Game-Theoretic Learning:

Regret Minimization vs. Utility Maximization

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Background

No-external-regret learning converges to the set of minimax equilibria, in zero-sum games. [e.g., Freund and Schapire 1996]

No-internal-regret learning converges to the set of correlated equilibria, in general-sum games. [e.g., Foster and Vohra 1997]

Foreground

1. Definitions
   - A continuum of no-regret properties, called no-Φ-regret.
   - A continuum of game-theoretic equilibria, called Φ-equilibria.
2. Existence Theorem
   - Constructive proof: No-Φ-regret learning algorithms exist, $\forall \Phi$.
3. Convergence Theorem
   - No-Φ-regret learning converges to the set of Φ-equilibria, $\forall \Phi$.
4. Surprising Result
   - No-internal-regret is the strongest form of no-Φ-regret learning.
   - Therefore, no no-Φ-regret algorithm learns Nash equilibria.

Outline

- Game Theory
- Single Agent Learning Model
- Multiagent Learning & Game-Theoretic Equilibria

Game Theory: A Crash Course

1. General-Sum Games
   - Nash Equilibrium
   - Correlated Equilibrium
2. Zero-Sum Games
   - Minimax Equilibrium

An Example

Prisoners' Dilemma

	C	D
C	$4, 4$	$0, 5$
D	$5, 0$	$1, 1$

C: Cooperate

D: Defect

One-Shot Games

A one-shot game is a 3-tuple $\Gamma = (I, (A_i, r_i)_{i \in I})$, where

- I is a set of players
- for all players $i \in I$,
  - a set of pure actions A_i with $a_i \in A_i$
  - a reward function $r_i : A \to \mathbb{R}$, where $A = \prod_{i \in I} A_i$ with $a \in A$

$\mathbb{R}$

$\mathbb{R}$

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  - a reward function $r_i : A \rightarrow \mathbb{R}$, where $A = \prod_{i \in I} A_i$ with $a \in A$

The players can employ randomized or mixed actions:

- for all players $i \in I$,
  - a set of mixed actions $Q_i = \{q_i \in \mathbb{R}^{A_i} | \sum_j q_{ij} = 1 \ \& \ q_{ij} \geq 0, \forall j\}$, with $q_i \in Q_i$
  - an expected reward function $r_i : Q \rightarrow \mathbb{R}$, where $Q = \prod_{i \in I} Q_i$ with $q \in Q$, *s.t.* $r_i(q) = \sum_{a \in A} q(a) r_i(a)$

Nash Equilibrium

Notation

Write $a = (a_i, a_{-i}) \in A$ for $a_i \in A_i$ and $a_{-i} \in A_{-i} = \prod_{j \neq i} A_j$.

Write $q = (q_i, q_{-i}) \in Q$ for $q_i \in Q_i$ and $q_{-i} \in Q_{-i} = \prod_{j \neq i} Q_i$.

Definition

A Nash equilibrium is a mixed action profile q^* s.t. $r_i(q^*) \geq r_i(q_i, q^*_{-i})$, for all players i and for all mixed actions $q_i \in Q_i$.

Theorem [Nash 51]

Every finite strategic form game has a mixed strategy Nash equilibrium.

Correlated Equilibrium

Chicken

	L	R
T	6,6	2,7
B	7,2	0,0

CE

	L	R
T	1/2	1/4
B	1/4	0

$$\max 12\pi_{TL} + 9\pi_{TR} + 9\pi_{BL} + 0\pi_{BR}$$

subject to

$$\pi_{TL} + \pi_{TR} + \pi_{BL} + \pi_{BR} = 1$$
$$\pi_{TL}, \pi_{TR}, \pi_{BL}, \pi_{BR} \geq 0$$

$$\begin{aligned}
6\pi_{L|T} + 2\pi_{R|T} &\geq 7\pi_{L|T} + 0\pi_{R|T} \\
7\pi_{L|B} + 0\pi_{R|B} &\geq 6\pi_{L|B} + 2\pi_{R|B} \\
6\pi_{T|L} + 2\pi_{B|L} &\geq 7\pi_{T|L} + 0\pi_{B|L} \\
7\pi_{T|R} + 0\pi_{B|R} &\geq 6\pi_{T|R} + 2\pi_{B|R}
\end{aligned}$$

