A Survey of Models of Network Formation: Stability and Efficiency *

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January 2003
Revision: May 21, 2003

Abstract

I survey the recent literature on the formation of networks. I provide definitions of network games, a number of examples of models from the literature, and discuss some of what is known about the (in)compatibility of overall societal welfare with individual incentives to form and sever links.

JEL Classification Numbers: A14, C71, C72

Keywords: Networks, Network Games, Network Formation, Game Theory, Allocation Rules

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*This was written for the volume *Group Formation in Economics: Networks, Clubs, and Coalitions*, edited by Gabrielle Demange and Myrna Wooders, which is forthcoming from Cambridge University Press: Cambridge.

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1 Introduction

The set of economic situations where network structures play an important role is wide and varied. For instance, personal contacts play critical roles in obtaining information about job opportunities.[1] Such networks of relationships also underlie the trade and exchange of goods in non-centralized markets[2], the provision of mutual insurance in developing countries[3], research and development and collusive alliances among corporations[4], and international alliances and trading agreements[5]; to mention just a few examples.

Given both the prevalence of situations where networks of relationships play a role, and their importance in determining the outcome of the interaction, it is essential to have theories about both how such networks structures matter and how they form. To get a feeling for what kinds of issues arise and why we might be interested, let me briefly discuss an example. We know from extensive research in both sociology literature and the labor economics literature that that social connections are the leading source of information about jobs and ultimately many (and in some professions most) jobs are obtained through personal contacts.[6] The reason that we might care about this is that the structure of the social network then turns out to be a key determinant of (i) who gets which jobs, which has implications for social mobility, (ii) how patterns of unemployment relate to ethnicity, education, geography, and other variables, and for instance why there might be persistent differences in employment between races, (iii) whether or not jobs are being efficiently filled, and (iv) the incentives that individuals have to educate themselves and to participate in the workforce. Related to all of these issues are what the impact of these things are on how people "network" or what social

[1]See, for example, Rees (1966), Granovetter (1973, 1974), Boorman (1975), Montgomery (1991), Topa (2000), Arrow and Borzekowski (2001), Calvo-Armengol (2000), Calvo-Armengol and Jackson (2001, 2001b), Cahuc and Fontaine (2002), and Ioannides and Datcher Loury (2002).

[2]See, for example, Tesfatsion (1997, 1998), Corominas-Bosch (1999), Weisbuch, Kirman and Herreiner (2000), Charness, Corominas-Bosch, and Frechette (2001), Kranton and Minehart (2002), and Wang and Watts (2002).

[3]See Fafchamps and Lund (2000) and De Weerdt (2002).

[4]See Bloch (2001), Belleflamme and Bloch (2002), Goyal and Moraga (2001), Goyal and Joshi (2000), and Billand and Bravard (2002).

[5]See Goyal and Joshi (2001), Casella and Rauch (2001), and Furusawa and Konishi (2002).

[6]The introduction in Montgomery (1991) provides a nice and quick overview of some of the studies on this. Some of the seminal references are Granovetter (1973, 1974), who found that over 50% of surveyed residents of a Massachusetts town had obtained their jobs through social contacts, and Rees (1966) who found over 60% in a similar study.

ties they maintain, and ultimately whether the resulting labor markets work efficiently, and how different policies (for instance, affirmative action, subsidization of education, etc.) will impact labor markets and how they might be best structured.[7] While this is quite a list of issues to consider, it makes clear why understanding how networks operate is of importance.

At this point it is useful to crudely divide situations where networks are important into two different categories, to make clear what the scope of this survey will be. In one category, the network structure is a distribution or service network that is the choice of a single actor. For instance, the routing of planes by an airline[8] falls into this category, as do many routing, transmission, and distribution network problems. In the other category of situations where networks are critical, the network structure connects different individuals and the formation of the network depends on the decisions of many participants. This includes the examples mentioned above of labor markets, political alliances, and generally any social network. It is this second category of network problems, where the networks connect a number of individuals, that I survey here.[9]

The recent and rapidly growing literature on network formation among individuals addresses various questions. I concentrate on the following three:

(i) How are such network relationships important in determining the outcome of economic interaction?

(ii) How can we predict which networks are likely to form when individuals have the discretion to choose their connections?

(iii) How efficient are the networks that form and how does that depend on the way that the value of a network is allocated among the individuals?

In terms of answering question (i), I will mostly just provide some examples (see Section 3), and the primary focus will be on questions (ii) and (iii).

Beyond the literature surveyed here, there is a well-established and vast literature in sociology on social networks.[10] That literature makes clear the importance of social networks in many contexts, and provides a detailed look at many issues associated

[7] See Calvo-Armengol and Jackson (2001, 2001b) for a look at some of these issues.

[8] See, for instance, Starr and Stinchcombe (1992) and Hendricks, Piccione, and Tan (1995).

[9] There is some overlap, as for instance in the case of the choices of internet backbone providers that affects the structure of the internet (e.g., see Badasyan and Chakrabarti (2003)).

