Confluence Theorem for Simply-Typed Lambda Calculus

Stephen Tse

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Contents

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1 Syntax

1.1 Variables

x ::= ···   x

1.2 Types

t ::= ···   t

1.3 Terms

e ::= ···   e

e ::= <>   (unit)

e ::= x   (var)

e ::= λx:t.e   (fun)

e ::= e e   (app)

2 Reductions

2.1 Variable unequalities

$$\boxed{\mathtt{x} \neq \mathtt{x}} \qquad \boxed{\text{ne}}$$

2.2 Substitutions

$$\boxed{\mathtt{e}\{\mathtt{e}/\mathtt{x}\} = \mathtt{e}} \qquad \boxed{\text{sub}}$$

$$\mathtt{x}\{\mathtt{e}/\mathtt{x}\} = \mathtt{e} \qquad \text{(sub-var1)}$$

$$\frac{\mathtt{x}_1 \neq \mathtt{x}_2}{\mathtt{x}_1\{\mathtt{e}/\mathtt{x}_2\} = \mathtt{x}_1} \qquad \text{(sub-var2)}$$

$$\frac{\mathtt{e}_1\{\mathtt{e}/\mathtt{x}_2\} = \mathtt{e}_2}{(\lambda \mathtt{x}_1 : \mathtt{t}.\mathtt{e}_1)\{\mathtt{e}/\mathtt{x}_2\} = (\lambda \mathtt{x}_1 : \mathtt{t}.\mathtt{e}_2)} \qquad \text{(sub-fun)}$$

$$\frac{\mathtt{e}_1\{\mathtt{e}/\mathtt{x}\} = \mathtt{e}_3 \quad \mathtt{e}_2\{\mathtt{e}/\mathtt{x}\} = \mathtt{e}_4}{(\mathtt{e}_1\ \mathtt{e}_2)\{\mathtt{e}/\mathtt{x}\} = (\mathtt{e}_3\ \mathtt{e}_4)} \qquad \text{(sub-app)}$$

2.3 Regular reductions

$$\boxed{\mathtt{e} \longrightarrow \mathtt{e}} \qquad \boxed{\text{r}}$$

$$\frac{\mathtt{e}_1\{\mathtt{e}_2/\mathtt{x}\} = \mathtt{e}_3}{((\lambda \mathtt{x} : \mathtt{t}.\mathtt{e}_1)\ \mathtt{e}_2) \longrightarrow \mathtt{e}_3} \qquad \text{(rbeta)}$$

$$\frac{\mathtt{e}_1 \longrightarrow \mathtt{e}_2}{(\lambda \mathtt{x} : \mathtt{t}.\mathtt{e}_1) \longrightarrow (\lambda \mathtt{x} : \mathtt{t}.\mathtt{e}_2)} \qquad \text{(rfun)}$$

$$\frac{\mathtt{e}_1 \longrightarrow \mathtt{e}_3}{(\mathtt{e}_1\ \mathtt{e}_2) \longrightarrow (\mathtt{e}_3\ \mathtt{e}_2)} \qquad \text{(rapp1)}$$

$$\frac{\mathtt{e}_2 \longrightarrow \mathtt{e}_3}{(\mathtt{e}_1\ \mathtt{e}_2) \longrightarrow (\mathtt{e}_1\ \mathtt{e}_3)} \qquad \text{(rapp2)}$$

2.4 Multi-step regular reductions

$$\boxed{\mathtt{e} \longrightarrow^* \mathtt{e}} \qquad \boxed{\text{rs}}$$

$$\mathtt{e} \longrightarrow^* \mathtt{e} \qquad \text{(rs-x)}$$

$$\frac{\mathtt{e}_1 \longrightarrow \mathtt{e}_2 \quad \mathtt{e}_2 \longrightarrow^* \mathtt{e}_3}{\mathtt{e}_1 \longrightarrow^* \mathtt{e}_3} \qquad \text{(rs-r)}$$

2.5 Parallel reductions

$$\boxed{\mathtt{e} \Longrightarrow \mathtt{e}} \qquad \boxed{\text{p}}$$

$$\frac{\mathtt{e}_1 \Longrightarrow \mathtt{e}_3 \quad \mathtt{e}_2 \Longrightarrow \mathtt{e}_4 \quad \mathtt{e}_3\{\mathtt{e}_4/\mathtt{x}\} = \mathtt{e}_5}{((\lambda \mathtt{x}:\mathtt{t}.\mathtt{e}_1)\ \mathtt{e}_2) \Longrightarrow \mathtt{e}_5} \quad \text{(pbeta)}$$

$$\frac{\mathtt{e}_1 \Longrightarrow \mathtt{e}_2}{(\lambda \mathtt{x}:\mathtt{t}.\mathtt{e}_1) \Longrightarrow (\lambda \mathtt{x}:\mathtt{t}.\mathtt{e}_2)} \quad \text{(pfun)}$$

$$\frac{\mathtt{e}_1 \Longrightarrow \mathtt{e}_3 \quad \mathtt{e}_2 \Longrightarrow \mathtt{e}_4}{(\mathtt{e}_1\ \mathtt{e}_2) \Longrightarrow (\mathtt{e}_3\ \mathtt{e}_4)} \quad \text{(papp)}$$

$$\mathtt{x} \Longrightarrow \mathtt{x} \quad \text{(pvar)}$$

2.6 Multi-step parallel reductions

$$\boxed{\mathtt{e} \Longrightarrow^* \mathtt{e}} \qquad \boxed{\text{ps}}$$

$$\mathtt{e} \Longrightarrow^* \mathtt{e} \quad \text{(ps-x)}$$

$$\frac{\mathtt{e}_1 \Longrightarrow \mathtt{e}_2 \quad \mathtt{e}_2 \Longrightarrow^* \mathtt{e}_3}{\mathtt{e}_1 \Longrightarrow^* \mathtt{e}_3} \quad \text{(ps-p)}$$