Correlated Equilibrium

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subject to

$$\pi_{TL} + \pi_{TR} + \pi_{BL} + \pi_{BR} = 1$$
$$\pi_{TL}, \pi_{TR}, \pi_{BL}, \pi_{BR} \geq 0$$

$$\begin{aligned}
6\pi_{TL} + 2\pi_{TR} &\geq 7\pi_{TL} + 0\pi_{TR} \\
7\pi_{BL} + 0\pi_{BR} &\geq 6\pi_{BL} + 2\pi_{BR} \\
6\pi_{TL} + 2\pi_{BL} &\geq 7\pi_{TL} + 0\pi_{BL} \\
7\pi_{TR} + 0\pi_{BR} &\geq 6\pi_{TR} + 2\pi_{BR}
\end{aligned}$$

Correlated Equilibrium

Definition

A mixed action profile $q^* \in Q$ is a correlated equilibrium iff for all pure actions $j, k \in A_i$,

$$\sum_{a_{-i} \in A_{-i}} q(j, a_{-i}) \left(r_i(j, a_{-i}) - r_i(k, a_{-i})\right) \geq 0 \tag{1}$$

Observe

Every Nash equilibrium is a correlated equilibrium $\Rightarrow$
Every finite normal form game has a correlated equilibrium.

Zero-Sum Games

Matching Pennies

	H	T
H	$-1, 1$	$1, -1$
T	$1, -1$	$-1, 1$

Rock-Paper-Scissors

	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$

$\sum_{i \in I} r_i(a) = 0$, for all $a \in A$

$\sum_{i \in I} r_i(a) = c$, for all $a \in A$, for some $c \in \mathbb{R}$

Minimax Equilibrium

Example

	L	R
T	1	2
B	4	3

Definition

A mixed action profile $(q_1^*, q_2^*) \in Q$ is a minimax equilibrium in a two-player, zero-sum game iff

- $r_1(q_1^*, q_2^*) \geq r_1(j, q_2^*)$, $\forall j \in A_1$
- $l_2(q_1^*, q_2^*) \leq l_2(q_1^*, k)$, $\forall k \in A_2$

Single Agent Learning Model

- set of actions $N = \{1, \ldots, n\}$
- for all times t,
  - mixed action vector $q^t \in Q = \{q \in \mathbb{R}^n | \sum_i q_i = 1 \ \& \ q_i \geq 0, \forall i\}$
  - pure action vector $a^t = e_i$ for some pure action i
  - reward vector $r^t = (r_1, \ldots, r_n) \in [0, 1]^n$

A learning algorithm $\mathcal{A}$ is a sequence of functions $q^t : \text{History}^{t-1} \rightarrow Q$, where a History is a sequence of action-reward pairs $(a^1, r^1), (a^2, r^2), \ldots$.

Transformations

Mixed Transformations

$$\Phi_{\text{LINEAR}} = \{\phi : Q \to Q\}$$
$$= \text{the set of all linear transformations}$$
$$= \text{the set of all row stochastic matrices}$$

$$\Phi_{\text{SWAP}} = \{\phi : Q \to Q \mid \phi \text{ deterministic}\} \subset \Phi_{\text{LINEAR}}$$

Pure Transformations

$$\mathcal{F}_{\text{SWAP}} = \{F : N \to N\}$$
$$= \text{the set of all pure transformations}$$

Isomorphism

The operation of elements of $\mathcal{F}_{\text{SWAP}}$ on $N \cong$ the operation of elements of Φ_{SWAP} on Q

$$\phi_{ij} = \delta_{F(i)=j} \quad (2)$$

$$\forall k \quad e_k\phi = e_{F(k)} \quad (3)$$

Example If $n = 4$ and $F = \{1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 4, 4 \mapsto 1\}$, then

$$\phi = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $\langle q_1, q_2, q_3, q_4 \rangle \phi = \langle q_4, q_1, q_2, q_3 \rangle$, for all $\langle q_1, q_2, q_3, q_4 \rangle \in Q$.

External Regret Matrices

$\mathcal{F}_{\text{EXT}} = \{F^j \in \mathcal{F}_{\text{SWAP}} | j \in N\}$, where $F^j(k) = j$
$\Phi_{\text{EXT}} = \{\phi^j \in \Phi_{\text{SWAP}} | j \in N\}$, where $e_k \phi^j = e_j$

Example If $n = 4$, then

$$\phi^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Thus, $\langle q_1, q_2, q_3, q_4 \rangle \phi = \langle 0, 1, 0, 0 \rangle$, for all $\langle q_1, q_2, q_3, q_4 \rangle \in Q$.