[10] An excellent and broad introductory text to the social networks literature is Wasserman and Faust (1994).

with social networks, ranging from measuring power and centrality to understanding the roles of different sorts of social ties. While that literature provides a wealth of knowledge of the workings of network interactions, largely missing from that literature are strategic models of how networks are formed and in particular an understanding of the relationship between individual incentives overall societal welfare. The development of game theoretic reasoning over past decades and its influx into economic models, has now come together with a realization that network relationships play important roles in many economic interactions. This has resulted in the birth of a literature that uses game theoretic reasoning to develop such models of self-organizing network relationships. This is a rapidly growing literature on a wide-open landscape with numerous important questions to be addressed and a huge variety of potential applications. As such, I cannot hope to cover all of the burgeoning literature here. I have the more modest aim of providing a look at some of the modeling approaches, a feeling for the tension between individual incentives to form links and societal welfare, and a glimpse of some of the applications of the developing theories.

2 Defining Network Games

Network relationships come in many shapes and sizes, and so there is no single model which encompasses them all. Here I focus one way of modeling networks that will be fairly broad and flexible enough to capture a multitude of applications. As we proceed, I will try to make clear what is being admitted and what is being ruled out.

Players

$N = \{1, \ldots, n\}$ is a set of players or individuals who are connected in some network relationship.

For this survey, I will refer to the individuals as "players" with the idea that they may be individual people, they may be firms or other organizations, and they might even be countries.

These players will be the nodes or vertices in a graph that will describe the network relationships.

A common aspect to the papers in the literature surveyed here is that they model situations where each player has discretion in forming his or her links in the network relationship. These may be people deciding on whom they wish to be friends with, or contract with, or pass job information to; these may be firms deciding on which partnerships to engage in; or these may be countries deciding on which trade or defense alliances to enter into.

Networks

Depending on the context the network relationship may take different forms. The simplest form is a non-directed graph, where two players are either connected or not. For instance in a network where links represent direct family relationships, the network is naturally a non-directed network. Two players are either related to each other or not, but it cannot be that one is related to the second without the second being related to the first. This is generally true of many social and/or economic relationships, such as partnerships, friendships, alliances, acquaintances, etc.. This sort of network will be central to the discussion below. However, there are other situations that are also discussed below that are modeled as directed networks, where one player may be connected to a second without the second being connected to the first. For instance, a network that keeps track of which authors reference which other authors, or which web sites have links to which others would naturally be a directed network.

The distinction between directed and non-directed networks is not a mere technicality. It is fundamental to the analysis as the applications and modeling are quite different. In particular, when links are necessarily reciprocal, then it will generally be the case that joint consent is needed to establish and/or maintain the link. For instance, in order to form a trading partnership, both partners need to agree. To maintain a friendship the same is generally true, as is maintaining a business relationship, alliance, etc. In the case of directed networks, one individual may direct a link at another without the other's consent. These differences result in some basic differences in the modeling network formation.

Most economic applications fall into the reciprocal link (and mutual consent) framework, and as such non-directed networks will be our central focus. Nevertheless, directed networks are also of interest and I will return to discuss them briefly at the end of this survey.

In many situations links might also have some intensity associated with them. For instance, if links represent friendships, some might be stronger than others and this might have consequences, such as affecting the chance that information passes through a given link. Much of the literature on network formation to date has been restricted to the case where links are either present or not, and do not have intensities associated with them.[11] This makes representing networks a bit easier, as we can just keep track

[11]For exceptions, see Calvo, Lasaga, and van den Nouweland (1999), Calvo-Armengol and Jackson

of which links are present.

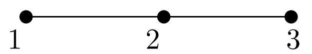
While the focus on 0-1 links is restrictive, it is still of significant interest for at least two reasons. First, much of the insight obtained in this framework is fairly robust, and so this is a useful starting point. Second, the fact that the value and costs that are generated by links may differ across links already allows for substantial heterogeneity and admits enough flexibility so that a large number of interesting applications are captured.

A network g is a list of which pairs of players are linked to each other. A network is then a list of unordered pairs of players $\{i, j\}$.

For any pair of players i and j, $\{i, j\} \in g$ indicates that i and j are linked under the network g.

For simplicity, write ij to represent the link $\{i, j\}$, and so $ij \in g$ indicates that i and j are linked under the network g.

For instance, if $N = \{1, 2, 3\}$ then $g = \{12, 23\}$ is the network where there is a link between players 1 and 2, a link between players 2 and 3, but no link between players 1 and 3.



Let g^N be the set of all subsets of N of size 2. $G = \{g \subset g^N\}$ denotes the set of all possible networks or graphs on N.

The network g^N is referred to as the "complete" network.

Another prominent network structure is that of a "star" network, which is a network where there exists some player i such that every link in the network involves player i. In this case i is referred to as the center of the star.

A shorthand notation for the network obtained by adding link ij to an existing network g is $g + ij$, and for the network obtained by deleting link ij from an existing network g is $g - ij$.

Let

$$g|_S = \{ij \ : \ ij \in g \text{ and } i \in S, j \in S\}.$$

Thus $g|_S$ is the network found deleting all links except those that are between players in S.

(2001, 2001b), and Goyal and Moraga (2001).

For any network g, let $N(g)$ be the set of players who have at least one link in the network g. That is, $N(g) = \{i \mid \exists j \text{ s.t. } ij \in g\}$.

Paths and Components

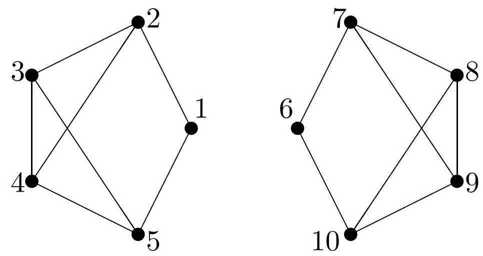
A *path* in a network $g \in G$ between players i and j is a sequence of players $i_1, \ldots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \ldots, K-1\}$, with $i_1 = i$ and $i_K = j$.