3 Lemmas

3.1 Transitivity of regular reductions

$$\frac{e_1 \longrightarrow^* e_2 \quad e_2 \longrightarrow^* e_3}{e_1 \longrightarrow^* e_3} \quad \boxed{\text{rsrsrs}}$$

(rsrsrs-x)

1 :	$e_1 \longrightarrow^* e_1$	*given
2 :	$e_1 \longrightarrow^* e_2$	*given

(rsrsrs-r)

1 :	$e_4 \longrightarrow^* e_2$	*given
2 :	$e_4 \longrightarrow e_1$	↓rs-r: 1
3 :	$e_1 \longrightarrow^* e_2$	↓rs-r: 1
4 :	$e_2 \longrightarrow^* e_3$	*given
5 :	$e_1 \longrightarrow^* e_3$	rsrsrs: 3,4
6 :	$e_4 \longrightarrow^* e_3$	rs-r: 2,5

3.2 Congruence of regular reductions

$$\frac{e_1 \longrightarrow^* e_2}{(\lambda x{:}t.e_1) \longrightarrow^* (\lambda x{:}t.e_2)} \quad \boxed{\text{rsfun}}$$

(rsfun-x)

1 :	$e \longrightarrow^* e$	*given
2 :	$(\lambda x{:}t.e) \longrightarrow^* (\lambda x{:}t.e)$	rs-x

(rsfun-r)

1 :	$e_3 \longrightarrow^* e_2$	*given
2 :	$e_3 \longrightarrow e_1$	↓rs-r: 1
3 :	$(\lambda x{:}t.e_3) \longrightarrow (\lambda x{:}t.e_1)$	rfun: 2
4 :	$e_1 \longrightarrow^* e_2$	↓rs-r: 1
5 :	$(\lambda x{:}t.e_1) \longrightarrow^* (\lambda x{:}t.e_2)$	rsfun: 4
6 :	$(\lambda x{:}t.e_3) \longrightarrow^* (\lambda x{:}t.e_2)$	rs-r: 3,5

$$\frac{e_1 \longrightarrow^* e_3}{(e_1\ e_2) \longrightarrow^* (e_3\ e_2)} \quad \boxed{\text{rsapp1}}$$

(rsapp1-x)

1 :	$e_1 \longrightarrow^* e_1$	*given
2 :	$(e_1\ e_2) \longrightarrow^* (e_1\ e_2)$	rs-x

(rsapp1-r)

1 :	$e_4 \longrightarrow^* e_2$	*given
2 :	$e_4 \longrightarrow e_1$	↓rs-r: 1
3 :	$(e_4\ e_3) \longrightarrow (e_1\ e_3)$	rapp1: 2
4 :	$e_1 \longrightarrow^* e_2$	↓rs-r: 1
5 :	$(e_1\ e_3) \longrightarrow^* (e_2\ e_3)$	rsapp1: 4
6 :	$(e_4\ e_3) \longrightarrow^* (e_2\ e_3)$	rs-r: 3,5

$$\frac{e_2 \longrightarrow^* e_3}{(e_1\ e_2) \longrightarrow^* (e_1\ e_3)} \quad \boxed{\text{rsapp2}}$$

(rsapp2-x)

1 :	$e_1 \longrightarrow^* e_1$	*given
2 :	$(e_2\ e_1) \longrightarrow^* (e_2\ e_1)$	rs-x

(rsapp2-r)

1 :	$e_4 \longrightarrow^* e_2$	*given
2 :	$e_4 \longrightarrow e_1$	↓rs-r: 1
3 :	$(e_3\ e_4) \longrightarrow (e_3\ e_1)$	rapp2: 2
4 :	$e_1 \longrightarrow^* e_2$	↓rs-r: 1
5 :	$(e_3\ e_1) \longrightarrow^* (e_3\ e_2)$	rsapp2: 4
6 :	$(e_3\ e_4) \longrightarrow^* (e_3\ e_2)$	rs-r: 3,5

3.3 Reflexivity of parallel reductions

$$\frac{}{e \Longrightarrow e} \qquad \boxed{\text{prefl}}$$

(prefl-var)

1 :	$x \Longrightarrow x$	pvar

(prefl-fun)

1 :	$e \Longrightarrow e$	prefl
2 :	$(\lambda x{:}t.e) \Longrightarrow (\lambda x{:}t.e)$	pfun: 1

(prefl-app)

1 :	$e_2 \Longrightarrow e_2$	prefl
2 :	$e_1 \Longrightarrow e_1$	prefl
3 :	$(e_2\ e_1) \Longrightarrow (e_2\ e_1)$	papp: 1,2

3.4 Equivalence of regular and parallel reductions

$$\frac{e_1 \longrightarrow e_2}{e_1 \Longrightarrow e_2} \qquad \boxed{\text{rp}}$$

(rp-beta)

