Network Formation Games

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Problem statement

Costs of equilbria

- Price of anarchy
- Price of stability
- Reachable equilibria

Setting - informal

Imagine a group of people want to use an existing network and will buy the right to use certain communication channels (edges).

To simplify, each wants to connect to a specific target destination from a specific source.

Players who use the same channel (edge) will share its cost.

Setting

Formally:

- A graph (directed or undirected) $G = (V, E)$;
- Each edge e has an associated cost c_e
- n players $1, \ldots n$; player i wants to buy a path from a node s_i to a node t_i, both in V.
- For player i the set of available strategies is the set of paths $\mathcal{P}_i$ from s_i to t_i. Each player chooses a path $P_i \in \mathcal{P}_i$.
- Players who use edge e divide the cost of e according to some mechanism. A player i pays

$$\sum_{e \in P_i} f_i(e),$$

where $f_i(e)$ is i's share of the cost of edge e.

Egalitarian cost sharing

If n_e players use edge $e \in E$, each of them pays an equal share: c_e / n_e.

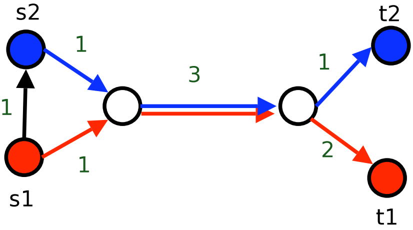


Figure: Cost sharing example

1 pays $1 + 1.5. + 2 = 4.5$;
2 pays $1 + 1.5 + 1 = 3.5$.

Optimal central solution

Let player i use path P_i to connect s_i to t_i. The total cost of a solution S is:

$$c(S) = \sum_i \sum_{e \in P_i} c_e / n_e = \sum_{e \in \bigcup P_i} \sum_{j=1}^{n_e} c_e / n_e = \sum_{e \in \bigcup P_i} c_e.$$

The optimal solution is the smallest subgraph T of G s.t. $\forall i$ there exists a path from s_i to t_i in T.

For undirected graphs: the Steiner tree on $X = \bigcup s_i \cup \bigcup t_i$.

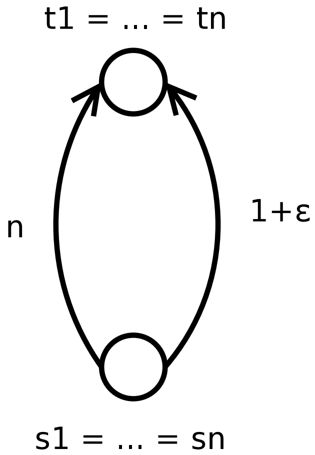
Definition

Recall:

$$\frac{\text{Cost of worst Nash equilibrium}}{\text{Cost of social optimum}}.$$

How bad will the cost of the network be if there is no central authority?

Price of anarchy for network formation is $\Omega(n)$



All n players take the edge with cost n (each player pays 1).

No player will move alone to the light edge and pay $1 + \epsilon$.

Cost of the equilibrium is n. Optimum solution has cost $1 + \epsilon$.

The price of anarchy is *arbitrarily close to n*.

Definition and motivation

Does a bad price of anarchy mean that the situation is hopeless?

One way to surmount the problem: suppose we have a central authority with limited power. Specifically the power to set up the initial network.

Then players are free to change their paths. The game will stabilize at the *local* Nash equilibrium.

We are interested in the ratio

$$\frac{\text{Cost of best Nash equilibrium}}{\text{Cost of optimal solution}}$$

Lower bound [ADK+04]

We start with an example of $\Omega(\log n)$ price of stability.

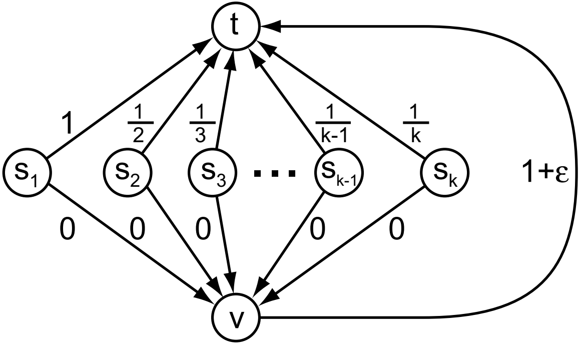


Figure: $\Omega(\log n)$ price of stability

Lower bound, contd.

