quad (\text{ltgelt-s})$$

$$\boxed{\frac{n_1 < n_2}{n_1 \neq n_2}} \quad (\text{ltne})$$

$$\frac{0 < n + 1}{0 \neq n + 1} \text{ne-zs} \quad (\text{ltne-z})$$

$$\frac{\frac{n_1 + 1 < n_2 + 1}{n_1 < n_2} \downarrow \text{lt-s} \quad \text{ltne}}{\frac{n_1 \neq n_2}{n_1 + 1 \neq n_2 + 1} \text{ne-ss}} \quad (\text{ltne-s})$$

$$\boxed{\frac{n_1 \neq n_2 \quad n_1 \geq n_2}{n_2 < n_1}} \quad (\text{nege})$$

$$\frac{n + 1 \neq 0 \quad \begin{array}{c} n + 1 \geq 0 \\ \vdots \end{array}}{0 < n + 1} \text{lt-z} \quad (\text{nege-zs})$$

$$\frac{\frac{n_1 + 1 \neq n_2 + 1}{n_1 \neq n_2} \downarrow \text{ne-ss} \quad \frac{n_1 + 1 \geq n_2 + 1}{n_1 \geq n_2} \downarrow \text{ge-ss}}{\frac{n_2 < n_1}{n_2 + 1 < n_1 + 1} \text{lt-s}} \text{nege} \quad (\text{nege-ss})$$

$$\boxed{\frac{n_1 < n_2}{n_2 - 1 = n_3}} \quad (\text{ltpred})$$

$$\frac{0 < n + 1}{n + 1 - 1 = n} \text{pred-s} \quad (\text{ltpred-z})$$

$$\begin{array}{c}
n_1 + 1 < n_2 + 1 \\
\vdots \\
\frac{n_1 + 1 < n_2 + 1}{n_1 < n_2} \downarrow \text{lt-s} \\
\hline
n_2 + 1 - 1 = n_2 \quad \text{pred-s}
\end{array} \quad (\text{ltpred-s})$$

4.2 Independence

$$\boxed{\frac{\uparrow_n e_1 = e_2}{n \notin \text{fv}(e_2)}} \quad \boxed{\text{upindep}}$$

$$\frac{\uparrow_n \langle \rangle = \langle \rangle}{n \notin \text{fv}(\langle \rangle)} \text{indep-unit} \quad (\text{upindep-unit})$$

$$\frac{\frac{\uparrow_{n_2} n_1 = n_1}{n_1 < n_2} \downarrow \text{up-var1}}{n_1 \neq n_2} \text{ltne}}{n_2 \notin \text{fv}(n_1)} \text{indep-var} \quad (\text{upindep-var1})$$

$$\frac{\frac{\uparrow_n \lambda.:t.e_1 = \lambda.:t.e_2}{\uparrow_{n+1} e_1 = e_2} \downarrow \text{up-fun}}{\frac{n+1 \notin \text{fv}(e_2)}{n \notin \text{fv}(\lambda.:t.e_2)} \text{upindep}} \text{indep-fun} \quad (\text{upindep-fun})$$

$$\frac{\frac{\uparrow_n e_3 \ e_1 = e_4 \ e_2}{\uparrow_n e_3 = e_4} \downarrow \text{up-app} \quad \frac{\uparrow_n e_3 \ e_1 = e_4 \ e_2}{\uparrow_n e_1 = e_2} \downarrow \text{up-app}}{\frac{n \notin \text{fv}(e_4)}{n \notin \text{fv}(e_4 \ e_2)} \text{upindep} \quad \frac{n \notin \text{fv}(e_2)}{n \notin \text{fv}(e_2)} \text{upindep}} \text{indep-app} \quad (\text{upindep-app})$$

$$\boxed{\frac{\uparrow_{n_1} e_1 = e_2 \quad n_2 \notin \text{fv}(e_1) \quad n_2 \geq n_1}{n_2 + 1 \notin \text{fv}(e_2)}} \quad \boxed{\text{up2indep}}$$

$$\frac{\uparrow_{n_1} \langle \rangle = \langle \rangle \quad n_2 \notin \text{fv}(\langle \rangle) \quad n_2 \geq n_1}{n_2 + 1 \notin \text{fv}(\langle \rangle)} \text{indep-unit} \quad (\text{up2indep-unit})$$

$$\frac{\frac{\uparrow_{n_3} n_1 = n_1}{n_1 < n_3} \downarrow \text{up-var1} \quad n_2 \geq n_3 \quad \text{ltgelt}}{\frac{\frac{n_1 < n_2}{n_1 < n_2 + 1} \text{Its}}{n_1 \neq n_2 + 1} \text{ltne}} \text{indep-var} \quad (\text{up2indep-var1})$$

$$\frac{\hat{n}_2 \ n_1 = n_1 + 1 \quad \frac{\hat{n}_2 \ n_1 = n_1 + 1}{n_1 \geq n_2} \downarrow \text{up-var2} \quad n_3 \geq n_2}{\dots} \quad \text{(up2indep-var2)}$$

$$\frac{\frac{\hat{n}_1 \ \lambda.:t.e_1 = \lambda.:t.e_2}{\hat{n}_1 + 1 \ e_1 = e_2} \downarrow \text{up-fun}}{\dots} \quad \text{(up2indep-fun)}$$

$$\frac{\frac{\hat{n}_1 \ e_3 \ e_1 = e_4 \ e_2}{\hat{n}_1 \ e_3 = e_4} \downarrow \text{up-app} \quad \frac{\hat{n}_1 \ e_3 \ e_1 = e_4 \ e_2}{\hat{n}_1 \ e_1 = e_2} \downarrow \text{up-app}}{\dots} \quad \text{(up2indep-app)}$$

$$\boxed{\frac{e_1 \{e_2/n\} = e_3 \quad n \notin \text{fv}(e_2)}{n \notin \text{fv}(e_3)}} \quad \text{subindep}$$

$$\frac{\langle e/n \rangle = \langle e \rangle \quad n \notin \text{fv}(e)}{\dots} \quad \text{(subindep-unit)}$$