1 :	$e_1 \Longrightarrow e_1$	prefl
2 :	$e_2 \Longrightarrow e_2$	prefl
3 :	$((\lambda x{:}t.e_1)\ e_2) \longrightarrow e_3$	*given
4 :	$e_1\{e_2/x\} = e_3$	↓rbeta: 3
5 :	$((\lambda x{:}t.e_1)\ e_2) \Longrightarrow e_3$	pbeta: 1,2,4

(rp-fun)

1 :	$(\lambda x{:}t.e_1) \longrightarrow (\lambda x{:}t.e_2)$	*given
2 :	$e_1 \longrightarrow e_2$	↓rfun: 1
3 :	$e_1 \Longrightarrow e_2$	rp: 2
4 :	$(\lambda x{:}t.e_1) \Longrightarrow (\lambda x{:}t.e_2)$	pfun: 3

(rp-app1)

1 :	$(e_2\ e_1) \longrightarrow (e_3\ e_1)$	*given
2 :	$e_2 \longrightarrow e_3$	↓rapp1: 1
3 :	$e_2 \Longrightarrow e_3$	rp: 2
4 :	$e_1 \Longrightarrow e_1$	prefl
5 :	$(e_2\ e_1) \Longrightarrow (e_3\ e_1)$	papp: 3,4

(rp-app2)

1 :	$e_3 \Longrightarrow e_3$	prefl

2 :	$(e_3\ e_1) \longrightarrow (e_3\ e_2)$	*given
3 :	$e_1 \longrightarrow e_2$	↓rapp2: 2
4 :	$e_1 \Longrightarrow e_2$	rp: 3
5 :	$(e_3\ e_1) \Longrightarrow (e_3\ e_2)$	papp: 1,4

$$\boxed{\frac{e_1 \Longrightarrow e_2}{e_1 \longrightarrow^* e_2}}$$

prs

(prs-beta)

1 :	$((\lambda x{:}t.e_1)\ e_2) \Longrightarrow e_5$	*given
2 :	$e_1 \Longrightarrow e_3$	↓pbeta: 1
3 :	$e_1 \longrightarrow^* e_3$	prs: 2
4 :	$(\lambda x{:}t.e_1) \longrightarrow^* (\lambda x{:}t.e_3)$	rsfun: 3
5 :	$((\lambda x{:}t.e_1)\ e_2) \longrightarrow^* ((\lambda x{:}t.e_3)\ e_2)$	rsapp1: 4
6 :	$e_2 \Longrightarrow e_4$	↓pbeta: 1
7 :	$e_2 \longrightarrow^* e_4$	prs: 6
8 :	$((\lambda x{:}t.e_3)\ e_2) \longrightarrow^* ((\lambda x{:}t.e_3)\ e_4)$	rsapp2: 7
9 :	$((\lambda x{:}t.e_1)\ e_2) \longrightarrow^* ((\lambda x{:}t.e_3)\ e_4)$	rsrsrs: 5,8
10 :	$e_3\{e_4/x\} = e_5$	↓pbeta: 1
11 :	$((\lambda x{:}t.e_3)\ e_4) \longrightarrow e_5$	rbeta: 10
12 :	$e_5 \longrightarrow^* e_5$	rs-x
13 :	$((\lambda x{:}t.e_3)\ e_4) \longrightarrow^* e_5$	rs-r: 11,12
14 :	$((\lambda x{:}t.e_1)\ e_2) \longrightarrow^* e_5$	rsrsrs: 9,13

(prs-fun)

1 :	$(\lambda x{:}t.e_1) \Longrightarrow (\lambda x{:}t.e_2)$	*given
2 :	$e_1 \Longrightarrow e_2$	↓pfun: 1
3 :	$e_1 \longrightarrow^* e_2$	prs: 2
4 :	$(\lambda x{:}t.e_1) \longrightarrow^* (\lambda x{:}t.e_2)$	rsfun: 3

(prs-app)

1 :	$(e_1\ e_2) \Longrightarrow (e_3\ e_4)$	*given
2 :	$e_1 \Longrightarrow e_3$	↓papp: 1
3 :	$e_1 \longrightarrow^* e_3$	prs: 2
4 :	$(e_1\ e_2) \longrightarrow^* (e_3\ e_2)$	rsapp1: 3
5 :	$e_2 \Longrightarrow e_4$	↓papp: 1
6 :	$e_2 \longrightarrow^* e_4$	prs: 5
7 :	$(e_3\ e_2) \longrightarrow^* (e_3\ e_4)$	rsapp2: 6
8 :	$(e_1\ e_2) \longrightarrow^* (e_3\ e_4)$	rsrsrs: 4,7

(prs-var)

1 :	$x \Longrightarrow x$	*given
2 :	$x \longrightarrow^* x$	rs-x

$$\boxed{\frac{e_1 \longrightarrow^* e_2}{e_1 \Longrightarrow^* e_2}}$$

rsps

(rsps-x)

1 :	$e \longrightarrow^* e$	*given
2 :	$e \Longrightarrow^* e$	ps-x

(rsps-r)

1 :	$e_3 \longrightarrow^* e_2$	*given
2 :	$e_3 \longrightarrow e_1$	↓rs-r: 1
3 :	$e_3 \Longrightarrow e_1$	rp: 2
4 :	$e_1 \longrightarrow^* e_2$	↓rs-r: 1
5 :	$e_1 \Longrightarrow^* e_2$	rsps: 4
6 :	$e_3 \Longrightarrow^* e_2$	ps-p: 3,5

$$\boxed{\frac{e_1 \Longrightarrow^* e_2}{e_1 \longrightarrow^* e_2}}$$ psrs

(psrs-x)

