Pebbles and Branching Programs

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Joint work with

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“Perhaps the principal embarrassment of complexity theory at the present time is its failure to provide techniques for proving non-trivial lower bounds on the complexity of some of the commonest combinatorial and arithmetic problems.”
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Les Valiant, STOC 1975
On Non-Linear Lower Bounds in Computational Complexity

(Constructs linear size superconcentrators)
Complexity Classes

\[
\text{AC}^0(6) \subseteq \text{NC}^1 \subseteq \text{L} \subseteq \text{NL} \subseteq \text{LogCFL}
\subseteq \text{AC}^1 \subseteq \text{NC}^2 \subseteq \text{P} \subseteq \text{NP} \subseteq \text{PH}
\]

As far as is known, \text{AC}^0(6) cannot determine whether a majority of its input bits are ones.

Yet it is open whether \text{AC}^0(6) = \text{PH}.

Here we introduce the

**Tree Evaluation Problem (TEP)**

We show TEP is in \text{LogDCFL}.

We are trying to prove TEP \notin \text{L}
(and TEP \notin \text{NL})
Tree Evaluation Problem
(Generalizes a problem in [Taitslin05])

Tree of height $h = 3$ with heap numbering

$T_d^h$: Balanced $d$-ary tree of height $h$
DEF AULT: $d = 2$
$[k] = \{1, \ldots, k\}$

TEP($h, k$) Applies to $T_2^h$. Assume $h, k \geq 2$

Input:
$v_i \in [k]$ for each leaf $i$
Function $f_i : [k] \times [k] \rightarrow [k]$ for each internal node $i$
(Thus every node $i$ gets a value $v_i \in [k]$)

Output: root value $v_1 \in [k]$

Decision Problem: Does $v_1 = 1$?

Claim: TEP($h, k$) $\in$ LogDCFL
Space-efficient algorithms for TEP come from pebbling

Deterministic algorithms come from ‘black’ pebbling. [Paterson/Hewitt70]

**Rules:**
- Place a pebble on any leaf.
- If both children of node $i$ are pebbled, slide one of them to the parent.
- Remove any pebble at any time.

**Goal:** Pebble the root using a minimum number of pebbles.

**Easy Theorem:** $T^h_2$ requires exactly $h$ pebbles.
Recall: $T^h_2$ requires exactly $h$ pebbles.

Corollary: $\text{TEP}(h, k) \in \text{DSPACE}(h \log k)$

This is NOT a log space algorithm.

Input size $n = (2^h - 1)k^2 \log k$
$log n = \Theta(h + \log k)$
A \( k \)-way BP \( B \) solving TEP(\( h, k \)) is a directed multigraph with nodes called \textit{states}. Each non-final state \( q \) is labeled either with a leaf node \( i \), with \( k \) outedges from \( q \) labeled 1, ..., \( k \) indicating the possible values for \( v_i \), or labeled with \((i, x, y)\) where \( i \) is an internal node and the out-edges are labeled with the possible values for \( f_i(x, y) \). Each final state has a label from \([k]\) indicating the output \( v_1 \).

\( \text{Size}(B) \) is the number of states in \( B \).

A Turing machine \( M \) solving TEP(\( h, k \)) in space \( s(h, k) \) can be simulated by a family of BPs of size \( 2^{O(s(h, k))} \) (the number of possible configurations of \( M \)).
\( \text{Size}(h, k) \) is the number of states in the smallest deterministic BP solving \( \text{TEP}(h, k) \).

\( \text{Size}_h(k) = \text{Size}(h, k) \) for fixed \( h \).

**Lemma** \( \text{Size}_h(k) = O(k^h) \)

**Proof:** \( h \) pebbles suffice to pebble \( T^h_2 \), and for fixed \( h \), the number of steps in the pebbling of \( T^h_2 \) is constant.

This is the best upper bound known for the order of \( \text{Size}_h(k) \).

**Lemma:** A lower bound of \( \text{Size}_h(k) = \Omega(k^{r(h)}) \) for some unbounded function \( r(h) \) implies \( L \neq \text{LogDCFL} \).
Recall best known upper bound:
\[ \text{Size}_h(k) = O(k^h) \]

**Best known lower bounds:**
\[ \text{Size}_h(k) = \Omega(k^3) \] for each \( h \geq 3 \).
(Tight bounds are known for \( h = 2 \) and \( h = 3 \))

\( h = 2 \): \( \text{Size}_2(k) = \Omega(k^2) \)
This is obvious because each state of the BP can only make one query of the form \((i, x, y)\), and there are \( k^2 \) possible values for \((x, y)\).

\( h = 3 \): \( \text{Size}_3(k) = \Omega(k^3) \)
This is *not* obvious, because the number of input variables is only \( O(k^2) \).

**Proof I:** Use Nečiporuk’s method
**Proof II:** Use the “state sequence” method.

Nečiporuk’s method counts the number of BPs on \( s \) states and compares this with the number of functions obtainable by various restrictions of TEP\(_h\)(\( k \)). This method cannot beat \( \Omega(n^2) \) states, and so cannot show TEP \( \notin L \).
**Theorem:** \( \text{Size}_3(k) \geq k^3 \)

**Proof:** ("State Sequence" method)

For \( r, s \in [k] \) let \( E^{r,s} \) be the set of inputs \( I \) s.t.
- \( f^I_1(x, y) = (x + y) \mod k \)
- \( f^I_2(x, y) = f^I_3(x, y) = 0 \) for all \( (x, y) \neq (r, s) \)
- \( v^I_4 = v^I_6 = r \) and \( v^I_5 = v^I_7 = s \)

Thus \( |E^{r,s}| = k^2 \) because each \( I \in E^{r,s} \) determined by \( v^I_2, v^I_3 \).