In the unique Nash equilibrium each player uses the direct edge to t:

- Each player can use either its direct edge or the path through v;
- Assume k players use the path through v:
  - at least one of them will default to using its direct edge of cost $1/k$ rather than the v path with share $(1+\epsilon)/k$;
  - by induction, all will default.

The optimal solution has cost $1+\epsilon$. The Nash equilibrium has cost $H(n) = 1 + 1/2 + \ldots + 1/n$. The price of stability is $\Omega(\log n)$.

Upper bound on price of stability

Our goal is to show that the $\Omega(\log n)$ bound is tight: there is a matching $O(\log n)$ upper bound on price of stability.

Exact potential function

A function Φ s.t. if player i improves her cost by m, $\Phi' = \Phi - m$.
For the network formation game, for a solution S:

$$\Phi(S) = \sum_{e \in \bigcup P_i} H(n_e) c_e$$

Easy to check that if only a single player changes her strategy the change in Φ matches the change in her cost.
Consequences:

- the game has a pure Nash equilibrium;
- the best response dynamics converge.

Properties of Φ

Intuition: Φ is directly related to both the cost $c(S)$ and to the Nash equilibrium. It can be the bridge between the cost of an optimal solution and the cost of a Nash equilibrium.

Φ has the following properties:

- $$\forall e \in \bigcup P_i : H(n_e) \geq 1$$
  $$\Rightarrow \Phi(S) = \sum_{e \in \bigcup P_i} H(n_e) c_e \geq \sum_{e \in \bigcup P_i} c_e = c(S).$$

- $$\forall e : H(n_e) \leq H(n)$$
  $$\Rightarrow \Phi(S) = \sum_{e \in \bigcup P_i} H(n_e) c_e \leq \sum_{e \in \bigcup P_i} H(n) c_e = H(n) c(S)$$

Price of stability upper bound [ADK+04]

We will now show an $O(\log n)$ price of stability for the network formation game.

Let S^* be an optimal solution and S be a Nash equilibrium we can reach from S^*.

- A Nash equilibrium is a local minimum for Φ or otherwise a player can improve her strategy;
- $c(S) \le \Phi(S) \le \Phi(S^*) \le H(n)c(S^*) = O(\log n)c(S^*)$.

An open problem: undirected graphs

The upper bound is tight for directed graphs, but our lower bound does not extend to undirected graphs.

For two players and a single source the price of stability is $4/3$, while $H(2) = 3/2$. [ADK$^+$04]

With a single source and players in all vertexes the price of stability is $O(\log \log n)$ [FKLO06].

Problem:
Find a tight bound on the price of stability for general undirected networks.

Price of anarchy, revisited

We showed that an equilibrium which can be reached from the optimal solution is not too bad. But:

- we need to trust a third party to set up the network;
- The optimal solution can be NP-Hard to compute.

What if we just restrict the rules of the game?

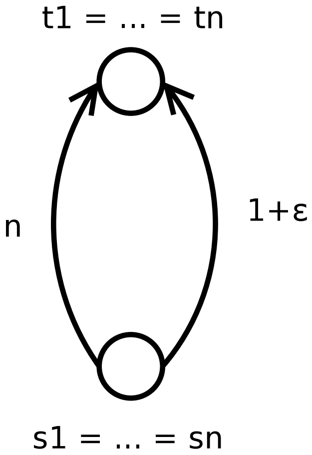
Consider an alternative scenario [CCLE+06]:

- **First phase**: We let players join in one by one in an arbitrary order; each chooses the best path available so far. We start from an empty network (no players just the infrastructure).
- **Second phase**: We let players change their path until reaching an equilibrium.

Formal problem statement

- Undirected graph $G = (V, E)$.
- A single source $s \in V$ and n terminal nodes $t_1, \ldots, t_n \in V$ (not necessarily distinct).
- **First phase**: Players $1, \ldots, n$ arrive in an arbitrary order. Upon arriving, player i chooses the cheapest path from t_i to s.
- **Second phase**:
  - A scheduler picks an arbitrary player i;
  - i picks the current cheapest path from t_i to s;
  - Continue until no player wants to change his path (Nash equilibrium).