1 :	$e \Longrightarrow^* e$	*given
2 :	$e \longrightarrow^* e$	rs-x

(psrs-p)

1 :	$e_1 \Longrightarrow^* e_3$	*given
2 :	$e_1 \Longrightarrow e_2$	↓ps-p: 1
3 :	$e_1 \longrightarrow^* e_2$	prs: 2
4 :	$e_2 \Longrightarrow^* e_3$	↓ps-p: 1
5 :	$e_2 \longrightarrow^* e_3$	psrs: 4
6 :	$e_1 \longrightarrow^* e_3$	rsrsrs: 3,5

3.5 Variable unequality[1]

$$\boxed{\frac{}{x_1 \neq x_2}}$$ nex

3.6 Totality of substitutions[2]

$$\boxed{\frac{}{e_1\{e_2/x\} = e_3}}$$ subx

$$\boxed{\frac{}{e_1\{e_2/x\} = e_1}}$$ subxx

3.7 Commutativity of substitutions

$$\boxed{e\{e/x\} = e \quad e\{e/x\} = e \quad e\{e/x\} = e}$$ sss

$$\frac{e_1\{e_4/x_2\} = e_6 \quad e_2\{e_4/x_2\} = e_7 \quad e_6\{e_7/x_1\} = e_5}{e_1\{e_4/x_2\} = e_6 \quad e_2\{e_4/x_2\} = e_7 \quad e_6\{e_7/x_1\} = e_5}$$ (sssx)

$$\boxed{\frac{e_1\{e_2/x_1\} = e_3 \quad e_3\{e_4/x_2\} = e_5 \quad x_1 \neq x_2}{e_1\{e_4/x_2\} = e_6 \quad e_2\{e_4/x_2\} = e_7 \quad e_6\{e_7/x_1\} = e_5}}$$ subsub

(subsub-var1)

1 :	$x_1\{e_1/x_1\} = e_1$	*given
2 :	$x_1 \neq x_2$	*given
3 :	$x_1\{e_2/x_2\} = x_1$	sub-var2: 2
4 :	$e_1\{e_2/x_2\} = e_3$	*given
5 :	$x_1\{e_3/x_1\} = e_3$	sub-var1
6 :	$x_1\{e_2/x_2\} = x_1 \quad e_1\{e_2/x_2\} = e_3 \quad x_1\{e_3/x_1\} = e_3$	sssx: 3,4,5

(subsub-var2)

1 :	$x_2\{e_3/x_1\} = x_2$	*given
2 :	$x_2 \neq x_1$	↓sub-var2: 1
3 :	$x_1 \neq x_2$	*given

[1]Special axiom used only for psub-beta on page 9: because of Barendregt Variable Convention, variables at substitutions of functions are renamed to be different.

[2]Special axiom (subx): the output of the substitution is unconstrained. Special axiom (subxx) used only for subsub-var2 on page 8: because of term well-formedness, the input of the substitution is free of the variable.

4 :	$x_2\{e_1/x_2\} = e_1$	*given
5 :	$e_3\{e_1/x_2\} = e_2$	subx
6 :	$e_1\{e_2/x_1\} = e_1$	subxx
7 :	$x_2\{e_1/x_2\} = e_1 \quad e_3\{e_1/x_2\} = e_2 \quad e_1\{e_2/x_1\} = e_1$	sssx: 4,5,6

(subsub-var3)

1 :	$x_2\{e_1/x_1\} = x_2$	*given
2 :	$x_2 \neq x_1$	↓sub-var2: 1
3 :	$x_1 \neq x_1$	*given
4 :	$x_2\{e_2/x_1\} = x_2$	*given
5 :	$e_1\{e_2/x_1\} = e_3$	subx
6 :	$x_2 \neq x_1$	↓sub-var2: 4
7 :	$x_2\{e_3/x_1\} = x_2$	sub-var2: 6
8 :	$x_2\{e_2/x_1\} = x_2 \quad e_1\{e_2/x_1\} = e_3 \quad x_2\{e_3/x_1\} = x_2$	sssx: 4,5,7

(subsub-fun)

1 :	$(\lambda x_3{:}t.e_1)\{e_2/x_1\} = (\lambda x_3{:}t.e_3)$	*given
2 :	$e_1\{e_2/x_1\} = e_3$	↓sub-fun: 1
3 :	$(\lambda x_3{:}t.e_3)\{e_4/x_2\} = (\lambda x_3{:}t.e_5)$	*given
4 :	$e_3\{e_4/x_2\} = e_5$	↓sub-fun: 3
5 :	$x_1 \neq x_2$	*given
6 :	$e_1\{e_4/x_2\} = e_6 \quad e_2\{e_4/x_2\} = e_7 \quad e_6\{e_7/x_1\} = e_5$	subsub: 2,4,5
7 :	$e_1\{e_4/x_2\} = e_6$	↓sssx: 6
8 :	$(\lambda x_3{:}t.e_1)\{e_4/x_2\} = (\lambda x_3{:}t.e_6)$	sub-fun: 7
9 :	$e_2\{e_4/x_2\} = e_7$	↓sssx: 6
10 :	$e_6\{e_7/x_1\} = e_5$	↓sssx: 6
11 :	$(\lambda x_3{:}t.e_6)\{e_7/x_1\} = (\lambda x_3{:}t.e_5)$	sub-fun: 10
12 :	$(\lambda x_3{:}t.e_1)\{e_4/x_2\} = (\lambda x_3{:}t.e_6) \quad e_2\{e_4/x_2\} = e_7 \quad (\lambda x_3{:}t.e_6)\{e_7/x_1\} = (\lambda x_3{:}t.e_5)$	sssx: 8,9,11