Internal Regret Matrices

$$\mathcal{F}_{\text{INT}} = \{F^{ij} \in \mathcal{F}_{\text{SWAP}} | ij \in N\}, \text{ where } F^{ij}(k) = \begin{cases} j & \text{if } k = i \\ k & \text{otherwise} \end{cases}$$

$$\Phi_{\text{INT}} = \{\phi^{ij} \in \Phi_{\text{SWAP}} | ij \in N\}, \text{ where } e_k\phi^{ij} = \begin{cases} e_j & \text{if } k = i \\ e_k & \text{otherwise} \end{cases}$$

Example If $n = 4$, then

$$\phi^{23} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, $\langle q_1, q_2, q_3, q_4 \rangle \phi = \langle q_1, 0, q_2 + q_3, q_4 \rangle$, for all $\langle q_1, q_2, q_3, q_4 \rangle \in Q$.

Regret Vector $\rho \in \mathbb{R}^{\Phi}$

Observed Regret Vector $\tilde{\rho}_\phi(r, a) = r \cdot a\phi - r \cdot a$

Expected Regret Vector
$$\begin{aligned}\hat{\rho}_\phi(r, q) &= \mathbb{E}[\rho_\phi(r, a) \mid a \sim q] \\ &= \rho_\phi(r, \mathbb{E}[a \mid a \sim q]) \\ &= r \cdot q\phi - r \cdot q\end{aligned}$$

No Observed Φ-Regret $\limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=1}^{t} \tilde{\rho}_\phi(r^\tau, a^\tau) \leq 0$, for all $\phi \in \Phi$, a.s.

No Expected Φ-Regret $\limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=1}^{t} \hat{\rho}_\phi(r^\tau, q^\tau) \leq 0$, for all $\phi \in \Phi$

Approachability

$U \subseteq V$ is said to be approachable iff there exists learning algorithm $\mathcal{A} = q^1, q^2, \ldots$ *s.t.* for any sequence of rewards $r^1, r^2, \ldots,$

$$\lim_{t \to \infty} d(U, \bar{\rho}^t) = \lim_{t \to \infty} \inf_{u \in U} d(u, \bar{\rho}^t) = 0$$

a.s., where $\bar{\rho}^t$ denotes the average value of ρ through time t.

A Φ-no-regret learning algorithm is one whose observed regret approaches the negative orthant $\mathbb{R}^{\Phi}_{-}$.

Blackwell's Theorem

The negative orthant $\mathbb{R}_-^\Phi$ is approachable iff there exists a learning algorithm $\mathcal{A} = q^1, q^2, \ldots$ *s.t.* for any sequence of rewards $r^1, r^2, \ldots$,

$$\rho(r^{t+1}, q^{t+1}) \cdot (\bar{\rho}^t)^+ \leq 0 \tag{4}$$

for all times t, where $x^+ = \max\{x, 0\}$.

Moreover, this procedure can be used to approach the negative orthant $\mathbb{R}_-^\Phi$:

- if $\bar{\rho}^t \in \mathbb{R}_-^\Phi$, play arbitrarily;
- if $\bar{\rho}^t \in V \setminus \mathbb{R}_-^\Phi$, play according to $\mathcal{A}$.

Regret Matching Algorithm

Given Φ
Given $Y \in \mathbb{R}_{+}^{\Phi}$

If $\sum_{\phi\in\Phi} Y_\phi = 0$, play arbitrarily
If $\sum_{\phi\in\Phi} Y_\phi > 0$, define stochastic matrix

$$A \equiv A(\Phi, Y) = \frac{\sum_{\phi\in\Phi} \phi Y_\phi}{\sum_{\phi\in\Phi} Y_\phi} \tag{5}$$

play mixed strategy $q = qA$

Regret Matching Theorem

Regret matching satisfies the generalized Blackwell condition:

$$\rho(r, q) \cdot Y = 0$$

Proof

$$
\begin{aligned}
\rho(r,q)\cdot Y &= \sum_{\phi\in\Phi} \rho_\phi(r,q) Y_\phi && (6)\\
&= \sum_{\phi\in\Phi} (r\cdot q\phi - r\cdot q) Y_\phi && (7)\\
&= \sum_{\phi\in\Phi} r\cdot (q\phi Y_\phi - q Y_\phi) && (8)\\
&= r\cdot\left(q\sum_{\phi\in\Phi}\phi Y_\phi - q\sum_{\phi\in\Phi} Y_\phi\right) && (9)\\
&= \left(\sum_{\phi\in\Phi} Y_\phi\right) r\cdot\left(q\frac{\sum_{\phi\in\Phi}\phi Y_\phi}{\sum_{\phi\in\Phi} Y_\phi} - q\right) && (10)\\
&= \left(\sum_{\phi\in\Phi} Y_\phi\right) r\cdot (qA - q) && (11)\\
&= \left(\sum_{\phi\in\Phi} Y_\phi\right) r\cdot (q - q) && (12)\\
&= 0 && (13)
\end{aligned}
$$

Generic Regret Matching Algorithm (Φ, g)

for $t = 1, \ldots, T$

1. play mixed strategy q^t
2. realize pure action i
3. observe rewards r^t
4. for all $\phi \in \Phi$
   - compute instantaneous regret
     * observed $\rho^t_\phi \equiv \rho_\phi(r^t, e_i) = r^t \cdot e_i\phi - r^t \cdot e_i$
     * expected $\rho^t_\phi \equiv \rho_\phi(r^t, q^t) = r^t \cdot q^t\phi - r^t \cdot q^t$
   - update cumulative regret vector $X^t_\phi = X^{t-1}_\phi + \rho^t_\phi$
5. compute $Y = g(X^t)$
6. compute $A = \frac{\sum_{\phi\in\Phi} \phi Y_\phi}{\sum_{\phi\in\Phi} Y_\phi}$
7. solve for the fixed point $q^{t+1} = q^{t+1}A$

Special Cases of Regret Matching

Foster and Vohra 97 (Φ_{INT})
Hart and Mas-Colell 00 (Φ_{EXT})
Choose $G(X) = \frac{1}{2}\sum_k (X_k^+)^2$ so that $g_k(X) = X_k^+$

Freund and Schapire 95 (Φ_{EXT})
Cesa-Bianchi and Lugosi 03 (Φ_{INT})
Choose $G(X) = \frac{1}{\eta} \ln \left(\sum_k e^{\eta X_k}\right)$ so that $g_k(X) = e^{\eta X_k} / \sum_k e^{\eta X_k}$

Multiagent Model

- a set of players I ($i \in I$)
- for all players i,
  - a set of pure actions A_i with $a_i \in A_i$
  - a set of mixed actions Q_i with $q_i \in Q_i$
  - a reward function $r_i : A \to [0, 1]$, where $A = \prod_i A_i$ with $a \in A$
  - an expected reward function $r_i : Q \to [0, 1]$, where $Q = \prod_i Q_i$ with $q \in Q$ *s.t.* $r_i(q) = \sum_{a \in A} q(a) r_i(a)$
  - a set Φ_i ($\phi_i \in \Phi_i$)

Φ-Equilibrium

An mixed action profile $q \in Q$ is a Φ-equilibrium iff $r_i(\phi_i(q)) \leq r_i(q)$, for all players i and for all $\phi_i \in \Phi_i$.

Examples

Correlated Equilibrium: $\Phi_i = \Phi_{\text{INT}}$, for all players i

Generalized Minimax Equilibrium: $\Phi_i = \Phi_{\text{EXT}}$, for all players i

Convergence Theorem

If all players i play via some no-Φ_i-regret algorithm, then the joint empirical distribution of play converges to the set of Φ-equilibria, almost surely.

Proof

For all players i, for all $\phi_i \in \Phi_i$,

$$
\begin{aligned}
& \limsup_{t\to\infty} r_i(\phi_i(z^t)) - r_i(z^t) && (14)\\
= \ & \limsup_{t\to\infty} \frac{1}{t}\sum_{\tau=1}^{t} r_i(\phi_i(a_i^\tau), a_{-i}^\tau) - \frac{1}{t}\sum_{\tau=1}^{t} r_i(a_i^\tau, a_{-i}^\tau) && (15)\\
\leq \ & 0 && (16)
\end{aligned}
$$

almost surely.