Looking at the path relationships in a network naturally partitions a network into different connected subgraphs that are commonly referred to as components.

A *component* of a network g, is a nonempty subnetwork $g' \subset g$, such that

- if $i \in N(g')$ and $j \in N(g')$ where $j \neq i$, then there exists a path in g' between i and j, and
- if $i \in N(g')$ and $ij \in g$, then $ij \in g'$.

Thus, the components of a network are the distinct connected subgraphs of a network. In the figure below there are two components with five agents each.



The set of components of g is denoted $C(g)$. Note that $g = \cup_{g' \in C(g)} g'$.

Note that under this definition of component, a completely isolated player who has no links is not considered a component. If one wants to have a definition of component that includes isolated nodes as a special case, then one can consider the partition induced by the network.

Components of a network partition the players into groups within which players are connected. Let $\Pi(g)$ denote the partition of N induced by the network g.[12]

Value Functions

The network structure is the key determinant of the level of productivity or utility to the society of players involved. For instance, a buyer's expected utility from trade

[12]That is, $S \in \Pi(g)$, if and only if either there exists $h \in C(g)$ such that $S = N(h)$, or there exists $i \notin N(g)$ such that $S = \{i\}$.

may depend on how many sellers that buyer is negotiating with, and how many other buyers they are connected to, etc. (as in Corominas-Bosch (1999) and Kranton and Minehart (2002)). Similarly, a network where players have very few acquaintances with whom they share information will result in different employment patterns than one where players have many such acquaintances (as in Calvo-Armengol and Jackson (2001, 2001b)).

Methods of keeping track of the overall value generated by a particular network, as well as how it is allocated across players, are through a value function and an allocation rule. These are the natural extensions of the notions of characteristic function and imputation rule from cooperative game theory. In cooperative game theory these would depend just on the set of players involved, while here in the network setting they depend on the full network structure rather than simply a coalition. In fact, in the special case where the value generated only depends on connected components rather than network structure, the value function and allocation rule reduce to a characteristic function (for a cooperative game in partition function form) and imputation rule.

A *value function* is a function $v : G \rightarrow I\!R$.

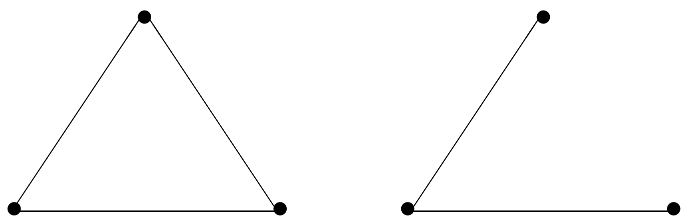
For simplicity, in what follows I maintain the normalization that $v(\emptyset) = 0$.
The set of all possible value functions is denoted $\mathcal{V}$.

A prominent subclass of value functions is the set of component additive ones.
A value function v is *component additive* if $\sum_{h \in C(g)} v(h) = v(g)$.
Component additivity is a condition that rules out externalities across components, but still allows them within components. It is quite natural in some contexts, for instance social interactions, and not in others, for instance in an oligopoly setting where links are alliances of some sort and different firms compete with each other.

Another prominent subclass of value functions is the set of anonymous ones.
Given a permutation of players π (a bijection from N to N) and any $g \in G$, let $g^\pi = \{\pi(i)\pi(j) | ij \in g\}$.
Thus, g^π is a network that shares the same architecture as g but with the specific players permuted.
A value function is *anonymous* if for any permutation of the set of players π, $v(g^\pi) = v(g)$.
Anonymity says that the value of a network is derived from the structure of the network and not the labels of the players who occupy various positions.

It is important to note that different networks that connect the same players may lead to different values. This makes a value function a much richer object than a characteristic function used in cooperative game theory. For instance, a society $N = \{1, 2, 3\}$ may have a different value depending on whether it is connected via the network $g = \{12, 23\}$ or the network $g^N = \{12, 23, 13\}$.

The special case where the value function depends only on the groups of players that are connected, but not how they are connected, corresponds to the communication networks (or cooperation structures) first considered by Myerson (1977) and surveyed in the chapter by van den Nouweland (2003). To be precise, Myerson started with a transferable utility cooperative game in characteristic function form, and layered on top of that network structures that indicated which players could communicate. A coalition could only generate value if its members were connected via paths in the network. But, the particular structure of the network did not matter, as long as the coalition's members were connected somehow.



Is the Value Necessarily the Same?

The approach surveyed here follows Jackson and Wolinsky (1996), who defined the value as a function that is allowed to depend on the specific network structure. A special case is where $v(g)$ only depends on the coalitions induced by the component structure of g, which corresponds to the communication games. In most applications, however, there may be some cost to links and thus some difference in total value across networks even if they connect the same sets of players, and so this more general and flexible formulation is more powerful and encompasses more applications.

It is also important to note that the value function can incorporate costs to links as well as benefits. It allows for quite general ways in which costs and benefits may vary across networks. This means that a value function allows for externalities both within and across components of a network.

Network Games

A *network game* is a pair (N, v) where N is the set of players and v is a value function on networks among those players.

