Approximation Bound Refinement of KLS

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1 Overview

Below is the description of a refinement on Lemmas 3 and 4 of KLS, leading to a lower representation size and computation complexity. The resulting versions of those lemmas follow.

Lemma 1 *Let the mixed strategies $\vec{p}$, $\vec{q}$ for $(G, \mathcal{M})$ satisfy $|p_i - q_i| \leq \tau/2$ for all i. Then*

$$|M_i(\vec{p}) - M_i(\vec{q})| \leq ((1+\tau)^k - 1)/2.$$

Lemma 2 *Let $\vec{p}$ be a Nash equilibrium for $(G, \mathcal{M})$ and let $\vec{q}$ be the nearest (in L_1 metric) mixed strategy on the τ-grid. Then, $\vec{q}$ is a $((1+\tau)^k - 1)$-Nash equilibrium for $(G, \mathcal{M})$.*

2 Bound revision

Let us first introduce some notation. As in the original expression of the bound, let $k \equiv |N_G(i)|$. Let the mixed strategies $\vec{p}$ and $\vec{q}$ be such that $\forall i, p_i = q_i + \Delta_i$, and their largest coordinate-wise difference $\Delta \equiv \max_i |\Delta_i|$. Since we will concentrate on the neighborhood of player i, we index the players in the neighborhood by $j \in N_G(i)$. Also, denote the set of players for which $\vec{p}$ and $\vec{q}$ differ by $D \equiv \{i : \Delta_i \neq 0\}$, those set of players in the local neighborhood of i by $D_i \equiv N_G(i) \bigcap D$, and the number of differing local players by $k' \equiv |D_i|$. For any $s \in \{1, \ldots, k'\}$, we index the set of all subsets of size s in D_i by $J_s \in \{\{j_1, \ldots, j_s\} \subseteq D_i\}$ and denote its complement $J_s^c \equiv N_G(i) \backslash J_s$. . We will also denote $(J_s, J_s^c) \equiv N_G(i)$ and $(\vec{x}^{J_s}, \vec{x}^{J_s^c}) \equiv \vec{x}$. Consider the expected payoff of player i under $\vec{p}$

