Non-Bayesian Social Learning

Presented by Arastoo Fazeli

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Learning in Complex Networks: Model and Abstractions

- Each vertex represents an agent
- Each edge represents information flow between two agents
- Agents have access to their neighbors’ information
Learning in Complex Networks: Model and Abstractions

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\[ \Theta \] parameter space

\[ \theta^* \in \Theta \] the unobservable true state of the world

\[ s_t = (s_t^1, \ldots, s_t^n) \] random signals observed by the agents
Bayesian Learning over Networks

\[ \mu_{i,t}(\theta) = \mathbb{P}[\theta = \theta^* | F_{i,t}] \]

where

\[ F_{i,t} = \sigma \left( s_1^i, \ldots, s_t^i, \{\mu_{j,k} : j \in N_i, k \leq t \} \right) \]

is the information available to agent \( i \) up to time \( t \).
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[Computationally hard!]
The Problem with Bayesian Learning

1. Incomplete network information
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2. Incomplete information about other agents’ signal structures
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3. Higher order beliefs matter
The Problem with Bayesian Learning

1. Incomplete network information
2. Incomplete information about other agents’ signal structures
3. Higher order beliefs matter
4. The source of each piece of information is not immediately clear

Intractable and not local.
Non-Bayesian Social Learning

Need a local and computationally tractable update, which hopefully delivers asymptotic social learning.
Non-Bayesian Social Learning

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Agent $i$ is

- Bayesian when it comes to her observation
- non-Bayesian when incorporating others information
Model

\[ N = \{1, 2, \ldots, n\} \] individuals in the society
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\[ G = (N, E) \quad \text{social network} \]
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signal space

\( \ell(s|\theta) \)  
global signal structure
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\[ \ell_i(s^i|\theta) \] agent \( i \)'s signal structure
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_{i,t}(\theta)$</td>
<td>time $t$ beliefs of agent $i$ (a probability measure on $\Theta$)</td>
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**Model**

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(time \( t \) beliefs of agent \( i \) 
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\[ \mathbb{P}^* = \bigotimes_{t=1}^{\infty} \ell(\cdot | \theta^*) \]  
the true probability measure
Model

\[ \mu_{i,t}(\theta) \] \hspace{1cm} \text{time } t \text{ beliefs of agent } i \\
\text{(a probability measure on } \Theta) \\
\mu_{i,0}(\theta) \hspace{1cm} \text{agent } i \text{'s prior belief} \\
\mathbb{P}^* = \bigotimes_{t=1}^{\infty} \mathcal{L}(\cdot | \theta^*) \hspace{1cm} \text{the true probability measure} \\

Agent \( i \)'s time \( t \) forecasts of the next observation:

\[ m_{i,t}(s^i_{t+1}) = \int_{\Theta} \mathcal{L}_i(s^i_{t+1} | \theta) d\mu_{i,t}(\theta) \]
Definition
The Forecasts of agent $i$ are eventually correct on a path \( \{s_t\}_{t=1}^{\infty} \) if, along that path,

\[
m_{i,t}(\cdot) \to \ell_i(\cdot|\theta^*) \quad \text{as} \quad t \to \infty.
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What Do We Mean by Learning?

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Agent $i$ asymptotically learns the true parameter $\theta^*$ on a path $\{s_t\}_{t=1}^{\infty}$ if, along that path,

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- Asymptotic learning, in this setup, is stronger.
Model: Belief Update

\[
\mu_{i,t+1}(\theta) = a_{ii} \text{BU}(\mu_{i,t}; s_{i,t+1})(\theta) + \sum_{j \in \mathcal{N}_i} a_{ij} \mu_{j,t}(\theta)
\]

where

\[
\text{BU}(\mu_{i,t}; s_{i,t+1})(\theta) = \mu_{i,t}(\theta) \frac{\ell_i(s_{i,t+1}|\theta)}{m_{i,t}(s_{i,t+1})}
\]

\[
a_{ij} \geq 0 \quad , \quad \sum_{j \in \mathcal{N}_i} a_{ij} = 1
\]