This notion of network game might be thought of as the analog of a cooperative game (rather than a non-cooperative game), as the allocation of values among players is not specified. The use of such games will involve both cooperative and non-cooperative perspectives, as they will be the basis for network formation. The augmenting of a network game by an allocation rule, which we turn to next, will be what allows one to model the formation of the network.

Allocation Rules

Beyond knowing how much total value is generated by a network, it is critical to keep track of how that value is allocated or distributed among the players in the society. This is captured by the concept of an allocation rule.

An *allocation rule* is a function $Y : G \times \mathcal{V} \to I\!R^N$ such that $\sum_i Y_i(g, v) = v(g)$ for all v and g.

Note that balance, $\sum_i Y_i(g, v) = v(g)$, is made part of this definition of allocation rule.

Generally, there will be some natural way in which the value is allocated in a given network situation. This might simply be the utility that the players directly receive, accounting for both the costs and benefits of maintaining their links, for instance in a social network. This might also be the result of some bargaining about the terms of trade, for instance in a network of international trading relationships. Beyond, the allocations that come naturally with the network, we might also be interested in designing the allocation rule; that is, re-allocating value using taxes, subsidies and other transfers. This might be motivated in a number of ways, including trying to affect the incentives of players to form networks, or more simply for fairness reasons. Regardless of the perspective taken, an allocation rule captures either an allocation that arises naturally or an allocation of value that is imposed.

It is important to note that an allocation rule depends on both g and v. This allows an allocation rule to take full account of a player i's role in the network. This includes not only what the network configuration is, but also and how the value generated depends on the overall network structure. For instance, consider a network $g = \{12, 23\}$ in a situation where the value generated is 1 ($v(g) = 1$). Player 2's allocation might be very different depending on what the value of other networks are. For instance, if $v(\{12, 23, 13\}) = 0 = v(\{13\})$, then 2 is essential to the network and may receive a large allocation. If on the other hand $v(g') = 1$ for all networks, then 2's role is not particularly special. This information can be relevant, especially in bargaining situations, which is why the allocation rule is allowed to depend on it. I return to discuss this in more detail below.

Before moving on, I note two properties of allocation rules that will come up repeatedly in what follows.

Component Balance

An allocation rule Y is *component balanced* if $\sum_{i \in S} Y_i(g, v) = v(g|_S)$ for each component additive v, $g \in G$ and $S \in \Pi(g)$.

Component balance requires that the value of a given component of a network is allocated to the members of that component in cases where the value of the component is independent of how other components are organized. This would tend to arise naturally. It also is a condition that an intervening planner or government would like to respect if they wish to avoid secession by components of the network.

Anonymity of an Allocation Rule

Given a permutation π, let v^π be defined by $v^\pi(g) = v(g^{\pi^{-1}})$ for each $g \in G$.

An allocation rule Y is *anonymous* if for any $v \in \mathcal{V}$, $g \in G$, and permutation of the set of players π, $Y_{\pi(i)}(g^\pi, v^\pi) = Y_i(g, v)$, where the value function v^π is defined by $v^\pi(g) = v(g^{\pi^{-1}})$ for each $g \in G$.

Anonymity of an allocation rule requires that if all that has changed is the labels of the players and the value generated by networks has changed in an exactly corresponding fashion, then the allocation only change according to the relabeling.[13]

3 Some Examples

In order to fix some ideas and illustrate the above definitions, I now describe a few examples of network situations that have been analyzed in the literature.

Example **1** *The Connections Model* (Jackson and Wolinsky (1996))

In this model, links represent social relationships between players; for instance friendships. These relationships offer benefits in terms of favors, information, etc., and also involve some costs. Moreover, players also benefit from indirect relationships. A

[13]Note that the definition does not require that v be anonymous or that g be symmetric.

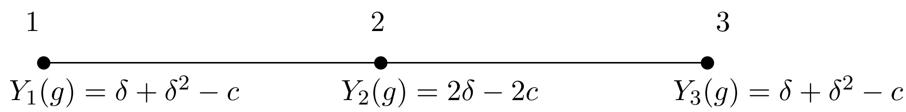
"friend of a friend" also results in some benefits, although of a lesser value than a "friend," as do "friends of a friend of a friend" and so forth. The benefit deteriorates in the "distance" of the relationship. For instance, in the network $g = \{12, 23, 34\}$ player 1 gets a benefit of δ from the direct connection with player 2, an indirect benefit of δ^2 from the indirect connection with player 3, and an indirect benefit of δ^3 from the indirect connection with player 4. For $\delta < 1$ this leads to a lower benefit from an indirect connection than a direct one. Players only pay costs, however, for maintaining their direct relationships. These payoffs and benefits may be relation specific, and so are indexed by ij.

Formally, the payoff player i receives from network g is

$$Y_i(g) = \sum_{j \neq i} \delta_{ij}^{d(i,j)} - \sum_{j:ij \in g} c_{ij},$$

where $d(i,j)$ is the number of links in the shortest path ("the geodesic") between i and j (setting $d(i,j) = \infty$ if there is no path between i and j). The value function in the connections model of a network g is simply $v(g) = \sum_i Y_i(g)$.

The case where there are common δ and c such that $\delta_{ij} = \delta$ and $c_{ij} = c$ for all i and j is referred to as the "symmetric connections model".