$$\frac{n\{e/n\} = e}{\dots} \quad \text{(subindep-var1)}$$

$$\frac{\frac{n_1 \{e/n_2\} = n_1}{n_1 \neq n_2} \downarrow \text{sub-var2}}{n_2 \notin \text{fv}(n_1)} \quad \text{(subindep-var2)}$$

$$\begin{array}{c}
\bullet \frac{(\lambda_{-}:\mathbf{t}.\mathbf{e}_1)\{\mathbf{e}_4/0\} = \lambda_{-}:\mathbf{t}.\mathbf{e}_3}{\uparrow_0 \mathbf{e}_4 = \mathbf{e}_2} \downarrow_{\text{sub-fun}} \\
\bullet 0 \notin \text{fv}(\mathbf{e}_4) \\
\bullet \frac{\overline{0 \geq 0}}{\text{ge-zz}} \\
\frac{(\lambda_{-}:\mathbf{t}.\mathbf{e}_1)\{\mathbf{e}_4/0\} = \lambda_{-}:\mathbf{t}.\mathbf{e}_3}{\mathbf{e}_1\{\mathbf{e}_2/0+1\} = \mathbf{e}_3} \downarrow_{\text{sub-fun}} \frac{\overline{0+1 \notin \text{fv}(\mathbf{e}_2)}}{0+1 \notin \text{fv}(\mathbf{e}_2)} \text{up2indep} \\
\frac{\overline{0+1 \notin \text{fv}(\mathbf{e}_3)}}{0 \notin \text{fv}(\lambda_{-}:\mathbf{t}.\mathbf{e}_3)} \text{indep-fun} \quad \text{subindep}
\end{array} \quad (\text{subindep-fun1})$$

$$\begin{array}{c}
\bullet \frac{(\lambda_{-}:\mathbf{t}.\mathbf{e}_1)\{\mathbf{e}_4/n+1\} = \lambda_{-}:\mathbf{t}.\mathbf{e}_3}{\uparrow_0 \mathbf{e}_4 = \mathbf{e}_2} \downarrow_{\text{sub-fun}} \\
\bullet n+1 \notin \text{fv}(\mathbf{e}_4) \\
\bullet \frac{\overline{n+1 \geq 0}}{\text{ge-sz}} \\
\frac{(\lambda_{-}:\mathbf{t}.\mathbf{e}_1)\{\mathbf{e}_4/n+1\} = \lambda_{-}:\mathbf{t}.\mathbf{e}_3}{\mathbf{e}_1\{\mathbf{e}_2/n+1+1\} = \mathbf{e}_3} \downarrow_{\text{sub-fun}} \frac{\overline{n+1+1 \notin \text{fv}(\mathbf{e}_2)}}{n+1+1 \notin \text{fv}(\mathbf{e}_2)} \text{up2indep} \\
\frac{\overline{n+1+1 \notin \text{fv}(\mathbf{e}_3)}}{n+1 \notin \text{fv}(\lambda_{-}:\mathbf{t}.\mathbf{e}_3)} \text{indep-fun} \quad \text{subindep}
\end{array} \quad (\text{subindep-fun2})$$

$$\begin{array}{c}
\frac{(\mathbf{e}_4 \mathbf{e}_1)\{\mathbf{e}_2/n\} = \mathbf{e}_1 \mathbf{e}_3}{\mathbf{e}_4\{\mathbf{e}_2/n\} = \mathbf{e}_1} \downarrow_{\text{sub-app}} \frac{\overline{n \notin \text{fv}(\mathbf{e}_2)}}{n \notin \text{fv}(\mathbf{e}_2)} \text{subindep} \quad \frac{(\mathbf{e}_4 \mathbf{e}_1)\{\mathbf{e}_2/n\} = \mathbf{e}_1 \mathbf{e}_3}{\mathbf{e}_1\{\mathbf{e}_2/n\} = \mathbf{e}_3} \downarrow_{\text{sub-app}} \frac{\overline{n \notin \text{fv}(\mathbf{e}_2)}}{n \notin \text{fv}(\mathbf{e}_2)} \text{subindep} \\
\frac{\overline{n \notin \text{fv}(\mathbf{e}_1)}}{n \notin \text{fv}(\mathbf{e}_1)} \quad \frac{\overline{n \notin \text{fv}(\mathbf{e}_3)}}{n \notin \text{fv}(\mathbf{e}_3)} \text{indep-app} \\
\frac{\overline{n \notin \text{fv}(\mathbf{e}_1 \mathbf{e}_3)}}{n \notin \text{fv}(\mathbf{e}_1 \mathbf{e}_3)} \text{indep-app}
\end{array} \quad (\text{subindep-app})$$

$$\boxed{\frac{\overline{n \notin \text{fv}(\mathbf{e}_1)}}{\downarrow_n \mathbf{e}_1 = \mathbf{e}_2}}$$

indepdown

$$\begin{array}{c}
n \notin \text{fv}(\langle \rangle) \\
\vdots \\
\downarrow_n \langle \rangle = \langle \rangle \quad \text{down-unit}
\end{array}$$

(indepdown-unit)

$$\begin{array}{c}
\frac{\overline{n_2 \notin \text{fv}(\mathbf{n}_1)}}{n_2 \notin \text{fv}(\mathbf{n}_1)} \frac{\overline{n_1 \neq n_2}}{n_1 \neq n_2} \downarrow_{\text{indep-var}} \\
\vdots \\
\frac{\overline{n_1 \diamond n_2}}{n_1 < n_2} \text{ltgex} \\
\downarrow_{n_2} n_1 = n_1 \quad \text{down-var1}
\end{array}$$

(indepdown-var1)

$$\begin{array}{c}
\frac{\overline{n_1 \notin \text{fv}(\mathbf{n}_2)}}{n_1 \notin \text{fv}(\mathbf{n}_2)} \downarrow_{\text{indep-var}} \frac{\overline{n_2 \diamond n_1}}{n_2 \geq n_1} \text{ltgex} \\
\downarrow_{n_2} n_1 = n_1 \quad \text{down-var1} \quad \frac{\overline{n_2 \geq n_1}}{n_2 \geq n_1} \text{nege} \\
\frac{\overline{n_2 \diamond n_1}}{n_2 \geq n_1} \text{ltgex} \quad \frac{\overline{n_1 < n_2}}{n_2 - 1 = n_3} \text{ltpred} \\
\downarrow_{n_1} n_2 = n_3 \quad \text{down-var2}
\end{array}$$