Zero-Sum Games

Matching Pennies

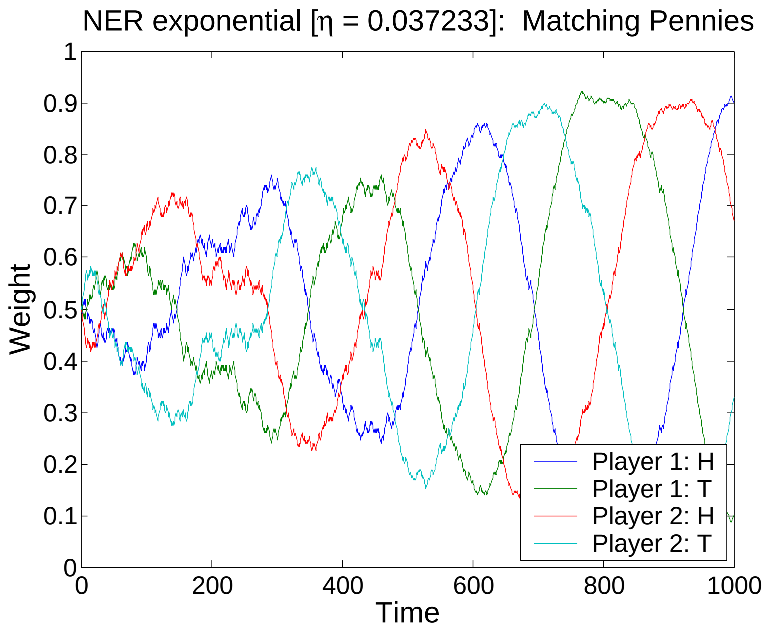
	H	T
H	$-1, 1$	$1, -1$
T	$1, -1$	$-1, 1$

Rock-Paper-Scissors

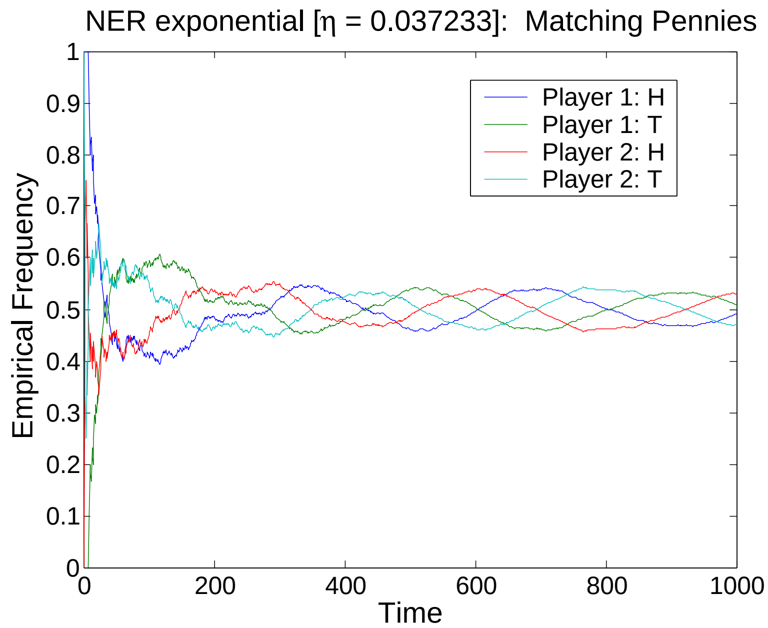
	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$