(subsub-app)

1 :	$(e_8\ e_1)\{e_2/x_1\} = (e_9\ e_3)$	*given
2 :	$e_8\{e_2/x_1\} = e_9$	↓sub-app: 1
3 :	$(e_9\ e_3)\{e_4/x_2\} = (e_{10}\ e_5)$	*given
4 :	$e_9\{e_4/x_2\} = e_{10}$	↓sub-app: 3
5 :	$x_1 \neq x_2$	*given
6 :	$e_8\{e_4/x_2\} = e_{11} \quad e_2\{e_4/x_2\} = e_7 \quad e_{11}\{e_7/x_1\} = e_{10}$	subsub: 2,4,5
7 :	$e_8\{e_4/x_2\} = e_{11}$	↓sssx: 6
8 :	$e_1\{e_2/x_1\} = e_3$	↓sub-app: 1
9 :	$e_3\{e_4/x_2\} = e_5$	↓sub-app: 3
10 :	$e_1\{e_4/x_2\} = e_6 \quad e_2\{e_4/x_2\} = e_7 \quad e_6\{e_7/x_1\} = e_5$	subsub: 8,9,5
11 :	$e_1\{e_4/x_2\} = e_6$	↓sssx: 10
12 :	$(e_8\ e_1)\{e_4/x_2\} = (e_{11}\ e_6)$	sub-app: 7,11
13 :	$e_2\{e_4/x_2\} = e_7$	↓sssx: 10
14 :	$e_{11}\{e_7/x_1\} = e_{10}$	↓sssx: 6
15 :	$e_6\{e_7/x_1\} = e_5$	↓sssx: 10
16 :	$(e_{11}\ e_6)\{e_7/x_1\} = (e_{10}\ e_5)$	sub-app: 14,15
17 :	$(e_8\ e_1)\{e_4/x_2\} = (e_{11}\ e_6) \quad e_2\{e_4/x_2\} = e_7 \quad (e_{11}\ e_6)\{e_7/x_1\} = (e_{10}\ e_5)$	sssx: 12,13,16

3.8 Type preservation of substitutions – asserted

$$\frac{(\lambda x{:}t.e_1) \Longrightarrow (\lambda x{:}t.e_2) \quad e_3 \Longrightarrow e_4 \quad e_1\{e_3/x\} = e_5 \quad e_2\{e_4/x\} = e_6 \quad e_5 \Longrightarrow e_6}{(\lambda x{:}t.e_5) \Longrightarrow (\lambda x{:}t.e_6)} \quad \boxed{\text{tsub}}$$

3.9 Reduction preservation of substitutions

$$\frac{e_1 \Longrightarrow e_2 \quad e_3 \Longrightarrow e_4 \quad e_1\{e_3/x\} = e_5 \quad e_2\{e_4/x\} = e_6}{e_5 \Longrightarrow e_6} \quad \boxed{\text{psub}}$$

		(psub-beta)
1 :	$((\lambda x_2 : t.e_7)\ e_1) \Longrightarrow e_{11}$	*given
2 :	$e_7 \Longrightarrow e_8$	↓pbeta: 1
3 :	$e_3 \Longrightarrow e_4$	*given
4 :	$((\lambda x_2 : t.e_7)\ e_1)\{e_3/x_1\} = ((\lambda x_2 : t.e_9)\ e_5)$	*given
5 :	$(\lambda x_2 : t.e_7)\{e_3/x_1\} = (\lambda x_2 : t.e_9)$	↓sub-app: 4
6 :	$e_7\{e_3/x_1\} = e_9$	↓sub-fun: 5
7 :	$e_8\{e_2/x_2\} = e_{11}$	↓pbeta: 1
8 :	$e_{11}\{e_4/x_1\} = e_{12}$	*given
9 :	$x_2 \neq x_1$	nex
10 :	$e_8\{e_4/x_1\} = e_{10} \quad e_2\{e_4/x_1\} = e_6 \quad e_{10}\{e_6/x_2\} = e_{12}$	subsub: 7,8,9
11 :	$e_8\{e_4/x_1\} = e_{10}$	↓sssx: 10
12 :	$e_9 \Longrightarrow e_{10}$	psub: 2,3,6,11
13 :	$e_1 \Longrightarrow e_2$	↓pbeta: 1
14 :	$e_1\{e_3/x_1\} = e_5$	↓sub-app: 4
15 :	$e_2\{e_4/x_1\} = e_6$	↓sssx: 10
16 :	$e_5 \Longrightarrow e_6$	psub: 13,3,14,15
17 :	$e_{10}\{e_6/x_2\} = e_{12}$	↓sssx: 10
18 :	$((\lambda x_2 : t.e_9)\ e_5) \Longrightarrow e_{12}$	pbeta: 12,16,17