Example **2** *The Spatial Connections Model* [Johnson and Gilles (2000)]

An interesting version of the connections model is studied by Johnson and Gilles (2000). This is a version with spatial costs, where there is a geography to locations and c_{ij} is related to distance. For instance, if players are spaced equally on a line and i's location is at the point i, then costs are proportional to $|i - j|$.

This variation of the connections model introduces natural asymmetries among the players and yields interesting variations on the networks that form and the ones that are most efficient from society's perspective.

Example **3** *Free-Trade Networks* [Furusawa and Konishi (2002)]

Furusawa and Konishi (2002) consider a model where the players in the network are countries. A link between two countries is interpreted as a free-trade agreement which means that the goods produced in either of the countries can be traded without any tariff to consumers in the other country. In the absence of a link, goods are traded with some tariff. A link between two countries has direct effects in the trade between those two countries, as there will be a greater flow of goods (possibly in both directions) in the absence of any tariffs. There are also indirect effects from links. Countries that are not directly involved in a link still feel some effects, as the relative prices for goods imported from a country changes as a free-trade agreement (link) between two other countries is put into place.

Once demands of the consumers, the production possibilities, and the tariffs for imports from other countries (in the absence of links) are specified for each country, then one can calculate the payoffs to each country as a function of the network of free trade agreements, as well as the total value generated by all countries. Thus, one ends up with a well-defined value function and allocation rule, and one can then study the incentives for countries to form free-trade agreements.

Example **4** *Market Sharing Agreements* [Belleflamme and Bloch (2002)]

In this model, the n players are firms who each have a home market for their goods. Firms are symmetric to start with and so any asymmetries that arise will come from the network structure that is formed. In this model a link represents an agreement between two firms. In the absence of any agreement between firms i and j, firm i will sell goods on firm j's market and vice versa. If firms i and j form a link, then that is interpreted as a market sharing agreement where firm i refrains from selling on j's market and vice versa.

The profits that a firm makes from selling its goods on any market are given by a function which is the profit to a firm who is selling on market j as a function of the n_j, number of firms selling on market j. Once this profit function is specified, then one can calculate the payoff to each firm as a function of the network structure in place.

Example **5** *Labor Markets* [Calvo-Armengol and Jackson (2001, 2001b)]

In this model each worker maintains social ties with some other workers. Over time, workers randomly lose jobs and new job opportunities randomly arrive. As information about a new job comes to a given worker, they might do several things with it. First, if they are unemployed or the new job opportunity looks more attractive than their current job, they might take the job themselves (or at least apply and obtain the job with some probability). Second, in the event that the job does not right for them personally, they may pass that information on to one or more of their friends; that is, they might pass the information about the job onto to some of those players to whom they are linked in the network. This passing of information leads to some probabilities that some of their friends might obtain the job. The model can also allow for the fact that these friends might further pass the information on, and so forth.

The set of possibilities for how information might be passed through the network can be quite complicated. However, all that really matters in this model is what the probability is that each player ends up getting a (new) job as a function of what the current status of all the players in the network are. Once this is specified one has a well-defined random (Markov) process, where one can calculate the probability distribution of any worker's employment and wage status at any given date given any information about the state of the network and employment statuses and wages at some previous date. The structure of the network and the initial starting state then provide predictions for the future expected discounted stream of wages of any worker.

With this network of information passing in place, and the predictions it yields for the stream of wages of workers, one can ask what the incentives of the workers are to maintain their position in the network versus drop out of the network. One might also ask what their incentives are to maintain or sever links, etc.

EXAMPLE **6** *The Co-Author Model* [Jackson and Wolinsky (1996)]

In the co-author model, each player is a researcher who spends time working on research projects. If two researchers are connected, then they are working on a project together. Each player has a fixed amount of time to spend on research, and so the time that researcher i spends on a given project is inversely related to the number of projects, n_i, that he is involved in. The synergy between two researchers depends on how much time they spend together, and this is captured by a term $\frac{1}{n_i n_j}$. Here the more projects a researcher is involved with, the lower the synergy that is obtained per project.

Player i's payoff is represented by

$$Y_i(g) = \sum_{j:ij \in g} \frac{1}{n_i} + \frac{1}{n_j} + \frac{1}{n_i n_j}$$

for $n_i > 0$, and $Y_i(g) = 1$ if $n_i = 0$.

So the value generated by any given research project is proportional to the sum of the time that i puts into the project, the time that j puts into it, and a synergy that is dependent on an interaction between the time that the two researchers put into the project.

The total value generated by all researchers is $v(g) = \sum_i Y_i(g)$.

Note that in the co-author model there are no directly modeled costs to links. Costs come indirectly in terms of diluted synergy in interaction with co-authors.

Example **7** *Organizations and Externalities* [Currarini (2002)]

While the general model of a network game allows for arbitrary forms of externalities, it is useful to understand how the particular structure of externalities matters in determining which networks form. We can gain some insight from looking at various models. For instance, we see positive externalities to other players (not involved in a new link) when a player forms a new link in the connections model, and we see negative externalities to other players when a given player forms a new link in the co- author model. While these allow us to see some effects of different sorts of externalities, the models differ on too many dimensions to be able to disentangle exactly what the impact of different forms of externalities are.

An approach to studying how different forms of externalities impact network formation is to specify a model which has a flexible enough structure so that we can include positive and negative externalities as special cases, and yet at the same time we need the model be specialized enough so we can make pointed predictions. Currarini's (2002) model is motivated in this way.