(indepdown-var2)

$$\begin{array}{c}
\frac{\overline{n \notin \text{fv}(\lambda_{-}:\mathbf{t}.\mathbf{e}_1)}}{n+1 \notin \text{fv}(\mathbf{e}_1)} \downarrow_{\text{indep-fun}} \\
\bullet \frac{\overline{\downarrow_{n+1} \mathbf{e}_1 = \mathbf{e}_2}}{\downarrow_n \lambda_{-}:\mathbf{t}.\mathbf{e}_1 = \lambda_{-}:\mathbf{t}.\mathbf{e}_2} \text{down-fun}
\end{array}$$

(indepdown-fun)

$$\begin{array}{c}
\frac{n \notin \text{fv}(e_3 \ e_1)}{n \notin \text{fv}(e_3)} \downarrow \text{indep-app} \\
\bullet \frac{\downarrow_n e_3 = e_4}{\text{indepdown}} \\
\\
\frac{n \notin \text{fv}(e_3 \ e_1)}{n \notin \text{fv}(e_1)} \downarrow \text{indep-app} \\
\bullet \frac{\downarrow_n e_1 = e_2}{\text{indepdown}} \\
\downarrow_n e_3 \ e_1 = e_4 \ e_2 \quad \text{down-app}
\end{array}$$

(indepdown-app)

4.3 Shifting

$$\boxed{\frac{\uparrow_{n_1} e_1 = e_2 \quad \uparrow_{n_2} e_2 = e_3 \quad \uparrow_{n_2} e_1 = e_4 \quad n_2 < n_1}{\uparrow_{n_1+1} e_4 = e_3}}$$

upup

- $\uparrow_{n_1} \langle \rangle = \langle \rangle$
- $\uparrow_{n_2} \langle \rangle = \langle \rangle$
- $\uparrow_{n_2} \langle \rangle = \langle \rangle$
- $n_2 < n_1$
- \vdots

$$\frac{}{\uparrow_{n_1+1} \langle \rangle = \langle \rangle} \text{up-unit}$$

(upup-unit)

- $\uparrow_{n_3} n_1 = n_1$
- $\frac{\uparrow_{n_3} n_1 = n_1}{n_1 < n_3} \downarrow \text{up-var1}$
- $\uparrow_{n_3} n_1 = n_1$
- $\frac{\uparrow_{n_3} n_1 = n_1}{n_1 < n_3} \downarrow \text{up-var1}$
- $n_3 < n_2$
- \vdots

$$\frac{\frac{\uparrow_{n_2} n_1 = n_1}{n_1 < n_2} \downarrow \text{up-var1}}{n_1 < n_2 + 1} \text{lts} \\
\frac{}{\uparrow_{n_2+1} n_1 = n_1} \text{up-var1}$$

(upup-var1)

- $\uparrow_{n_3} n_1 = n_1 + 1$
- $\frac{\uparrow_{n_3} n_1 = n_1 + 1}{n_1 \geq n_3} \downarrow_{\text{up-var2}}$
- $\uparrow_{n_3} n_1 = n_1 + 1$
- $\frac{\uparrow_{n_3} n_1 = n_1 + 1}{n_1 \geq n_3} \downarrow_{\text{up-var2}}$
- $n_3 < n_2$
- \vdots
- $\frac{\uparrow_{n_2} n_1 = n_1}{n_1 < n_2} \downarrow_{\text{up-var1}}$
- $\frac{n_1 + 1 < n_2 + 1}{\text{lt-s}}$
- $\frac{\uparrow_{n_2+1} n_1 + 1 = n_1 + 1}{\text{up-var1}}$

(upup-var2)

- $\uparrow_{n_2} n_1 + 1 = n_1 + 1 + 1$
- $\frac{\uparrow_{n_2} n_1 + 1 = n_1 + 1 + 1}{n_1 + 1 \geq n_2} \downarrow_{\text{up-var2}}$
- $\uparrow_{n_2} n_1 = n_1 + 1$
- $\frac{\uparrow_{n_2} n_1 = n_1 + 1}{n_1 \geq n_2} \downarrow_{\text{up-var2}}$
- $n_2 < n_3$
- $\frac{\uparrow_{n_2} n_1 + 1 = n_1 + 1 + 1}{n_1 + 1 \geq n_2} \downarrow_{\text{up-var2}}$
- $\frac{\uparrow_{n_2} n_1 = n_1 + 1}{n_1 \geq n_2} \downarrow_{\text{up-var2}}$
- \vdots
- $\frac{\uparrow_{n_3} n_1 = n_1 + 1}{n_1 \geq n_3} \downarrow_{\text{up-var2}}$
- $\frac{n_1 + 1 \geq n_3 + 1}{\text{ge-ss}}$
- $\frac{\uparrow_{n_3+1} n_1 + 1 = n_1 + 1 + 1}{\text{up-var2}}$

(upup-var3)

- $\frac{\uparrow_{n_1} \lambda.:t.e_1 = \lambda.:t.e_2}{\uparrow_{n_1+1} e_1 = e_2} \downarrow_{\text{up-fun}}$
- $\frac{\uparrow_{n_2} \lambda.:t.e_2 = \lambda.:t.e_3}{\uparrow_{n_2+1} e_2 = e_3} \downarrow_{\text{up-fun}}$
- $\frac{\uparrow_{n_2} \lambda.:t.e_1 = \lambda.:t.e_4}{\uparrow_{n_2+1} e_1 = e_4} \downarrow_{\text{up-fun}}$
- $\frac{n_2 < n_1}{n_2 + 1 < n_1 + 1} \text{lt-s}$
- $\frac{\uparrow_{n_1+1+1} e_4 = e_3}{\uparrow_{n_1+1} \lambda.:t.e_4 = \lambda.:t.e_3} \text{upup}$