- Individuals rationally update their beliefs after observing the signal
- exhibit a bias towards the average belief in the neighborhood
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- Does not require knowledge about the network.
- Does not require deduction about the beliefs of others.
- Does not require knowledge about other agents' signal structures.
- The update is local and tractable.
- If signals are uninformative, reduces to the model of DeGroot (1974).
- Reduces to the benchmark Bayesian case if agents assign weight zero to the beliefs of their neighbors.
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First Result: Correct Forecasts

\[ \mu_{i,t+1}(\theta) = a_{ii} \mu_{i,t}(\theta) \frac{\ell_i(s_{t+1}^i | \theta)}{m_{i,t}(s_{t+1}^i)} + \sum_{i \neq j} a_{ij} \mu_{j,t}(\theta) \quad \forall \theta \in \Theta \]

Proposition

Suppose that

(i) social network is strongly connected,

(ii) all agents have strictly positive self-confidence,

(iii) there exists an agent with strictly positive prior belief on \( \theta^* \).

Then, all agents eventually forecast their private observations accurately with \( \mathbb{P}^* \)-probability one.
Why Strong Connectivity?

What if the network has a directed spanning tree but is not strongly connected?

- $\mathcal{N} = \{B, N\}$
- $\Theta = \{\theta_1, \theta_2, \theta_3\}$
- $\theta^* = \theta_2$
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\[
\mu_{N,t+1}(\theta) = \lambda \mu_{N,t}(\theta) \frac{\ell_N(s_{t+1}^N|\theta)}{m_{N,t}(s_{t+1}^N)} + (1 - \lambda) \mu_{B,t}(\theta) \quad \forall \theta \in \Theta
\]

$N$ is mislead by listening to the less informed agent $B$. 
Convergence of Beliefs & Agreement

**Proposition**
Under the assumptions of previous proposition, the beliefs of all agents converge with \( \mathbb{P}^* \)-probability one.

**Corollary**
Under the assumptions of the proposition, all agents have asymptotically equal beliefs \( \mathbb{P}^* \)-almost surely.
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Consensus!
Social Learning: Information Aggregation

**Theorem**

Suppose that

(i) social network is strongly connected,

(ii) all agents have strictly positive self-confidence,

(iii) there exists an agent with strictly positive prior on \( \theta^* \).

(iv) for any agent \( i \) there exists a signal \( \hat{s}^i \in S_i \) such that

\[
\frac{l_i(\hat{s}^i|\theta)}{l_i(\hat{s}^i|\theta^*)} < 1
\]

\( \forall \theta \notin \bar{\Theta}_i \) where \( \bar{\Theta}_i = \{ \theta \in \Theta : l_i(s^i|\theta) = l_i(s^i|\theta^*), \ \forall s^i \in S^i \} \)

(v) there is no state \( \theta \neq \theta^* \) that is observationally equivalent to \( \theta^* \) from the point of view of all agents in the network, i.e.,

\[
\bar{\Theta}_1 \cap \ldots \cap \bar{\Theta}_n = \{ \theta^* \}
\]

Then, **all agents in the social network learn the true state of the world \( P^* \) almost surely; that is, \( \mu_i(\theta^*) \rightarrow 1 \) with \( P^* \) probability 1 \( \forall i \in \mathcal{N} \)**
Information Aggregation: An Example

\[ \Theta = \{ \theta^*, \theta_1, \theta_2, \ldots, \theta_7 \} \]

\[ S_i = \{ H, T \} \]

\[ \ell_i(H|\theta) = \begin{cases} 
\frac{i}{i+1} & \text{if } \theta = \theta_i \\
\frac{1}{(i+1)^2} & \text{otherwise}
\end{cases} \]

Agents learn as if they had access to all information and updated their beliefs rationally.
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Summary and Potential Future Directions

A non-Bayesian social learning model:

- **Local and tractable**
- No information about network topology or signal structures required
- Can handle repeated interactions and information flow over time

Remaining questions:

- The effect of network topology on the learning
- What if actions are observable, and not beliefs?
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