Let \( \Gamma^{r,s} \) be the set of states which query either \( f_2(r, s) \) or \( f_3(r, s) \). It suffices to show

\[
(*) \quad |\Gamma^{r,s}| \geq k \quad \text{for all} \quad r, s \in [k].
\]

Proof of \((*)\): \((\gamma^I, v^I_i)\) determines the output of \( C(I) \) (the computation on input \( I \)), where \( \gamma^I \) is the last state of \( C(I) \) in \( \Gamma^{r,s} \), and \( i \) is the node queried by \( \gamma^I \).
Thrifty Branching Programs

A deterministic BP solving $\text{TEP}_h(k)$ is *thrifty* if for every query $f_i(x, y)$ (for every input), $(x, y)$ are the values of the children of node $i$.

Thrifty BPs can implement black pebbling, and hence solve $\text{TEP}_h(k)$ with $O(k^h)$ states. It turns out that this is also a lower bound.

**Theorem:** Thrifty deterministic BPs solving $\text{TEP}_h(k)$ have $\Omega(k^h)$ states.

The proof is nontrivial.

Thus any BP beating the $O(k^h)$ upper bound must make queries $f_i(x, y)$ which are irrelevant to the value $v_i$ of the node $i$.

**Thrifty Hypothesis:** Thrifty BPs are optimal among deterministic BPs solving $\text{TEP}_h(k)$.

(Not true for solving the decision version of TEP)
Nondeterministic Branching Programs

Black/White Pebbling: A white pebble can be placed on any node at any time (representing a guess as to the value). The pebble can be removed if the node is a leaf, or both children have pebbles.

\( T_2^h \) can be B/W pebbled with \( \lceil h/2 \rceil + 1 \) pebbles. (This is optimal.)

Recall \( T_2^h \) requires \( h \) pebbles to black pebble it.
Nondeterministic Branching Programs

Black/White Pebbling: A white pebble can be placed on any node at any time (representing a guess as to the value). The pebble can be removed if the node is a leaf, or both children have pebbles.

\[ T_2^h \] can be B/W pebbled with \( \lceil h/2 \rceil + 1 \) pebbles. (This is optimal.)

Recall \( T_2^h \) requires \( h \) pebbles to black pebble it.

Nondeterministic BPs implement B/W pebbling, so \( NSize_h(k) = O(k^{\lceil h/2 \rceil + 1}) \).

For \( h = 3 \) this gives \( O(k^3) \) states, but best lower bound is \( k^{2.5} \) states (via both Nečiporuk and ‘state-sequence’ methods).

This led us to discover “fractional pebbling”.

\( T_2^3 \) can be B/W pebbled with 2.5 pebbles, so

\[ NSize_3(k) = \Theta(k^{2.5}) \].
Fractional Pebbling

Fractional pebbling is like B/W pebbling, except now a node $i$ can have a pair $(b(i), w(i))$ of real values, where

$$0 \leq b(i), w(i) \quad b(i) + w(i) \leq 1$$

If both children of node $i$ have total pebble value 1, then $w(i)$ can be set to 0, and any black fraction can be slid up from the children to increase $b(i)$.

The tree $T^3_2$ can be fractionally pebbled with 2.5 pebbles.

![Fractional Pebbling Diagram]

**Theorem** Thrifty nondeterministic BPs can implement fractional pebbling to solve $\text{TEP}_{h}(k)$.  

\[ 15 \]
**Theorem** Bounds on fractional pebbling.

\[ \# \text{FRpebbles}(T_2^3) = 2.5 \]
\[ \# \text{FRpebbles}(T_2^4) = 3 \]
\[ \frac{h}{2} - 1 \leq \# \text{FRpebbles}(T_2^h) \leq \frac{h}{2} + 1 \]

**Theorem** (Repeat) Thrifty nondeterministic BPs can implement fractional pebbling.

**Corollary** \( N\text{Size}_3(k) = \Theta(k^{2.5}) \)
\( N\text{Size}_4(k) = O(k^3) \)
\( N\text{Size}_h(k) = O(k^{h/2+1}), h \geq 2 \)

(All upper bounds use thrifty BPs)

**Theorem** \( \text{ThiftyNSize}_4(k) = \Theta(k^3) \)

**Open Question:** Can nondeterministic Thrifty BPs beat fractional pebbling bound for \( h > 4 \)?

(Recall that black pebbling is optimal for deterministic thrifty BPs.)
Thrifty Hypothesis: Thrifty BPs are optimal among deterministic $k$-way BPs solving $\text{TEP}_h(k)$.
(i.e. $\text{Size}_h(k) = \Omega(k^h)$.)

In other words, the black pebbling method is the most space-efficient deterministic method for solving $\text{TEP}_h(k)$.

A proof implies $L \neq \text{LogDCFL}$
(so $\text{NC}^1 \subsetneq \text{NC}^2$).

A disproof would involve a new space-efficient algorithm and would also be interesting (think superconcentrators).

Next Step: Prove or disprove $\text{Size}_4(k) = \Omega(k^4)$
(Best known bound: $\text{Size}_4(k) = \Omega(k^3)$.)

Separating $L$ from $P$ is important!