		(psub-fun)
1 :	$(\lambda x : t.e_1) \Longrightarrow (\lambda x : t.e_2)$	*given
2 :	$e_3 \Longrightarrow e_4$	*given
3 :	$(\lambda x : t.e_1)\{e_3/x\} = (\lambda x : t.e_5)$	*given
4 :	$e_1\{e_3/x\} = e_5$	↓sub-fun: 3
5 :	$(\lambda x : t.e_2)\{e_4/x\} = (\lambda x : t.e_6)$	*given
6 :	$e_2\{e_4/x\} = e_6$	↓sub-fun: 5
7 :	$e_1 \Longrightarrow e_2$	↓pfun: 1
8 :	$e_5 \Longrightarrow e_6$	psub: 7,2,4,6
9 :	$(\lambda x : t.e_5) \Longrightarrow (\lambda x : t.e_6)$	tsub: 1,2,4,6,8

		(psub-app)
1 :	$(e_7\ e_1) \Longrightarrow (e_8\ e_2)$	*given
2 :	$e_7 \Longrightarrow e_8$	↓papp: 1
3 :	$e_3 \Longrightarrow e_4$	*given
4 :	$(e_7\ e_1)\{e_3/x\} = (e_9\ e_5)$	*given
5 :	$e_7\{e_3/x\} = e_9$	↓sub-app: 4
6 :	$(e_8\ e_2)\{e_4/x\} = (e_{10}\ e_6)$	*given
7 :	$e_8\{e_4/x\} = e_{10}$	↓sub-app: 6
8 :	$e_9 \Longrightarrow e_{10}$	psub: 2,3,5,7
9 :	$e_1 \Longrightarrow e_2$	↓papp: 1
10 :	$e_1\{e_3/x\} = e_5$	↓sub-app: 4
11 :	$e_2\{e_4/x\} = e_6$	↓sub-app: 6
12 :	$e_5 \Longrightarrow e_6$	psub: 9,3,10,11
13 :	$(e_9\ e_5) \Longrightarrow (e_{10}\ e_6)$	papp: 8,12

		(psub-var1)
1 :	$x \Longrightarrow x$	*given
2 :	$x\{e_1/x\} = e_1$	*given
3 :	$x\{e_2/x\} = e_2$	*given
4 :	$e_1 \Longrightarrow e_2$	*given

		(psub-var2)
1 :	$e_1 \Longrightarrow e_2$	*given
2 :	$x_1\{e_1/x_2\} = x_1$	*given
3 :	$x_1 \neq x_2$	↓sub-var2: 2
4 :	$x_1\{e_2/x_2\} = x_1$	*given
5 :	$x_1 \Longrightarrow x_1$	*given

4 Church-Rosser Theorems

4.1 Parallel reductions

$$\boxed{e \Longrightarrow e \Longleftarrow e} \qquad \boxed{\text{pp}}$$

$$\frac{e_1 \Longrightarrow e_3 \quad e_2 \Longrightarrow e_3}{e_1 \Longrightarrow e_3 \Longleftarrow e_2} \qquad \text{(ppx)}$$

$$\boxed{\frac{e \Longrightarrow e_1 \quad e \Longrightarrow e_2}{e_1 \Longrightarrow e_3 \Longleftarrow e_2}} \qquad \boxed{\text{pcr}}$$

(pcr-bb)

1 :	$((\lambda x{:}t.e_5)\ e_4) \Longrightarrow e_3$	*given
2 :	$e_5 \Longrightarrow e_1$	↓pbeta: 1
3 :	$e_5 \Longrightarrow e_1$	↓pbeta: 1
4 :	$e_1 \Longrightarrow e_1 \Longleftarrow e_1$	pcr: 2,3
5 :	$e_1 \Longrightarrow e_1$	↓ppx: 4
6 :	$e_4 \Longrightarrow e_2$	↓pbeta: 1
7 :	$e_4 \Longrightarrow e_2$	↓pbeta: 1
8 :	$e_2 \Longrightarrow e_2 \Longleftarrow e_2$	pcr: 6,7
9 :	$e_2 \Longrightarrow e_2$	↓ppx: 8
10 :	$e_1\{e_2/x\} = e_3$	↓pbeta: 1
11 :	$e_1\{e_2/x\} = e_3$	↓pbeta: 1
12 :	$e_3 \Longrightarrow e_3$	psub: 5,9,10,11
13 :	$e_1 \Longrightarrow e_1$	↓ppx: 4
14 :	$e_2 \Longrightarrow e_2$	↓ppx: 8
15 :	$e_3 \Longrightarrow e_3$	psub: 13,14,10,11
16 :	$e_3 \Longrightarrow e_3 \Longleftarrow e_3$	ppx: 12,15

(pcr-ba)

1 :	$((\lambda x{:}t.e_9)\ e_7) \Longrightarrow e_5$	*given
2 :	$e_9 \Longrightarrow e_1$	↓pbeta: 1
3 :	$((\lambda x{:}t.e_9)\ e_7) \Longrightarrow ((\lambda x{:}t.e_{10})\ e_8)$	*given
4 :	$(\lambda x{:}t.e_9) \Longrightarrow (\lambda x{:}t.e_{10})$	↓papp: 3
5 :	$e_9 \Longrightarrow e_{10}$	↓pfun: 4
6 :	$e_1 \Longrightarrow e_2 \Longleftarrow e_{10}$	pcr: 2,5
7 :	$e_1 \Longrightarrow e_2$	↓ppx: 6
8 :	$e_7 \Longrightarrow e_3$	↓pbeta: 1
9 :	$e_7 \Longrightarrow e_8$	↓papp: 3
10 :	$e_3 \Longrightarrow e_4 \Longleftarrow e_8$	pcr: 8,9
11 :	$e_3 \Longrightarrow e_4$	↓ppx: 10
12 :	$e_1\{e_3/x\} = e_5$	↓pbeta: 1
13 :	$e_2\{e_4/x\} = e_6$	subx
14 :	$e_5 \Longrightarrow e_6$	psub: 7,11,12,13
15 :	$e_{10} \Longrightarrow e_2$	↓ppx: 6
16 :	$e_8 \Longrightarrow e_4$	↓ppx: 10
17 :	$((\lambda x{:}t.e_{10})\ e_8) \Longrightarrow e_6$	pbeta: 15,16,13
18 :	$e_5 \Longrightarrow e_6 \Longleftarrow ((\lambda x{:}t.e_{10})\ e_8)$	ppx: 14,17