In Currarini's model, the value of a network depends only on the partition of players induced by the components of the network. That is, any network partitions the players into different subsets, where two players are in the same subset if and only if there is some path in the network that includes them both. The simplifying assumption that he makes is that the value of a network g depends only on $\Pi(g)$, so we can write $v(\Pi(g))$.[14]

The reason that the network structure still plays an important role in Currarini's analysis, is that if some group of players change their links, the resulting network (and thus partition structure that results) depends on how they were connected to start

[14] To be careful, the v in Currarini's analysis is a richer object than the value function defined in this survey as Currarini's version specifies the value of each component as a function of the partition.

with. For instance, if player 2 severs all links in the network $\{12, 23\}$ the resulting partition of players is different from what happens when player 2 severs all links in the network $\{12, 23, 13\}$.

Currarini's definition of positive and negative externalities is based on whether value increases or decreases as the partition of players becomes finer.

EXAMPLE **8** *Unequal Connections* [Goyal and Joshi (2002)]

Goyal and Joshi (2002) provide a different model of externalities, but one that is similar to Currarini's in its spirit of having a specialized enough structure to allow for pointed predictions, but still sufficient flexibility to allow for both positive and negative externalities.

In the Goyal and Joshi model, the allocation to a given player as a function of the network can be written as

$$Y_i(g) = b_i(g) - n_i(g)c$$

where c is a cost, n_i is the number of links that i has, and b_i is a benefit function. In particular the benefit function is assumed to take one of two forms. In the "playing the field" version,

$$b_i(g + ij) - b_i(g) = \phi(n_i(g), \sum_{j \neq i} n_j(g_{-i}))$$

where ϕ is common to all players and g_{-i} is the network with all links to i removed. Under this assumption, players care only about how many links they have and how many links all other players have in total. Essentially, players benefit (or suffer) from others' links in symmetric ways regardless of the particulars of the path structure. The other version of the model that they consider is the "local spillovers" version, where there is a function ψ such that

$$b_i(g + ij) - b_i(g) = \psi(n_i(g), n_j(g)).$$

Here the marginal value of a link depends only on how connected the two players are, and not on the particulars of who they are connected to or other aspects of the network.

Under these assumptions on marginal benefits from links, Goyal and Joshi can then look at positive and negative spillovers by considering how these functions change with the n_i's and n_j's. Under different possible scenarios, they can compute the networks that will be formed and see how these vary with the scenario.

EXAMPLE **9** *A Bilateral Bargaining Model* [Corominas-Bosch (1999)]

Corominas-Bosch (1999) considers a bargaining model where buyers and sellers bargain over prices for trade. A link is necessary between a buyer and seller for a transaction to occur, but if a player has several links then there are several possibilities as to whom they might transact with. Thus, the network structure essentially determines bargaining power of various buyers and sellers.

More specifically, each seller has a single unit of an indivisible good to sell which has no value to the seller. Buyers have a valuation of 1 for a single unit of the good. If a buyer and seller exchange at a price p, then the buyer receives a payoff of $1-p$ and the seller a payoff of p. A link in the network represents the opportunity for a buyer and seller to bargain and potentially exchange a good.[15]

Corominas-Bosch models bargaining via the following variation on a Rubinstein bargaining protocol. In the first period sellers simultaneously each call out a price. A buyer can only select from the prices that she has heard called out by the sellers to whom she is linked. Buyers simultaneously respond by either choosing to accept some single price offer they received, or to reject all price offers they received. If there are several sellers who have called out the same price and/or several buyers who have accepted the same price, and there is any discretion under the given network connections as to which trades should occur, then there is a careful protocol for determining which trades occur (which is essentially designed to maximize the number of eventual transactions).

At the end of the period, trades are made and buyers and sellers who have traded are cleared from the market. In the next period the situation reverses and buyers call out prices. These are then either accepted or rejected by the sellers connected to them in the same way as described above. Each period the role of proposer and responder switches and this process repeats itself indefinitely, until all remaining buyers and sellers are not linked to each other. Buyers and sellers are impatient and discount according to a common discount factor $0 < \delta < 1$. So a transaction at price p in period t is worth $\delta^t p$ to a seller and $\delta^t(1-p)$ to a buyer.

Given this specification and some specification of costs of links, one can calculate the expected payoff to every buyer and seller (the allocation rule) as a function of the network structure.

[15]Note that in the Corominas-Bosch framework links can only form between buyers and sellers. This fits into the setting we are considering here where links can form between any players simply by having the value function and allocation rule ignore any links except those between buyers and sellers.

Example **10** *A Model of Buyer-Seller Networks* [Kranton and Minehart (2002)]

The Kranton and Minehart model of buyer-seller networks is similar to the Corominas-Bosch model described above except that the valuations of the buyers for a good are random and the determination of prices is made through an auction rather than alternating offers bargaining.

The Kranton and Minehart model is described as follows. Again, each seller has an indivisible object for sale. Buyers have independently and identically distributed utilities for the object, denoted u_i. Each buyer knows her own valuation, but only the distribution over other buyers' valuations, and similarly sellers know only the distribution of buyers' valuations.

Again, link patterns represent the potential transactions, however, the transactions and prices are determined by an auction rather than bargaining. In particular, prices rise simultaneously across all sellers. Buyers drop out when the price exceeds their valuation (as they would in an English or ascending oral auction). As buyers drop out, there emerge sets of sellers for whom the remaining buyers still linked to those sellers is no larger than the set of sellers. Those sellers transact with the buyers still linked to them. The exact matching of whom trades with whom given the link pattern is done carefully to maximize the number of transactions. Those sellers and buyers are cleared from the market, and the prices continue to rise among remaining sellers, and the process repeats itself.

