Safe Schedulability for Bounded-Rate Multi-Mode Systems

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Safe Schedulability for Bounded-Rate Multi-Mode Systems
Bounded-Rate Multi-Mode Systems

Safe Schedulability for Bounded-Rate Multi-Mode Systems
Variables = \{x, y\}
Variables = \{x, y\} (x, y): State

(y)  
\[ \rightarrow x \]

\((x, y): State\)
Bounded Convex Polytope
BMS: Dynamics

Safe Schedulability for Bounded-Rate Multi-Mode Systems
BMS: Dynamics

Extreme rate vectors

Safe Schedulability for Bounded-Rate Multi-Mode Systems
BMS: Dynamics

Safe Schedulability for Bounded-Rate Multi-Mode Systems
Sequence of modes and times

Schedule

UP
LEFT
RIGHT

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Sequence of modes and times

(RIGHT,1)
Sequence of modes and times

(RIGHT,1)
Sequence of modes and times

(SRIGHT,1), (UP,1)
Schedule

Sequence of modes and times

(RIGHT, 1), (UP, 1)
Sequence of modes and times

(RIGHT, 1), (UP, 1), (LEFT, 2)
Sequence of modes and times

(RIGHT,1), (UP,1), (LEFT,2)
Sequence of modes and times

(RIGHT,1), (UP,1), (LEFT,2)
Schedulability Problem

The diagram illustrates a cyclic relationship among the states UP, LEFT, and RIGHT. Arrows indicate the transitions between these states.
Schedulability Problem

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Schedulability Problem

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Schedulability problem

Given a BMS $\mathcal{H}$, a safety set $S$ as a convex polytope, and a starting state $X_0 \in S$, the (safe) schedulability problem is to decide whether there exists a non-Zeno schedule that always keeps the state of the system inside $S$. 

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Safe Schedulability for Bounded-Rate Multi-Mode Systems
Applications

- Modeling of multi-mode systems with uncertainty
- Green Scheduling [2]
- Temperature and humidity control in cloud servers
- Robot motion planning
- Autonomous vehicles navigation
Constant-Rate Multi-Mode Systems

Safe Schedulability for Bounded-Rate Multi-Mode Systems
Constant-Rate Multi-Mode Systems [1]

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Safe Schedulability for Bounded-Rate Multi-Mode Systems
Constant-Rate Multi-Mode Systems [1]
Theorem

Safe Schedulability problem for CMSs can be solved in polynomial time [1].
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\[ \text{temp} \quad \overset{\text{ON}}{\longrightarrow} \quad \overset{\text{OFF}}{\longrightarrow} \]

\[ \text{temp} \quad \overset{\text{ON}}{\longrightarrow} \quad \overset{X_0}{\longrightarrow} \]
Static vs. Dynamic Schedule

- **Static schedule**
  - Scheduler do not observe the state of the system
  - Related: open-loop control

- **Dynamic schedule**
  - Scheduler can observe the decisions of the environment so far
  - Related: closed-loop control

There is no static schedule for the multi-mode system with uncertainty
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If there exists an $\mathcal{H}$-closed polytope, then there exists a winning dynamic strategy for the scheduler.
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**Theorem**

If there exists an \( \mathcal{H} \)-closed polytope, then there exists a winning dynamic strategy for the scheduler.
Dynamic Schedule

ON

OFF

$\varepsilon$

$\varepsilon$

$x_0$
Dynamic Schedule

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Dynamic Schedule

\[ x_0 \]

\[ x_1, \text{\(OFF, t_1\)} \]

\[ x_2, \text{\(ON, t_2\)} \]
Dynamic Schedule

\[ x = \lambda x_1 + (1 - \lambda) x_2, \quad t = \max(\lambda t_1, (1 - \lambda) t_2) \]
$x_0 = \frac{1}{2} x_1 + \frac{1}{2} x_2, \ t = \max(\frac{1}{2}, \frac{1}{2}), \ mode = ON$
Dynamic Schedule

\[ x_0 = (-1, -0.5) \]

\[ (ON, \frac{1}{2}) \]

\[ (-1, \frac{1}{2}) \quad \ldots \quad (-1, 1) \]
Dynamic Schedule

\[ x_0 = (-1, -0.5) \]

\[ (ON, \frac{1}{2}) \]

\[ (OFF, \frac{5}{6}) \]

\[ (OFF, 1) \]

\[ (-1, \frac{1}{2}) \]

\[ ... \]

\[ (-1, 1) \]

\[ x = (-1, \frac{1}{2}) \rightarrow x = \frac{5}{6} x_1 + \frac{1}{6} x_2, \quad t = max(\frac{5}{6}, \frac{1}{6}), \quad mode = OFF \]
Dynamic Schedule