(upup-fun)

$$\begin{array}{c}
\bullet \frac{\uparrow_{n_1} e_5 \ e_1 = e_6 \ e_2}{\uparrow_{n_1} e_5 = e_6} \downarrow\text{up-app} \quad \bullet \frac{\uparrow_{n_1} e_5 \ e_1 = e_6 \ e_2}{\uparrow_{n_1} e_1 = e_2} \downarrow\text{up-app} \\
\bullet \frac{\uparrow_{n_2} e_6 \ e_2 = e_7 \ e_3}{\uparrow_{n_2} e_6 = e_7} \downarrow\text{up-app} \quad \bullet \frac{\uparrow_{n_2} e_6 \ e_2 = e_7 \ e_3}{\uparrow_{n_2} e_2 = e_3} \downarrow\text{up-app} \\
\bullet \frac{\uparrow_{n_2} e_5 \ e_1 = e_8 \ e_4}{\uparrow_{n_2} e_5 = e_8} \downarrow\text{up-app} \quad \bullet \frac{\uparrow_{n_2} e_5 \ e_1 = e_8 \ e_4}{\uparrow_{n_2} e_1 = e_4} \downarrow\text{up-app} \\
\bullet \frac{n_2 < n_1}{\uparrow_{n_1+1} e_8 = e_7} \text{upup} \quad \bullet \frac{n_2 < n_1}{\uparrow_{n_1+1} e_4 = e_3} \text{upup} \\
\frac{\uparrow_{n_1+1} e_8 \ e_4 = e_7 \ e_3}{\uparrow_{n_1+1} e_8 \ e_4 = e_7 \ e_3} \text{up-app} \quad \text{(upup-app)}
\end{array}$$

$$\boxed{\frac{\uparrow_n e_1 = e_2}{\downarrow_n e_2 = e_1}}$$

updown

$$\frac{\uparrow_n \langle \rangle = \langle \rangle}{\downarrow_n \langle \rangle = \langle \rangle} \text{down-unit}$$

(updown-unit)

$$\frac{\frac{\uparrow_{n_2} n_1 = n_1}{n_1 < n_2} \downarrow\text{up-var1}}{\downarrow_{n_2} n_1 = n_1} \text{down-var1}$$

(updown-var1)

$$\frac{\frac{\uparrow_{n_2} n_1 = n_1 + 1}{n_1 \geq n_2} \downarrow\text{up-var2}}{\frac{n_1 + 1 \geq n_2}{\downarrow_{n_2} n_1 + 1 = n_1} \text{ges}} \frac{\frac{n_1 + 1 - 1 = n_1}{\text{pred-s}}}{\text{down-var2}}$$

(updown-var2)

$$\frac{\frac{\uparrow_n \lambda. :t.e_1 = \lambda. :t.e_2}{\uparrow_{n+1} e_1 = e_2} \downarrow\text{up-fun}}{\frac{\uparrow_{n+1} e_1 = e_2}{\downarrow_{n+1} e_2 = e_1} \text{updown}} \downarrow\text{down-fun}$$

(updown-fun)

$$\frac{\frac{\uparrow_n e_3 \ e_1 = e_4 \ e_2}{\uparrow_n e_3 = e_4} \downarrow\text{up-app} \quad \frac{\uparrow_n e_3 \ e_1 = e_4 \ e_2}{\uparrow_n e_1 = e_2} \downarrow\text{up-app}}{\frac{\uparrow_n e_3 = e_4}{\downarrow_n e_4 = e_3} \text{updown} \quad \frac{\uparrow_n e_1 = e_2}{\downarrow_n e_2 = e_1} \text{updown}} \downarrow_n e_4 \ e_2 = e_3 \ e_1 \ \text{down-app}$$

(updown-app)

4.4 Totality

$$\boxed{\uparrow_n e_1 = e_2}$$

upx

$$\boxed{e_1 \{e_2/n\} = e_3}$$

subx

$$\boxed{\frac{\Gamma_1 [n_1] = t_1 \quad \uparrow_{n_2}^t \Gamma_1 = \Gamma_2 \quad n_1 < n_2}{\Gamma_2 [n_1] = t_1}}$$

uplt

- $\Gamma_1, \mathbf{t}_1 [0] = \mathbf{t}_1$
- $\uparrow_{n+1}^{t_2} \Gamma_1, \mathbf{t}_1 = \Gamma_2, \mathbf{t}_1$
- $\frac{\uparrow_{n+1}^{t_2} \Gamma_1, \mathbf{t}_1 = \Gamma_2, \mathbf{t}_1}{\uparrow_n^{t_2} \Gamma_1 = \Gamma_2} \downarrow_{\text{gup-s}}$
- $0 < n + 1$
- ⋮
- $\frac{\Gamma_2, \mathbf{t}_1 [0] = \mathbf{t}_1}{\text{map-z}}$

(uplt-z)

$$\frac{\frac{\Gamma_1, \mathbf{t}_3 [n_1 + 1] = \mathbf{t}_1}{\Gamma_1 [n_1] = \mathbf{t}_1} \downarrow_{\text{map-s}} \frac{\uparrow_{n_2+1}^{t_2} \Gamma_1, \mathbf{t}_3 = \Gamma_2, \mathbf{t}_3}{\uparrow_{n_2}^{t_2} \Gamma_1 = \Gamma_2} \downarrow_{\text{gup-s}} \frac{n_1 + 1 < n_2 + 1}{n_1 < n_2} \downarrow_{\text{lt-s}}}{\frac{\Gamma_2 [n_1] = \mathbf{t}_1}{\Gamma_2, \mathbf{t}_3 [n_1 + 1] = \mathbf{t}_1} \text{map-s}} \text{uplt}$$

