

Computational Game Theory (CIS 620/OPIM 952)  
Large Population Models  
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Notation:

$\bar{p}$  - population mixed strategy  
 $M_i(\bar{p})$  - expected payoff to player  $i$   
under  $\bar{p}$

$\bar{p}[i:b]$  -  $\bar{p}$  but with  $p_i$   
replaced by action  $b$

Potential Games (Monderer & Shapley)

Natural idea: a global function whose  
maxima correspond to N.E.

Assume  $n$  players, 2 actions each.

Say that  $F: \{0,1\}^n \rightarrow \mathbb{R}$  is a potential  
for the game if:

$\forall$  pure population strategy  $\bar{x} \in \{0,1\}^n$   
 $\forall$  player  $i, \forall b \in \{0,1\}$

$$M_i(\bar{x}[i:b]) - M_i(\bar{x}[i:\bar{x}_i]) > 0$$

$$\iff F(\bar{x}[i:b]) - F(\bar{x}[i:\bar{x}_i]) > 0$$

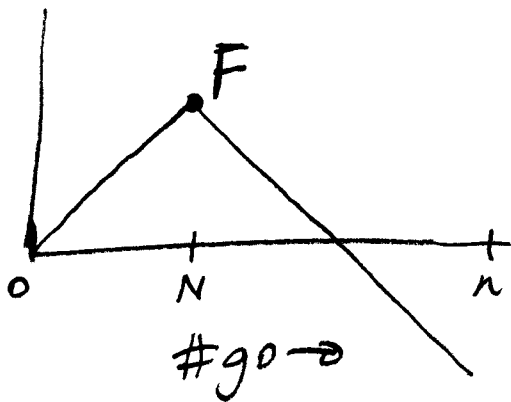
( $F$  independent of  $i$ !)

Some remarks:

- "Natural" choice  $F(\bar{x}) \triangleq \sum_i M_i(\bar{x})$   
may fail - improvement to  $M_i$  may be offset by reduction to  $M_j$
- Not necessarily a "team" or cooperative game

Example: Santa Fe Bar Problem

- $n$  potential patrons
- actions: go to bar or stay home
- stay always pays 0
- go pays  $\begin{cases} 1 & \text{if } \leq N \text{ people at bar} \\ -1 & \text{else} \end{cases}$
- define  $F(\bar{x}) = \begin{cases} \# \text{ of "go" if } \leq N \\ N - (\# \text{ go} - N) \text{ else} \\ \text{"excess"} \\ (= 2N - \# \text{ go}) \end{cases}$



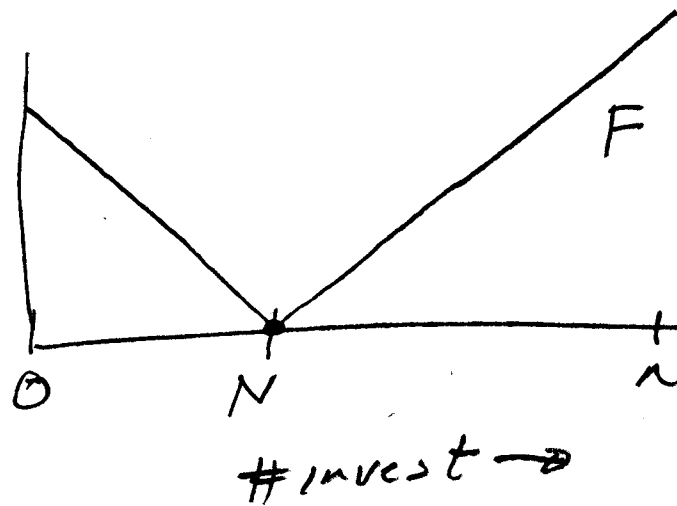
Example of  
a congestion game

## Example: Interdependent Security

- $n$  airlines
- actions: invest in security or don't invest
- recall model (homogeneous case)

if  $\# \text{invest} \leq N$ , don't pays better  
if  $\# \text{invest} > N$ , invest pays better

- define  $F(\bar{x}) = |\# \text{invest} - N|$



Note:  $F$  not nec. a measure of "goodness"

### Obvious Theorem:

Any maxima of  $F$  (over  $\{0, 1\}^n$ )  
is a pure N.E. of the game.

### Corollary:

Any potential game has a pure N.E.  
May miss mixed N.E.

## Computation?

- Can always move closer to N.E. by incrementally modifying  $\bar{x}$  to improve  $F$
- For SF Bar and IDS, this will find N.E. quickly
- But in general, "improvement path" may traverse all of  $\{0,1\}^n$   
(note players may reverse their choices w/o  $\bar{x}$  cycling)

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## Summarization Games

Assume global summarization function:

$$S: \{0,1\}^n \rightarrow [0,1]$$

Intuition:  $S$  summarizes the population action sufficient to compute payoffs

Voting example:  $S(\bar{x}) = \sum_i x_i$

or weighted:  $S(\bar{x}) = \sum_i u_i x_i$

Assume:  $M_i(\bar{x}[i:b]) = F_i^b(S(\bar{x}[i:b]))$

(so  $i$  has  $F_i^0 \neq F_i^b$ )

Assume the  $F_i^b$  are continuous & bounded derivatives  
(bound =  $\rho$ )

$\Rightarrow$  Assume  $S$  has bounded influence:

$$\zeta_i \triangleq \max_{\bar{x} \in \{0,1\}^n} \{ |S(\bar{x}[i:0]) - S(\bar{x}[i:1])| \}$$

$$\zeta \triangleq \max_i \{ \zeta_i \}$$

### Remarks

- Though  $F_i^0$  &  $F_i^1$  are smooth, can differ greatly  $\Rightarrow$  i's action has unbounded impact on i's payoff
- But i's action has only influence  $\leq \zeta$  on  $S$   
 $\Rightarrow$  i has limited impact on payoffs of others
- Smoothness of  $F_i^b$  prevents "readout" of bits from  $S$
- Crucial: often expect  $\zeta$  to decay with  $n$ !  
E.g. voting:  $\zeta = 1/n$

## Computational Result:

$\rho$  = bound on derivatives of  $F_i^b$

$\zeta$  = bound on influence of  $S$

$\varepsilon$  = input parameter

assume unit cost of evaluating  
 $S$  &  $F_i^b$

An algorithm whose

running time is poly in  $n, 1/\varepsilon$  and  $\rho$ ;

that outputs a pure

$$O(\varepsilon + \zeta\rho) - N.E.$$

(Algorithm & analysis:  
see K. & Mansour paper)