For each link pattern every player has a well-defined expected payoff from the above described process (from an ex-ante perspective before buyers know their u_i's). ¿From this expected payoff can be deducted costs of maintaining links to buyers and sellers to obtain a prediction of net payoffs as a function of the network structure, or in other words the allocation rule.

Example **11** *Buyer-Seller Networks with Quality Differentiated Products* [Wang and Watts (2002)]

The Wang and Watts model of buyer-seller networks enriches the above bargaining models in the following ways. First, sellers have a choice of selling goods of either high or low quality (which is observable to the buyer). Also, in addition to having links between buyers and sellers, buyers and sellers may link with each other to form buyers associations and sellers associations. The advantage of forming such associations is that they influence the bargaining power and the eventual prices that emerge. The disadvantage is that sales may be rationed among members of an association. For instance, if a sellers association has an excess number of members relative to the number of buyers who have linked with it, then the determination of who gets to sell is made by a randomization. This model brings together issues of how network structure affects bargaining power and how collective structures can influence such power.

The examples above provide an idea of how rich and varied the potential applications of network models are. This is only a subset of the models in the literature. Let us now turn to look at the ways in which the formation of networks has been analyzed.

4 Modeling Network Formation

The are many possible approaches modeling network formation.[16] An obvious one is simply to model it explicitly as a non-cooperative game, and let us start with this approach as the literature did as well.[17]

An Extensive Form Game

Aumann and Myerson (1988) were the first to model network formation explicitly as a game, and did so by describing an extensive form game for the formation of a network in the context of cooperative games with communication structures.[18] In their game, players sequentially propose links which are then accepted or rejected. The extensive form begins with an ordering over possibly links. Let this ranking be $(i_1j_1, \ldots, i_nj_n)$. The game is such that the pair of players i_kj_k decide on whether or not to form that link knowing the decisions of all pairs coming before them, and forecasting the play that will come after them. A decision to form a link is binding and cannot be undone. If a pair i_kj_k decide not to form a link, but some other pair coming after them forms a link, then i_kj_k are allowed to reconsider their decision. This feature allows player 1

[16]The discussion here focuses on situations where players decide to participate in a link or not, modeled in various ways. Brueckner (2003) examines an interesting alternative where players put in effort and then links are determined randomly, but with increasing probability in players' efforts.

[17]Another possibility is simply to specify some exogenous rule for adding and deleting links in a network environment and then to run simulations. That method of studying self-organizing networks is seen in some of the social networks literature. The idea of studying incentives and explicitly modeling the strategic aspects of network formation is to try to put more structure and understanding behind this process, to see which networks form and why.

[18]See van den Nouweland (2003) for a detailed discussion of this game.

to make a credible threat to 2 of the form "I will not form a link with 3 if you do not. But if you do form a link with 3, then I will also do so."

In terms of its usefulness as an approach to modeling network formation, this game has some nice features to it. However, the extensive form makes it difficult to analyze beyond very simple examples and the ordering of links can have a non-trivial impact on which networks emerge. These hurdles have prompted some other approaches.

A Simultaneous Move Game

Myerson (1991) suggests a different game for modeling network formation. It is in a way the simplest one that one could come up with, and as such is a natural one. It can be described as follows.[19] The strategy space of each player is the list of other players. So the strategy space of i is $S_i = 2^{N\setminus\{i\}}$. Players (simultaneously) announce which other players they wish to be connected to. If $s \in S_1 \times \cdot \times S_n$ is the set of strategies played, then link ij forms if and only if both $j \in s_i$ and $i \in s_j$.

This game has the advantage of being very simple and pretty directly capturing the idea of forming links. Unfortunately, it generally has a large multiplicity of Nash equilibria. For instance, $s_i = \emptyset$ for all i is *always* a Nash equilibrium, regardless of what the payoffs to various networks are. The idea is that no player suggests any links under the correct expectation that no players will reciprocate. This is especially unnatural in situations where links result in some positive payoff. This means that in order to make use of this game, one must really use some refinement of Nash equilibrium. Moreover, in order to really deal with the fact that it takes two players to form a link, one needs something beyond refinements like undominated Nash equilibrium or trembling hand perfection. One needs to employ concepts such as strong equilibrium or Coalition Proof Nash Equilibrium. Dutta and Mutuswami (1997) discuss such refinements in detail,[20] and the relationship of the equilibria to the concept of Pairwise Stability, which is the concept that I discuss next.

The fact that mutual consent is needed to form a link is generally a hurdle for trying use any off-the-shelf noncooperative game theoretic approach. In whatever game one specifies for link formation, requiring the consent of two players to form a link means that either some sort of coalitional equilibrium concept is required, or the game needs

[19]See van den Nouweland (2003) for a discussion of some results for this game.

[20]See also Dutta, van den Nouweland, and Tijs (1998), Harrison and Munoz (2002), and an earlier use of the game in Qin (1996). See also McBride (2002) for a variation of the game and a solution concept to allow for incomplete information of players regarding their payoffs and structure of the network.

to be an extensive form with a protocol for proposing and accepting links in some sequence. Another serious challenge to the off-the-shelf noncooperative game theoretic approach is that the game is necessarily ad hoc and fine details of the protocol (e.g., the ordering of who proposes links when, whether or not the game has a finite horizon, players are impatient, etc.) generally matter.