(uplt-s)

$$\boxed{\frac{\Gamma_1 [n_1] = \mathbf{t}_1 \quad \uparrow_{n_2}^{t_2} \Gamma_1 = \Gamma_2 \quad n_1 \geq n_2}{\Gamma_2 [n_1 + 1] = \mathbf{t}_1}} \quad \boxed{\text{upge}}$$

$$\frac{\Gamma_1, \mathbf{t}_1 [0] = \mathbf{t}_1 \quad \uparrow_0^{t_2} \Gamma_1, \mathbf{t}_1 = \Gamma_1, \mathbf{t}_1, \mathbf{t}_2 \quad \overline{0 \geq 0}^{\text{ge-zz}}}{\frac{\Gamma_1, \mathbf{t}_1 [0] = \mathbf{t}_1}{\Gamma_1, \mathbf{t}_1, \mathbf{t}_2 [0 + 1] = \mathbf{t}_1} \text{map-s}} \text{map-z}$$

(upge-zz)

$$\frac{\uparrow_0^{t_2} \Gamma = \Gamma, \mathbf{t}_2 \quad n + 1 \geq 0}{\frac{\Gamma [n + 1] = \mathbf{t}_1}{\Gamma, \mathbf{t}_2 [n + 1 + 1] = \mathbf{t}_1} \text{map-s}} \text{map-s}$$

(upge-sz)

$$\frac{\frac{\Gamma_1, \mathbf{t}_3 [n_1 + 1] = \mathbf{t}_1}{\Gamma_1 [n_1] = \mathbf{t}_1} \downarrow_{\text{map-s}} \frac{\uparrow_{n_2+1}^{t_2} \Gamma_1, \mathbf{t}_3 = \Gamma_2, \mathbf{t}_3}{\uparrow_{n_2}^{t_2} \Gamma_1 = \Gamma_2} \downarrow_{\text{gup-s}} \frac{n_1 + 1 \geq n_2 + 1}{n_1 \geq n_2} \downarrow_{\text{ge-ss}}}{\frac{\Gamma_2 [n_1 + 1] = \mathbf{t}_1}{\Gamma_2, \mathbf{t}_3 [n_1 + 1 + 1] = \mathbf{t}_1} \text{map-s}} \text{upge}$$

(upge-ss)

4.5 Preservation under up-shifting

$$\boxed{\frac{\Gamma_1 \vdash \mathbf{e}_1 : \mathbf{t} \quad \uparrow_n^{t_1} \Gamma_1 = \Gamma_2 \quad \uparrow_n \mathbf{e}_1 = \mathbf{e}_2}{\Gamma_2 \vdash \mathbf{e}_2 : \mathbf{t}}} \quad \boxed{\text{psup}}$$

$$\frac{\Gamma_1 \vdash \langle \rangle : \langle \rangle \quad \uparrow_n^t \Gamma_1 = \Gamma_2 \quad \uparrow_n \langle \rangle = \langle \rangle}{\Gamma_2 \vdash \langle \rangle : \langle \rangle} \text{ty-unit}$$

(psup-unit)

$$\frac{\frac{\Gamma_1 \vdash \mathbf{n}_1 : \mathbf{t}_1}{\Gamma_1 [\mathbf{n}_1] = \mathbf{t}_1} \downarrow \text{ty-var} \quad \frac{\overline{\mathbf{n}_1 \diamond \mathbf{n}_2} \text{ ltgex} \quad \dots \quad \uparrow_{\mathbf{n}_2}^{\mathbf{t}_2} \Gamma_1 = \Gamma_2 \quad \frac{\uparrow_{\mathbf{n}_2} \mathbf{n}_1 = \mathbf{n}_1}{\mathbf{n}_1 < \mathbf{n}_2} \downarrow \text{up-var1}}{\Gamma_2 [\mathbf{n}_1] = \mathbf{t}_1} \text{ ty-var} \quad \text{uplt}}{\Gamma_2 \vdash \mathbf{n}_1 : \mathbf{t}_1} \text{ ty-var} \quad (\text{psup-var1})$$

$$\frac{\frac{\Gamma_1 \vdash \mathbf{n}_1 : \mathbf{t}_1}{\Gamma_1 [\mathbf{n}_1] = \mathbf{t}_1} \downarrow \text{ty-var} \quad \frac{\overline{\mathbf{n}_1 \diamond \mathbf{n}_2} \text{ ltgex} \quad \dots \quad \uparrow_{\mathbf{n}_2}^{\mathbf{t}_2} \Gamma_1 = \Gamma_2 \quad \frac{\uparrow_{\mathbf{n}_2} \mathbf{n}_1 = \mathbf{n}_1 + 1}{\mathbf{n}_1 \geq \mathbf{n}_2} \downarrow \text{up-var2}}{\Gamma_2 [\mathbf{n}_1 + 1] = \mathbf{t}_1} \text{ ty-var} \quad \text{upge}}{\Gamma_2 \vdash \mathbf{n}_1 + 1 : \mathbf{t}_1} \text{ ty-var} \quad (\text{psup-var2})$$

- $\frac{\Gamma_1 \vdash \lambda. : \mathbf{t}_1. \mathbf{e}_1 : \mathbf{t}_1 \rightarrow \mathbf{t}_2}{\Gamma_1, \mathbf{t}_1 \vdash \mathbf{e}_1 : \mathbf{t}_2} \downarrow \text{ty-fun}$
- $\frac{\uparrow_{\mathbf{n}}^{\mathbf{t}_3} \Gamma_1 = \Gamma_2}{\uparrow_{\mathbf{n}+1}^{\mathbf{t}_3} \Gamma_1, \mathbf{t}_1 = \Gamma_2, \mathbf{t}_1} \text{ gup-s}$
- $\frac{\frac{\uparrow_{\mathbf{n}} \lambda. : \mathbf{t}_1. \mathbf{e}_1 = \lambda. : \mathbf{t}_1. \mathbf{e}_2}{\uparrow_{\mathbf{n}+1} \mathbf{e}_1 = \mathbf{e}_2} \downarrow \text{up-fun}}{\frac{\Gamma_2, \mathbf{t}_1 \vdash \mathbf{e}_2 : \mathbf{t}_2}{\Gamma_2 \vdash \lambda. : \mathbf{t}_1. \mathbf{e}_2 : \mathbf{t}_1 \rightarrow \mathbf{t}_2} \text{ ty-fun}} \text{ psup} \quad (\text{psup-fun})$