(pcr-ab)

1 :	$((\lambda x{:}t.e_9)\ e_7) \Longrightarrow ((\lambda x{:}t.e_{10})\ e_8)$	*given
2 :	$(\lambda x{:}t.e_9) \Longrightarrow (\lambda x{:}t.e_{10})$	↓papp: 1
3 :	$e_9 \Longrightarrow e_{10}$	↓pfun: 2
4 :	$((\lambda x{:}t.e_9)\ e_7) \Longrightarrow e_5$	*given
5 :	$e_9 \Longrightarrow e_1$	↓pbeta: 4
6 :	$e_{10} \Longrightarrow e_2 \Longleftarrow e_1$	pcr: 3,5
7 :	$e_{10} \Longrightarrow e_2$	↓ppx: 6
8 :	$e_7 \Longrightarrow e_8$	↓papp: 1
9 :	$e_7 \Longrightarrow e_3$	↓pbeta: 4
10 :	$e_8 \Longrightarrow e_4 \Longleftarrow e_3$	pcr: 8,9
11 :	$e_8 \Longrightarrow e_4$	↓ppx: 10

12 :	$e_2\{e_4/x\} = e_6$	subx
13 :	$((\lambda x{:}t.e_{10})\ e_8) \Longrightarrow e_6$	pbeta: 7,11,12
14 :	$e_1 \Longrightarrow e_2$	↓ppx: 6
15 :	$e_3 \Longrightarrow e_4$	↓ppx: 10
16 :	$e_1\{e_3/x\} = e_5$	↓pbeta: 4
17 :	$e_5 \Longrightarrow e_6$	psub: 14,15,16,12
18 :	$((\lambda x{:}t.e_{10})\ e_8) \Longrightarrow e_6 \Longleftarrow e_5$	ppx: 13,17

(pcr-aa)

1 :	$(e_5\ e_1) \Longrightarrow (e_6\ e_2)$	*given
2 :	$e_5 \Longrightarrow e_6$	↓papp: 1
3 :	$(e_5\ e_1) \Longrightarrow (e_7\ e_3)$	*given
4 :	$e_5 \Longrightarrow e_7$	↓papp: 3
5 :	$e_6 \Longrightarrow e_8 \Longleftarrow e_7$	pcr: 2,4
6 :	$e_6 \Longrightarrow e_8$	↓ppx: 5
7 :	$e_1 \Longrightarrow e_2$	↓papp: 1
8 :	$e_1 \Longrightarrow e_3$	↓papp: 3
9 :	$e_2 \Longrightarrow e_4 \Longleftarrow e_3$	pcr: 7,8
10 :	$e_2 \Longrightarrow e_4$	↓ppx: 9
11 :	$(e_6\ e_2) \Longrightarrow (e_8\ e_4)$	papp: 6,10
12 :	$e_7 \Longrightarrow e_8$	↓ppx: 5
13 :	$e_3 \Longrightarrow e_4$	↓ppx: 9
14 :	$(e_7\ e_3) \Longrightarrow (e_8\ e_4)$	papp: 12,13
15 :	$(e_6\ e_2) \Longrightarrow (e_8\ e_4) \Longleftarrow (e_7\ e_3)$	ppx: 11,14

(pcr-ff)

1 :	$(\lambda x{:}t.e_1) \Longrightarrow (\lambda x{:}t.e_2)$	*given
2 :	$e_1 \Longrightarrow e_2$	↓pfun: 1
3 :	$(\lambda x{:}t.e_1) \Longrightarrow (\lambda x{:}t.e_3)$	*given
4 :	$e_1 \Longrightarrow e_3$	↓pfun: 3
5 :	$e_2 \Longrightarrow e_4 \Longleftarrow e_3$	pcr: 2,4
6 :	$e_2 \Longrightarrow e_4$	↓ppx: 5
7 :	$(\lambda x{:}t.e_2) \Longrightarrow (\lambda x{:}t.e_4)$	pfun: 6
8 :	$e_3 \Longrightarrow e_4$	↓ppx: 5
9 :	$(\lambda x{:}t.e_3) \Longrightarrow (\lambda x{:}t.e_4)$	pfun: 8
10 :	$(\lambda x{:}t.e_2) \Longrightarrow (\lambda x{:}t.e_4) \Longleftarrow (\lambda x{:}t.e_3)$	ppx: 7,9

(pcr-var)

1 :	$x \Longrightarrow x$	*given
2 :	$x \Longrightarrow x \Longleftarrow x$	ppx: 1,1

4.2 Strip lemma

$$\boxed{e \Longrightarrow^* e \Longleftarrow e} \qquad \boxed{\text{psp}}$$

$$\frac{e_1 \Longrightarrow^* e_3 \quad e_2 \Longrightarrow e_3}{e_1 \Longrightarrow^* e_3 \Longleftarrow e_2} \quad \text{(pspx)}$$

$$\boxed{\frac{e \Longrightarrow e_1 \quad e \Longrightarrow^* e_2}{e_1 \Longrightarrow^* e_3 \Longleftarrow e_2}} \qquad \boxed{\text{ppscr}}$$