$x_0 = (-1, -0.5)$

\[\begin{align*}
  x_1, (OFF, 1) \\
  x_0 \\
  x_2, (ON, 1)
\end{align*}\]

\[\begin{align*}
  (-1, \frac{1}{2}) & \quad \text{...} & \quad (-1, 1) \\
  (OFF, \frac{5}{6}) \\
  (OFF, 1)
\end{align*}\]

\[\begin{align*}
  (-1, -\frac{1}{3}) & \quad \text{...} & \quad (-1, -\frac{1}{4}) \\
  (-1, 0) & \quad \text{...} & \quad (-1, -\frac{1}{2})
\end{align*}\]
Existence of $\mathcal{H}$-closed polytope

The diagram shows a plot with axes $\dot{x}$ and $\dot{y}$, with regions labeled "LEFT" and "RIGHT" in the quadrants.
Existence of \( \mathcal{H} \)-closed polytope

A CMS instance of BMS
Existence of $\mathcal{H}$-closed polytope

Extreme CMSs of $\mathcal{H}$
Existence of $\mathcal{H}$-closed polytope

Extreme CMSs of $\mathcal{H}$
Construction of $\mathcal{H}$-closed Polytope

BMS $\mathcal{H}$
Construction of $\mathcal{H}$-closed Polytope
Construction of $\mathcal{H}$-closed Polytope
Construction of $\mathcal{H}$-closed Polytope

$\mathcal{H}$-closed polytope
Construction of $\mathcal{H}$-closed Polytope
Schedulability of BMS

For every BMS $\mathcal{H}$ and the starting state in the interior of a convex and bounded safety set we have that scheduler has a winning strategy if and only if all Extreme-rate CMSs of $\mathcal{H}$ are schedulable.
Other results

**co-NP completeness**

The schedulability problem for BMSs and convex polytope safety sets is co-NP complete.

Schedulability problems for BMSs with convex polytope safety sets are in PTIME for systems with 2 variables.

**Discrete Schedulability**

Discrete schedulability problem is complete for EXPTIME
Summary

- Necessary and sufficient conditions for schedulability of BMSs
- Synthesizing winning strategy for scheduler if exists
- Co-NP completeness of the schedulability problem
- PTIME algorithm for BMSs with two variables
Conclusions and Future Work

Summary

- Necessary and sufficient conditions for schedulability of BMSs
- Synthesizing winning strategy for scheduler if exists
- Co-NP completeness of the schedulability problem
- PTIME algorithm for BMSs with two variables

Future Work

- Schedulability problem with respect to more expressive higher-level control objectives including temporal logic based specification
- Bounded-rate multi-mode systems with reward functions
R. Alur, A. Trivedi, and D. Wojtczak.
Optimal scheduling for constant-rate multi-mode systems.

Green scheduling of control systems for peak demand reduction.
In *IEEE CDC*, December 2011.
Questions?
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Complexity

co-NP completeness

The schedulability problem for BMSs and convex polytope safety sets is co-NP complete.

co-NP hardness

- Reduction from 3SAT
- For a 3SAT instance $\Phi$, construct a MMS $\mathcal{H}$
- $\Phi$ is satisfiable iff $\mathcal{H}$ is not schedulable
Co-NP Hardness

Satisfiable: $\Phi = (x_1 \lor x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2 \lor \neg x_3)$
Satisfiable: $\Phi = (x_1 \lor x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2 \lor \neg x_3)$

Unsatisfiable:

$\Phi = (x_1 \lor x_1 \lor x_1) \land (\neg x_1 \lor \neg x_1 \lor \neg x_1) \land (x_1 \lor x_2 \lor x_3)$
Schedulability problems for BMSs with convex polytope safety sets are in PTIME for systems with 2 variables.
BMS with two variables

Schedulability problems for BMSs with convex polytope safety sets are in PTIME for systems with 2 variables.

Lemma

Let $R$ be a set of vectors. There is $\vec{v}$ such that $\vec{v} \cdot \vec{r} > 0$ for all $\vec{r} \in R$ if and only if there are $\vec{u}$ and $\vec{r}_{\perp} \in R$ satisfying $\vec{u} \cdot \vec{r}_{\perp} = 0$ and for all $\vec{r} \in R$ either $\vec{u} \cdot \vec{r} > 0$ or $\vec{r} = p \cdot \vec{r}_{\perp}$ for some $p > 0$. 