- $\frac{\Gamma_1 \vdash \mathbf{e}_3 \ \mathbf{e}_1 : \mathbf{t}_3}{\Gamma_1 \vdash \mathbf{e}_3 : \mathbf{t}_1 \rightarrow \mathbf{t}_3} \downarrow \text{ty-app} \quad \bullet \quad \frac{\Gamma_1 \vdash \mathbf{e}_3 \ \mathbf{e}_1 : \mathbf{t}_3}{\Gamma_1 \vdash \mathbf{e}_1 : \mathbf{t}_1} \downarrow \text{ty-app}$
- $\uparrow_{\mathbf{n}}^{\mathbf{t}_2} \Gamma_1 = \Gamma_2 \quad \bullet \quad \uparrow_{\mathbf{n}}^{\mathbf{t}_2} \Gamma_1 = \Gamma_2$
- $\frac{\frac{\uparrow_{\mathbf{n}} \mathbf{e}_3 \ \mathbf{e}_1 = \mathbf{e}_4 \ \mathbf{e}_2}{\uparrow_{\mathbf{n}} \mathbf{e}_3 = \mathbf{e}_4} \downarrow \text{up-app}}{\Gamma_2 \vdash \mathbf{e}_4 : \mathbf{t}_1 \rightarrow \mathbf{t}_3} \text{ psup} \quad \bullet \quad \frac{\frac{\uparrow_{\mathbf{n}} \mathbf{e}_3 \ \mathbf{e}_1 = \mathbf{e}_4 \ \mathbf{e}_2}{\uparrow_{\mathbf{n}} \mathbf{e}_1 = \mathbf{e}_2} \downarrow \text{up-app}}{\Gamma_2 \vdash \mathbf{e}_2 : \mathbf{t}_1} \text{ psup}}{\Gamma_2 \vdash \mathbf{e}_4 \ \mathbf{e}_2 : \mathbf{t}_3} \text{ ty-app} \quad (\text{psup-app})$

4.6 Preservation under substitution

$$\frac{\Gamma_1 \vdash \mathbf{e}_1 : \mathbf{t} \quad \Gamma_1 [\mathbf{n}] = \mathbf{t}_0 \quad \Gamma_2 \vdash \mathbf{e}_0 : \mathbf{t}_0 \quad \Gamma_1 \{\mathbf{n}\} = \Gamma_2 \quad \uparrow_{\mathbf{n}} \mathbf{e}_0 = \mathbf{e}_2 \quad \mathbf{e}_1 \{\mathbf{e}_2/\mathbf{n}\} = \mathbf{e}_3 \quad \downarrow_{\mathbf{n}} \mathbf{e}_3 = \mathbf{e}_4}{\Gamma_2 \vdash \mathbf{e}_4 : \mathbf{t}}$$

psub

- $\Gamma_1 \vdash \langle \rangle : \langle \rangle$
- $\Gamma_1 [n] = t$
- $\Gamma_2 \vdash e_1 : t$
- $\Gamma_1 \{n\} = \Gamma_2$
- $\uparrow_n e_1 = e_2$
- $\langle \rangle \{e_2/n\} = \langle \rangle$
- $\downarrow_n \langle \rangle = \langle \rangle$
- \vdots

$$\frac{}{\Gamma_2 \vdash \langle \rangle : \langle \rangle} \text{ty-unit} \quad (\text{pssub-unit})$$

- $\Gamma_1 \vdash n : t$
- $\frac{\Gamma_1 \vdash n : t}{\Gamma_1 [n] = t} \downarrow_{\text{ty-var}}$
- $\frac{\Gamma_1 \vdash n : t}{\Gamma_1 [n] = t} \downarrow_{\text{ty-var}}$
- $\Gamma_1 \{n\} = \Gamma_2$
- $\uparrow_n e_1 = e_2$
- $n\{e_2/n\} = e_2$
- $\downarrow_n e_2 = e_1$
- \vdots

$$\Gamma_2 \vdash e_1 : t \quad (\text{pssub-var1})$$

- $\Gamma [n_2] = t_2$
- $\Gamma \vdash e_1 : t_2$
- $\Gamma \{n_2\} = \Gamma$
- $\uparrow_{n_2} e_1 = e_2$
- $n_1 \{e_2/n_2\} = n_1$
- $\frac{n_1 \{e_2/n_2\} = n_1}{n_1 \neq n_2} \downarrow_{\text{sub-var2}}$
- $\downarrow_{n_2} n_1 = n_1$
- \vdots