$$\begin{aligned} M_i(\vec{p}) &= \sum_{\vec{x} \in \{0,1\}^k} \prod_j p_j^{x_j} (1-p_j)^{1-x_j} M_i(\vec{p}[N_G(i) : \vec{x}]) \\ &= \sum_{\vec{x} \in \{0,1\}^k} \left[\prod_j \left(q_j^{x_j} (1-q_j)^{1-x_j} + (-1)^{1-x_j} \Delta_j \right) \right] M_i(\vec{p}[N_G(i) : \vec{x}]) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\vec{x}\in\{0,1\}^k} \left[\prod_j q_j^{x_j}(1-q_j)^{1-x_j} + \right. \\
&\quad \left. \sum_{s=1}^{k'} \sum_{J_s} \left(\prod_{j'\in J_s} (-1)^{1-x_{j'}} \Delta_{j'} \right) \left(\prod_{j\in J_s^c} q_j^{x_j}(1-q_j)^{1-x_j} \right) \right] M_i(\vec{p}[N_G(i):\vec{x}]) \\
&= \sum_{\vec{x}\in\{0,1\}^k} \left(\prod_j q_j^{x_j}(1-q_j)^{1-x_j} \right) M_i(\vec{p}[N_G(i):\vec{x}]) + \\
&\quad \sum_{\vec{x}\in\{0,1\}^k} \sum_{s=1}^{k'} \sum_{J_s} \left(\prod_{j'\in J_s} (-1)^{1-x_{j'}} \Delta_{j'} \right) \left(\prod_{j\in J_s^c} q_j^{x_j}(1-q_j)^{1-x_j} \right) M_i(\vec{p}[N_G(i):\vec{x}]) \\
&= M_i(\vec{q}) + \sum_{s=1}^{k'} \sum_{J_s} \sum_{\vec{x}^{J_s}\in\{0,1\}^s} \left(\prod_{j'\in J_s} (-1)^{1-x_{j'}} \Delta_{j'} \right) \\
&\quad \sum_{\vec{x}^{J_s^c}\in\{0,1\}^{k-s}} \left(\prod_{j\in J_s^c} q_j^{x_j}(1-q_j)^{1-x_j} \right) M_i(\vec{p}[N_G(i):\vec{x}]) \\
&= M_i(\vec{q}) + \sum_{s=1}^{k'} \sum_{J_s} \sum_{\vec{x}^{J_s}} \left(\prod_{j'\in J_s} (-1)^{1-x_{j'}} \Delta_{j'} \right) M_i(\vec{q}[J_s:\vec{x}^{J_s}]) \\
&= M_i(\vec{q}) + \sum_{s=1}^{k'} \sum_{J_s} \left(\prod_{j'\in J_s} \Delta_{j'} \right) \\
&\quad \left[\sum_{\vec{x}^{J_s}:\sum_{l=1}^{s} x_l^{J_s}\ \text{even}} M_i(\vec{q}[J^s:\vec{x}^{J_s}]) - \sum_{\vec{x}^{J_s}:\sum_{l=1}^{s} x_l^{J_s}\ \text{odd}} M_i(\vec{q}[J_s:\vec{x}^{J_s}]) \right] \\
&\le M_i(\vec{q}) + \sum_{s=1}^{k'} 2^{s-1} \sum_{J_s} \left(\prod_{j'\in J_s} |\Delta_{j'}| \right) \\
&\le M_i(\vec{q}) + \sum_{s=1}^{k'} 2^{s-1} \Delta^s \sum_{J_s} 1 \\
&= M_i(\vec{q}) + 2^{-1} \sum_{s=1}^{k'} \binom{k'}{s} (2\Delta)^s \\
&= M_i(\vec{q}) + 2^{-1}((1+2\Delta)^{k'} - 1) \\
&\le M_i(\vec{q}) + 2^{-1}(e^{2k'\Delta} - 1) \\
&\le M_i(\vec{q}) + k'\Delta(1+2k'\Delta) \\
&\le M_i(\vec{q}) + 2k'\Delta.
\end{aligned}
$$

The lower bound follows similarly. So we have, for any pair of mixed strategies $\vec{p}$ and $\vec{q}$ such that $\| \vec{p} - \vec{q} \|_1 \leq \Delta$, $|M_i(\vec{p}) - M_i(\vec{q})| \leq ((1 + 2\Delta)^{k'} - 1)/2$ (other simpler bounds are possible—see above).

Now consider a discretization scheme with τ-size grid. For any mixed strategy $\vec{p}$ there exists a mixed strategy $\vec{q}$ on the τ-grid at most $\tau/2$ away (in L_1 metric). In particular, if $\vec{p}$ is a NE, then for the nearest (in L_1 metric) mixed strategy $\vec{q}^{\,*}$ on the τ-grid is a $((1+\tau)^k - 1)$-NE for the game

$$\begin{aligned}
M_i(\vec{q}^{\,*}) + ((1+\tau)^k - 1)/2 &\geq M_i(\vec{p}) \\
&= \max_{a \in \{0,1\}} M_i(\vec{p}[i:a]) \\
&\geq \max_{a \in \{0,1\}} M_i(\vec{q}^{\,*}[i:a]) - ((1+\tau)^{k-1} - 1)/2 \\
M_i(\vec{q}^*) &\geq \max_{a \in \{0,1\}} M_i(\vec{q}^{\,*}[i:a]) - ((1+\tau)^k - 1).
\end{aligned}$$

Therefore, for an ϵ-NE, we require $\tau \leq (1+\epsilon)^{1/k} - 1 \leq \epsilon/(2k))$. Hence, the size of each (local) table is $\lceil 1/\tau \rceil^2 \leq (1/((1+\epsilon)^{1/k} - 1) + 1)^2 \leq (2k/\epsilon + 1)^2$ and computation is $O((1/\tau)^{2k}) = O((1/((1+\epsilon)^{1/k} - 1) + 1)^{2k}) = O((2k/\epsilon + 1)^{2k})$.