Matching Pennies

Weights

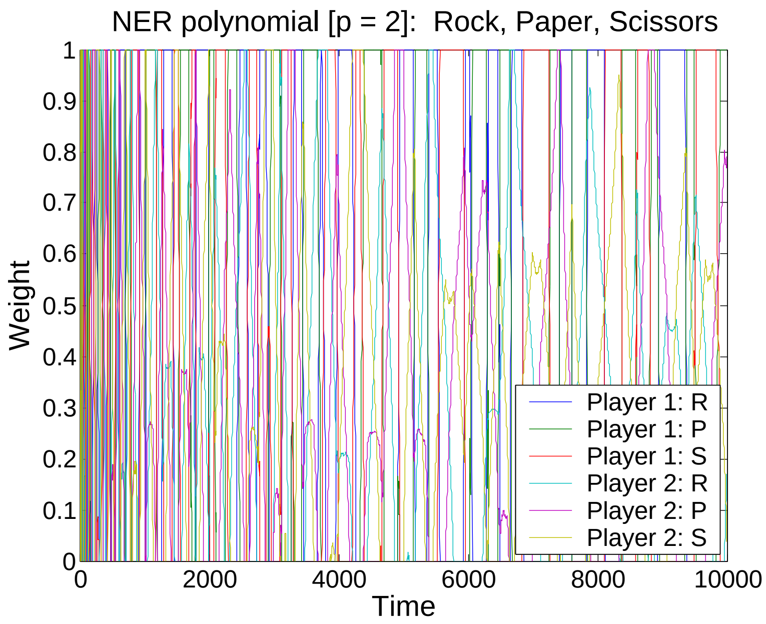


Frequencies



Rock-Paper-Scissors

Weights



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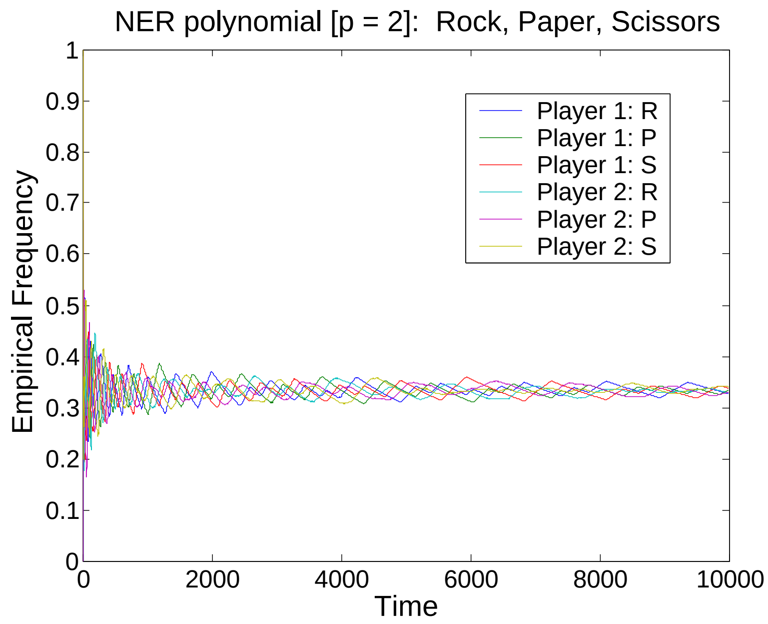


General-Sum Games

Shapley Game

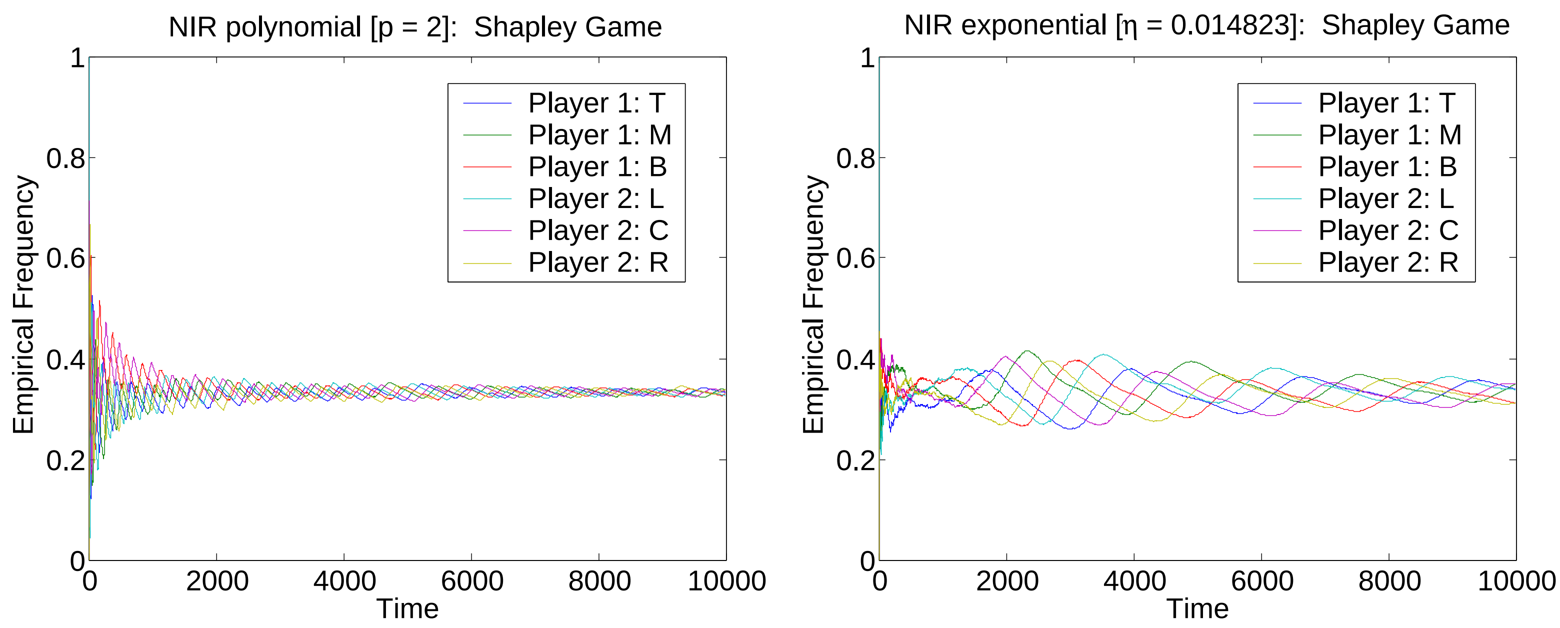
	L	C	R
T	$0,0$	$1,0$	$0,1$
M	$0,1$	$0,0$	$1,0$
B	$1,0$	$0,1$	$0,0$