$$\Gamma \vdash n_1 : t_1 \quad (\text{pssub-var2})$$

- $\frac{\Gamma_1 \vdash \lambda. : \mathbf{t}_1 . \mathbf{e}_1 : \mathbf{t}_1 \rightarrow \mathbf{t}_2}{\Gamma_1, \mathbf{t}_1 \vdash \mathbf{e}_1 : \mathbf{t}_2} \downarrow \text{ty-fun}$
- $\frac{\Gamma_1 \quad [\mathbf{n} + 1] = \mathbf{t}_3}{\Gamma_1, \mathbf{t}_1 \quad [\mathbf{n} + 1 + 1] = \mathbf{t}_3} \text{map-s}$
 - $\Gamma_2 \vdash \mathbf{e}_6 : \mathbf{t}_3$
 - $\frac{}{\uparrow_0^{\mathbf{t}_1} \Gamma_2 = \Gamma_2, \mathbf{t}_1} \text{gup-z}$
 - $\frac{}{\uparrow_0 \mathbf{e}_6 = \mathbf{e}_2} \text{upx}$
- $\frac{}{\Gamma_2, \mathbf{t}_1 \vdash \mathbf{e}_2 : \mathbf{t}_3} \text{psup}$
- $\frac{\Gamma_1 \{ \mathbf{n} + 1 \} = \Gamma_2}{\Gamma_1, \mathbf{t}_1 \{ \mathbf{n} + 1 + 1 \} = \Gamma_2, \mathbf{t}_1} \text{dis-s}$
 - $\uparrow_{\mathbf{n}+1} \mathbf{e}_6 = \mathbf{e}_7$
 - $\frac{(\lambda. : \mathbf{t}_1 . \mathbf{e}_1) \{ \mathbf{e}_7 / \mathbf{n} + 1 \} = \lambda. : \mathbf{t}_1 . \mathbf{e}_4}{\uparrow_0 \mathbf{e}_7 = \mathbf{e}_3} \downarrow \text{sub-fun}$
 - $\frac{}{\uparrow_0 \mathbf{e}_6 = \mathbf{e}_2} \text{upx}$
 - $\frac{}{0 < \mathbf{n} + 1} \text{lt-z}$
- $\frac{}{\uparrow_{\mathbf{n}+1+1} \mathbf{e}_2 = \mathbf{e}_3} \text{upup}$
- $\frac{(\lambda. : \mathbf{t}_1 . \mathbf{e}_1) \{ \mathbf{e}_7 / \mathbf{n} + 1 \} = \lambda. : \mathbf{t}_1 . \mathbf{e}_4}{\mathbf{e}_1 \{ \mathbf{e}_3 / \mathbf{n} + 1 + 1 \} = \mathbf{e}_4} \downarrow \text{sub-fun}$
- $\frac{\downarrow_{\mathbf{n}+1} \lambda. : \mathbf{t}_1 . \mathbf{e}_4 = \lambda. : \mathbf{t}_1 . \mathbf{e}_5}{\downarrow_{\mathbf{n}+1+1} \mathbf{e}_4 = \mathbf{e}_5} \downarrow \text{down-fun}$
- $\frac{}{\frac{\Gamma_2, \mathbf{t}_1 \vdash \mathbf{e}_5 : \mathbf{t}_2}{\Gamma_2 \vdash \lambda. : \mathbf{t}_1 . \mathbf{e}_5 : \mathbf{t}_1 \rightarrow \mathbf{t}_2} \text{ty-fun}} \text{pssub}$

(pssub-fun)

<ul style="list-style-type: none"> • $\frac{\Gamma_1 \vdash \mathbf{e}_6 \ \mathbf{e}_1 : \mathbf{t}_3}{\Gamma_1 \vdash \mathbf{e}_6 : \mathbf{t}_1 \rightarrow \mathbf{t}_3} \downarrow \text{ty-app}$ • $\Gamma_1 \quad [\mathbf{n}] = \mathbf{t}_2$ • $\Gamma_2 \vdash \mathbf{e}_2 : \mathbf{t}_2$ • $\Gamma_1 \{ \mathbf{n} \} = \Gamma_2$ • $\uparrow_{\mathbf{n}} \mathbf{e}_2 = \mathbf{e}_3$ • $\frac{(\mathbf{e}_6 \ \mathbf{e}_1) \{ \mathbf{e}_3 / \mathbf{n} \} = \mathbf{e}_1 \ \mathbf{e}_4}{\mathbf{e}_6 \{ \mathbf{e}_3 / \mathbf{n} \} = \mathbf{e}_1} \downarrow \text{sub-app}$ • $\frac{\downarrow_{\mathbf{n}} \mathbf{e}_1 \ \mathbf{e}_4 = \mathbf{e}_7 \ \mathbf{e}_5}{\downarrow_{\mathbf{n}} \mathbf{e}_1 = \mathbf{e}_7} \downarrow \text{down-app}$ • $\frac{}{\frac{\Gamma_2 \vdash \mathbf{e}_7 : \mathbf{t}_1 \rightarrow \mathbf{t}_3}{\Gamma_2 \vdash \mathbf{e}_7 \ \mathbf{e}_5 : \mathbf{t}_3} \text{pssub}} \text{ty-app}$ 	<ul style="list-style-type: none"> • $\frac{\Gamma_1 \vdash \mathbf{e}_6 \ \mathbf{e}_1 : \mathbf{t}_3}{\Gamma_1 \vdash \mathbf{e}_1 : \mathbf{t}_1} \downarrow \text{ty-app}$ • $\Gamma_1 \quad [\mathbf{n}] = \mathbf{t}_2$ • $\Gamma_2 \vdash \mathbf{e}_2 : \mathbf{t}_2$ • $\Gamma_1 \{ \mathbf{n} \} = \Gamma_2$ • $\uparrow_{\mathbf{n}} \mathbf{e}_2 = \mathbf{e}_3$ • $\frac{(\mathbf{e}_6 \ \mathbf{e}_1) \{ \mathbf{e}_3 / \mathbf{n} \} = \mathbf{e}_1 \ \mathbf{e}_4}{\mathbf{e}_1 \{ \mathbf{e}_3 / \mathbf{n} \} = \mathbf{e}_4} \downarrow \text{sub-app}$ • $\frac{\downarrow_{\mathbf{n}} \mathbf{e}_1 \ \mathbf{e}_4 = \mathbf{e}_7 \ \mathbf{e}_5}{\downarrow_{\mathbf{n}} \mathbf{e}_4 = \mathbf{e}_5} \downarrow \text{down-app}$ • $\frac{}{\frac{\Gamma_2 \vdash \mathbf{e}_5 : \mathbf{t}_1}{\Gamma_2 \vdash \mathbf{e}_5 \ \mathbf{t}_1} \text{pssub}} \text{ty-app}$
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(pssub-app)