How this helps



Observe: the $O(n)$ price of anarchy equilibrium cannot be reached

Upper bound

We will show:

- $O(\log^2 n)$ compatitive ratio for the first phase;
- $O(\log^3 n)$ bound on price of anarchy (for reachable equilibria) for the second phase.

Proof sketch

- We will model the first phase of the game as a linear program:
  - any outcome of the first phase should give a feasible solution to the LP;
  - the cost of the outcome of the game should be the same as the cost of the corresponding LP solution;
- We will construct a relaxation of the LP which is easier to analyze; an upper bound for the relaxation is an upper bound for the original LP;
- We will construct the dual of the relaxation; the cost of any feasible solution to the dual is an upper bound on the maximum cost of the primal;
- We will show that for a certain relaxation there is a feasible solution to its dual with cost $O(\log^2 n) \cdot OPT$, where OPT is the cost of the optimal Steiner tree.

Upper bound for the first phase

We claim that every outcome of the first phase is a feasible solution to the linear program:

$$\max \sum_{i=1}^{n} b(i) \text{s.t.} \tag{1}$$

$$s(j) - s(i) + b(i)/2 \leq d(i,j) : \forall 1 \leq i < j \leq n \tag{2}$$

$$\sum_i s(i) - \sum_i b(i) H(n) \leq 0 \tag{3}$$

$$s(0) = b(0) = 0 \tag{4}$$

$$s(i), b(i) \geq 0 : \forall 1 \leq i \leq n \tag{5}$$

$d(i,j)$ is the distance between i and j in G.

LP, contd.

We will construct a feasible solution for the LP from the outcome of the first phase of the game.

- s_i is the cost of P_i upon arrival of player i.
- b_i is the cost of the edges used for the first time by P_i.

Note that $c(S) = \sum_{e \in \bigcup P_i} c_e = \sum_i b_i$ - equivalence of $c(S)$ and (1).

LP, contd.

One option for player $j > i$ is to connect to t_i and follow P_i. Then

$$\begin{aligned} s_j &\le d(i,j) + \ c(P_i) \text{ when shared with } j \\ &\le d(i,j) + (s(i) - b(i)) + b(i)/2, \end{aligned}$$

as at least the edges used for the first time by i will be shared by j. This is conditions (2).

LP, contd.

Consider the sum of all s_i:

- For each edge e, the first player who useses it sees cost c_e, the second: $\frac{1}{2}c_e$, …, the n_e-th: $\frac{1}{n_e}c_e$.

$$\sum_i s(i) = \sum_{e \in \bigcup P_i} (c_e + \frac{c_e}{2} + \ldots + \frac{c_e}{n_e})$$

$$= \sum_{e \in \bigcup P_i} c_e H(n_e)$$

$$\leq \sum_{e \in \bigcup P_i} c_e H(n)$$

$$= \sum_i b(i) H(n).$$

This is condition (3).

The dual

Consider a relaxation of the LP.

Let T be a tree on $W = \{s, t_1, \ldots, t_n\}$, s.t. for any j, the parent of j arrived before j.
LP_T is the relaxation of the LP s.t. constraints (2) need to hold only for $i = p(j)$.

For any T a solution to the dual of LP_T is an upper bound on the maximum solution of the LP. We can construct a T' s.t. there exists a solution to $DLP_{T'}$ with cost $O(\log^2 n) OPT$.

Bound on the second phase

Let the outcome of the first phase be S. Follow the best response dynamics to a Nash equilibrium S'. Then

$$c(S') \leq \Phi(S') \leq \Phi(S) \leq O(\log n)c(S) = O(\log^3 n)OPT$$

Further questions

- What if some players are allowed to update their paths before all players have arrived?
- We know that for the multisource case the price of anarchy is $\Omega(\sqrt{n})$. Can we get a polylog upper bound under some restriction weaker than a single source?
  - The $\Omega(\sqrt{n})$ example has the number of players $n = O(|V|^2)$. What if $n = O(|V|)$?
  - Same as single source bound when all sources are equidistant from each other.
- Can we explore yet another solution concept? (E.g. price of total anarchy).

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