Correlated Equilibrium

	L	C	R
T	0	$1/6$	$1/6$
M	$1/6$	0	$1/6$
B	$1/6$	$1/6$	0

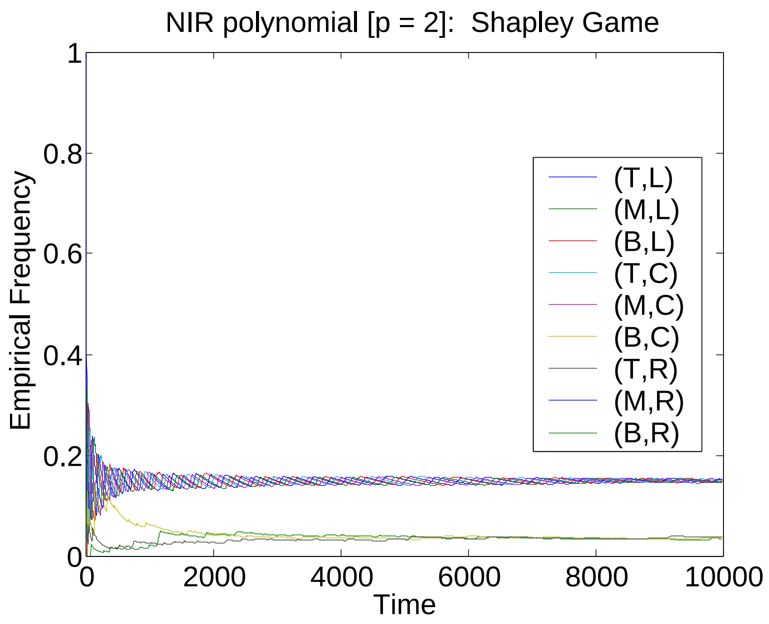
Shapley Game: No Internal Regret Learning