(ppscr-x)

1 :	$e_1 \Longrightarrow^* e_1$	*given
2 :	$e_2 \Longrightarrow^* e_2$	ps-x
3 :	$e_1 \Longrightarrow e_2$	*given
4 :	$e_2 \Longrightarrow^* e_2 \Longleftarrow e_1$	pspx: 2,3

(ppscr-p)

1 :	$e_5 \Longrightarrow e_6$	*given
2 :	$e_5 \Longrightarrow^* e_3$	*given
3 :	$e_5 \Longrightarrow e_1$	↓ps-p: 2
4 :	$e_6 \Longrightarrow e_2 \Longleftarrow e_1$	pcr: 1,3
5 :	$e_6 \Longrightarrow e_2$	↓ppx: 4
6 :	$e_1 \Longrightarrow e_2$	↓ppx: 4
7 :	$e_1 \Longrightarrow^* e_3$	↓ps-p: 2
8 :	$e_2 \Longrightarrow^* e_4 \Longleftarrow e_3$	ppscr: 6,7
9 :	$e_2 \Longrightarrow^* e_4$	↓pspx: 8
10 :	$e_6 \Longrightarrow^* e_4$	ps-p: 5,9
11 :	$e_3 \Longrightarrow e_4$	↓pspx: 8
12 :	$e_6 \Longrightarrow^* e_4 \Longleftarrow e_3$	pspx: 10,11

4.3 Multi-step parallel reductions

$$\boxed{e \Longrightarrow^* e \Longleftarrow^* e} \qquad \boxed{\text{psps}}$$

$$\frac{e_1 \Longrightarrow^* e_3 \quad e_2 \Longrightarrow^* e_3}{e_1 \Longrightarrow^* e_3 \Longleftarrow^* e_2} \quad \text{(pspsx)}$$

$$\boxed{\frac{e \Longrightarrow^* e_1 \quad e \Longrightarrow^* e_2}{e_1 \Longrightarrow^* e_3 \Longleftarrow^* e_2}} \qquad \boxed{\text{pscr}}$$

(pscr-x)

1 :	$e_1 \Longrightarrow^* e_1$	*given
2 :	$e_1 \Longrightarrow^* e_2$	*given
3 :	$e_2 \Longrightarrow^* e_2$	ps-x
4 :	$e_1 \Longrightarrow^* e_2 \Longleftarrow^* e_2$	pspsx: 2,3

(pscr-p)

1 :	$e_5 \Longrightarrow^* e_2$	*given
2 :	$e_1 \Longrightarrow^* e_2$	↓ps-p: 1
3 :	$e_5 \Longrightarrow e_1$	↓ps-p: 1
4 :	$e_5 \Longrightarrow^* e_6$	*given
5 :	$e_1 \Longrightarrow^* e_3 \Longleftarrow e_6$	ppscr: 3,4
6 :	$e_1 \Longrightarrow^* e_3$	↓pspx: 5
7 :	$e_2 \Longrightarrow^* e_4 \Longleftarrow^* e_3$	pscr: 2,6
8 :	$e_2 \Longrightarrow^* e_4$	↓pspsx: 7
9 :	$e_6 \Longrightarrow e_3$	↓pspx: 5
10 :	$e_3 \Longrightarrow^* e_4$	↓pspsx: 7
11 :	$e_6 \Longrightarrow^* e_4$	ps-p: 9,10
12 :	$e_2 \Longrightarrow^* e_4 \Longleftarrow^* e_6$	pspsx: 8,11

4.4 Multi-step regular reductions

$$\boxed{e \longrightarrow^* e \longleftarrow^* e} \qquad \boxed{\text{rsrs}}$$

$$\frac{e_1 \longrightarrow^* e_3 \quad e_2 \longrightarrow^* e_3}{e_1 \longrightarrow^* e_3 \longleftarrow^* e_2} \quad \text{(rsrsx)}$$

$$\boxed{\frac{e \longrightarrow^* e_1 \quad e \longrightarrow^* e_2}{e_1 \longrightarrow^* e_3 \longleftarrow^* e_2}} \qquad \boxed{\text{rscr}}$$

(rscrx)

1 :	$e_4 \longrightarrow^* e_3$	*given
2 :	$e_4 \Longrightarrow^* e_3$	rsps: 1
3 :	$e_4 \longrightarrow^* e_1$	*given
4 :	$e_4 \Longrightarrow^* e_1$	rsps: 3

5 :	$\mathsf{e}_3 \Longrightarrow^* \mathsf{e}_2 \Longleftarrow^* \mathsf{e}_1$	pscr: 2,4
6 :	$\mathsf{e}_3 \Longrightarrow^* \mathsf{e}_2$	↓pspsx: 5
7 :	$\mathsf{e}_3 \longrightarrow^* \mathsf{e}_2$	psrs: 6
8 :	$\mathsf{e}_1 \Longrightarrow^* \mathsf{e}_2$	↓pspsx: 5
9 :	$\mathsf{e}_1 \longrightarrow^* \mathsf{e}_2$	psrs: 8
10 :	$\mathsf{e}_3 \longrightarrow^* \mathsf{e}_2 \longleftarrow^* \mathsf{e}_1$	rsrsx: 7,9