5 Theorems

5.1 Preservation

$$\boxed{\frac{\Gamma \vdash e_1 : t \quad e_1 \longrightarrow e_2}{\Gamma \vdash e_2 : t}} \quad \boxed{\text{ps}}$$

$$\frac{\frac{\Gamma \vdash e_1 \ e_3 : t_2}{\Gamma \vdash e_1 : t_1 \rightarrow t_2} \downarrow \text{ty-app} \quad \frac{\frac{e_1 \ e_3 \longrightarrow e_2 \ e_3}{e_1 \longrightarrow e_2} \downarrow \text{ev-app1} \quad \text{ps}}{\Gamma \vdash e_2 : t_1 \rightarrow t_2} \quad \frac{\Gamma \vdash e_1 \ e_3 : t_2}{\Gamma \vdash e_3 : t_1} \downarrow \text{ty-app}}{\Gamma \vdash e_2 \ e_3 : t_2} \text{ty-app} \quad (\text{ps-app1})$$

$$\frac{\frac{\frac{\Gamma \vdash e_3 \ e_1 : t_2}{\Gamma \vdash e_3 : t_1 \rightarrow t_2} \downarrow \text{ty-app} \quad \frac{\frac{e_3 \ e_1 \longrightarrow e_3 \ e_2}{\text{val } e_3} \downarrow \text{ev-app2} \quad \dots}{\Gamma \vdash e_3 \ e_1 : t_2} \downarrow \text{ty-app} \quad \frac{\frac{e_3 \ e_1 \longrightarrow e_3 \ e_2}{e_1 \longrightarrow e_2} \downarrow \text{ev-app2} \quad \text{ps}}{\Gamma \vdash e_2 : t_1} \text{ty-app}}{\Gamma \vdash e_3 \ e_2 : t_2} \text{ty-app} \quad (\text{ps-app2})$$

$$\frac{\frac{\frac{(\lambda : t_1 . e_1) \ e_2 \longrightarrow e_5}{\text{val } e_2} \downarrow \text{ev-app3} \quad \dots}{\Gamma \vdash (\lambda : t_1 . e_1) \ e_2 : t_2} \downarrow \text{ty-app} \quad \frac{\Gamma \vdash \lambda : t_1 . e_1 : t_1 \rightarrow t_2}{\Gamma, t_1 \vdash e_1 : t_2} \downarrow \text{ty-fun}}{\bullet} \quad \frac{}{\Gamma, t_1 [0] = t_1} \text{map-z}$$

$$\bullet \quad \frac{\Gamma \vdash (\lambda : t_1 . e_1) \ e_2 : t_2}{\Gamma \vdash e_2 : t_1} \downarrow \text{ty-app}$$

$$\bullet \quad \frac{}{\Gamma, t_1 \{0\} = \Gamma} \text{dis-z}$$

$$\bullet \quad \frac{\frac{(\lambda : t_1 . e_1) \ e_2 \longrightarrow e_5}{\uparrow_0 \ e_2 = e_3} \downarrow \text{ev-app3}}{\bullet}$$

$$\bullet \quad \frac{\frac{(\lambda : t_1 . e_1) \ e_2 \longrightarrow e_5}{e_1 \{e_3 / 0\} = e_4} \downarrow \text{ev-app3}}{\bullet}$$

$$\bullet \quad \frac{\frac{(\lambda : t_1 . e_1) \ e_2 \longrightarrow e_5}{\downarrow_0 \ e_4 = e_5} \downarrow \text{ev-app3}}{\Gamma \vdash e_5 : t_2} \text{pssub} \quad (\text{ps-app3})$$

5.2 Value or evaluation

$$\boxed{\text{eval } e} \quad \boxed{\text{eval}}$$

$$\frac{\text{val } e}{\text{eval } e} \quad (\text{eval-v})$$

$$\frac{e \longrightarrow e_0}{\text{eval } e} \quad (\text{eval-e})$$

5.3 Progress

$$\boxed{\frac{\cdot \vdash e : t}{\text{eval } e}}$$

pg

$$\frac{\cdot \vdash \langle \rangle : \langle \rangle \quad \text{val-unit}}{\text{eval } \langle \rangle} \text{eval-v}$$

(pg-unit)

$$\frac{\cdot \vdash \lambda : t_1 . e : t_1 \rightarrow t_2 \quad \frac{\cdot \vdash \lambda : t_1 . e : t_1 \rightarrow t_2 \quad \cdot, t_1 \vdash e : t_2}{\text{val } \lambda : t_1 . e} \text{val-fun}}{\text{eval } \lambda : t_1 . e} \text{eval-v}$$

(pg-fun)

$$\frac{\cdot \vdash e_1 e_3 : t_2 \quad \cdot \vdash e_3 : t_1}{\text{eval } e_1 e_3} \text{eval-e}$$

(pg-app1)

$$\frac{\frac{\cdot \vdash e_3 e_1 : t_2 \quad \cdot \vdash e_3 : t_1 \rightarrow t_2}{\text{eval } e_3} \text{eval-v} \quad \frac{\cdot \vdash e_3 e_1 : t_2 \quad \cdot \vdash e_1 : t_1}{\text{eval } e_1} \text{eval-e}}{\text{eval } e_3 e_1} \text{eval-e}$$

(pg-app2)

$$\frac{\cdot \vdash (\lambda : t_2 . e_2) e_1 : t_3 \quad \cdot \vdash \lambda : t_2 . e_2 : t_1 \rightarrow t_3}{\text{eval } (\lambda : t_2 . e_2) e_1} \text{eval-e} \quad \frac{\cdot \vdash (\lambda : t_2 . e_2) e_1 : t_3 \quad \cdot \vdash \lambda : t_2 . e_2 : t_1 \rightarrow t_3}{\text{val } \lambda : t_2 . e_2} \text{val-fun}$$

$$\frac{\cdot \vdash (\lambda : t_2 . e_2) e_1 : t_3 \quad \cdot \vdash e_1 : t_1}{\text{eval } e_1} \text{eval-e}$$

$$\frac{}{\uparrow_0 e_1 = e_5} \text{upx}$$

$$\frac{}{e_2 \{e_5/0\} = e_3} \text{subx}$$

$$\frac{\frac{\frac{}{e_2 \{e_5/0\} = e_3} \text{subx} \quad \frac{\uparrow_0 e_1 = e_5}{0 \notin \text{fv}(e_5)} \text{upindep}}{0 \notin \text{fv}(e_3)} \text{subindep}}{\downarrow_0 e_3 = e_4} \text{indepdown} \quad \frac{}{\text{eval } (\lambda : t_2 . e_2) e_1} \text{eval-e}}{\text{eval } (\lambda : t_2 . e_2) e_1} \text{eval-e}$$

(pg-app3)

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