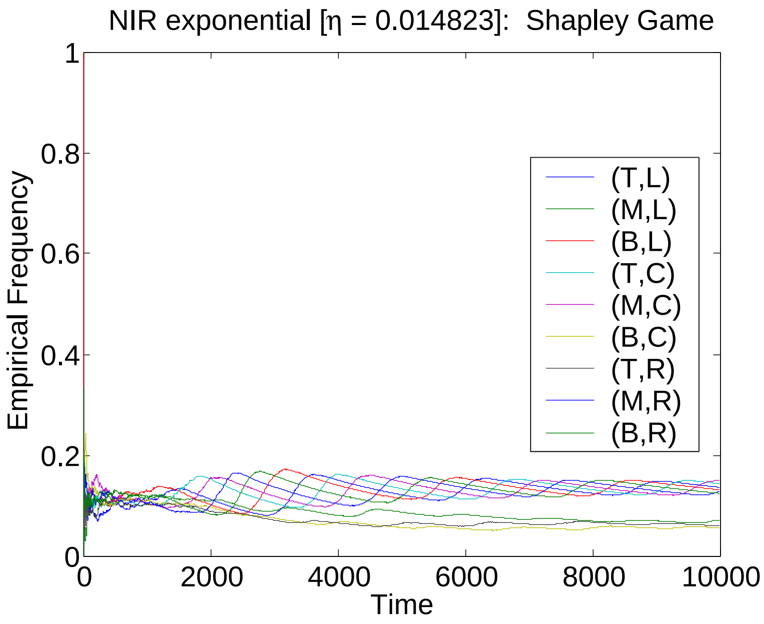
Frequencies



Shapley Game: No Internal Regret Learning

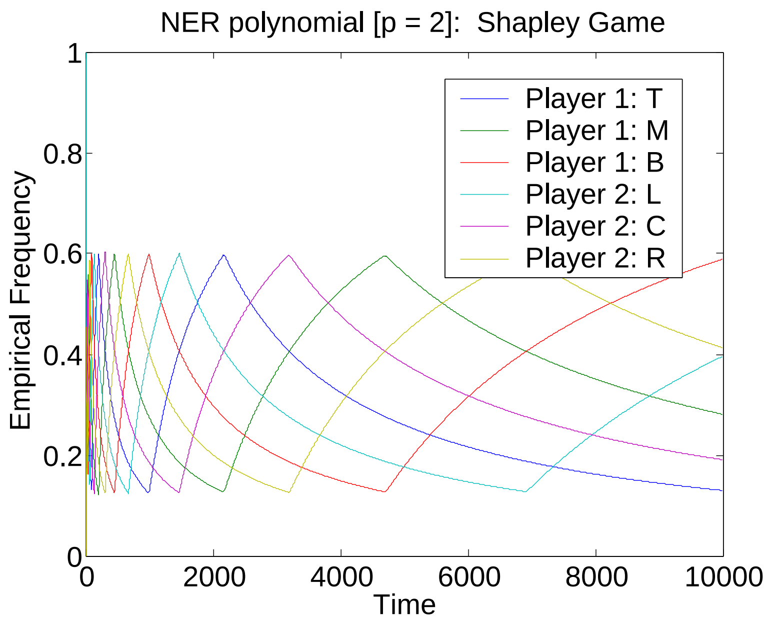
Joint Frequencies

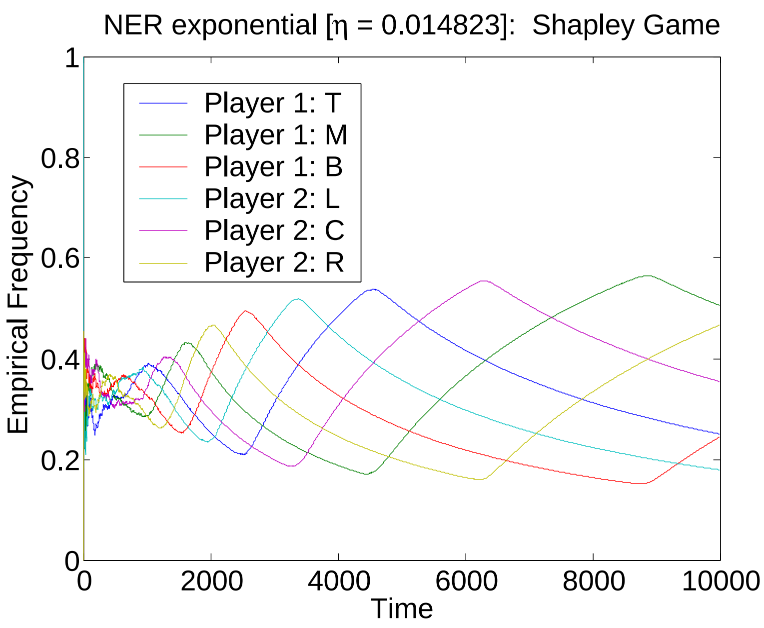




Shapley Game: No External Regret Learning

Frequencies





Summary

- No-external- and no-internal-regret can be defined along one continuum, no-Φ-regret.
- No-Φ-regret learning algorithms exist, $\forall \Phi$.
- No-Φ-regret learning converges to the set of Φ-equilibria, $\forall \Phi$.
- No-internal-regret learning is the strongest form of no-Φ-regret learning. Therefore, Nash equilibrium cannot be learned via no-Φ-regret learning.

"A little rationality goes a long way" [Hart 03]

Regret Minimization vs. Utility Maximization

- RM is easy to implement.
- RM justifies randomness in actions.
- Can RM be used to explain human behavior?