Pairwise Stability

A different approach to modeling network formation is to dispense with the specifics of a noncooperative game and to simply model a notion of what a stable network is directly. This is the approach that was taken by Jackson and Wolinsky (1996) and is captured in the following definition.

A network g is *pairwise stable* with respect to allocation rule Y and value function v if

(i) for all $ij \in g$, $Y_i(g, v) \geq Y_i(g - ij, v)$ and $Y_j(g, v) \geq Y_j(g - ij, v)$, and

(ii) for all $ij \notin g$, if $Y_i(g + ij, v) > Y_i(g, v)$ then $Y_j(g + ij, v) < Y_j(g, v)$.

The first part of the definition of pairwise stability requires that no player wish to delete a link that he or she is involved in. Implicitly, any player has the discretion to unilaterally terminate relationships that they are involved in. The second part of the definition requires that if some link is not in the network and one of the involved players would benefit from adding it, then it must be that the other player would suffer from the addition of the link. Here it is implicit that the consent of both players is needed for adding a link.[21] This seems to be an aspect that is pervasive in applications, and is thus important to capture in a solution concept.

While pairwise stability is natural and quite easy to work with, there are some limitations of the concept that deserve discussion.

First, it is a weak notion in that it only considers deviations on a single link at a time. This is part of what makes it easy to apply. However, if other sorts of deviations are viable and attractive, then pairwise stability may be too weak a concept.[22] For

[21] Jackson and Wolinsky (1996) also study another stability concept where side-payments are possible. That is, one player can pay another (or provide some sort of favors) so that new links form whenever the total benefit to the two players involved is positive. See that paper for details.

[22] One can augment pairwise stability by various extra considerations, for instance allowing players to sever many links at once. For a look at different such variations, see for instance, Belleflamme and Bloch (2001) and Goyal and Joshi (2001).

instance, it could be that a player would not benefit from severing any single link but would benefit from severing several links simultaneously, and yet the network would still be pairwise stable. Second, pairwise stability considers only deviations by at most a pair of players at a time. It might be that some group of players could all be made better off by some more complicated reorganization of their links, which is not accounted for under pairwise stability. To the extent that larger groups can coordinate their actions in making changes in a network, a stronger solution concept might be needed.

In both of these regards, pairwise stability might be thought of as a necessary but not sufficient requirement for a network to be stable over time. Nevertheless, pairwise stability still turns out to be quite useful and in particular often provides narrow predictions about the set of stable networks.

Strong Stability

Alternatives to pairwise stability that allow for larger coalitions than just pairs of players to deviate were first considered by Dutta and Mutuswami (1997).[23] The following definition is in that spirit, and is due to Jackson and van den Nouweland (2000). [24]

A network $g' \in G$ *is obtainable from* $g \in G$ *via deviations by* S if

(i) $ij \in g'$ and $ij \notin g$ implies $ij \subset S$, and

(ii) $ij \in g$ and $ij \notin g'$ implies $ij \cap S \neq \emptyset$.

The above definition identifies changes in a network that can be made by a coalition S, without the need of consent of any players outside of S. (i) requires that any new links that are added can only be between players in S. This reflects the fact that consent of both players is needed to add a link. (ii) requires that at least one player of any deleted link be in S. This reflects that fact that either player in a link can unilaterally sever the relationship.

A network g is *strongly stable* with respect to allocation rule Y and value function v if for any $S \subset N$, g' that is obtainable from g via deviations by S, and $i \in S$ such

[23]Core-based notions had been discussed in the exchange network literature, but mainly in terms of bargaining over value as adapted from the cooperative game theory literature. See Bienenstock and Bonacich (1997) for an overview.

[24]Konishi and Ünver (2003) develop an interesting variation of strong stability for matching games.

that $Y_i(g', v) > Y_i(g, v)$, there exists $j \in S$ such that $Y_j(g', v) < Y_j(g, v)$.[25]

Strong stability provides a powerful refinement of pairwise stability. The concept of strong stability mainly makes sense in smaller network situations where players have substantial information about the overall structure and potential payoffs and can coordinate their actions. Thus, for instance, it might be more applicable to agreements between firms in an oligopoly, than modeling friendships in a large society.

Strong stability also faces some high hurdles in terms of existence as it is a very demanding concept. In fact, one might argue that this is too demanding a concept as it might be that what appears to be an improving deviation might not be taken if one starts to forecast how the other players might react. That is an issue that is addressed in the recently developed notions of "farsighted" network formation, which is discussed in the chapter by Page (2003). Nevertheless, when strongly stable networks exist, they have very nice properties.

Forming a Network and Bargaining

The above mentioned methods of modeling network formation are such that the network formation process and the allocation of value among players in a network are separated. Currarini and Morelli (2000) provide an interesting approach where the allocation of value among players takes place simultaneously with the link formation, as players may bargain over their shares of value as they negotiate whether or not to add a link.[26]

The game that Currarini and Morelli analyze is described as follows. Players are ordered exogenously according to a function $\rho : N \to N$. Without loss of generality assume that this is in the order of their labels, so player 1 moves first, then player 2 and so forth. A player i announces the set of players with whom they are willing to be linked ($a_i \in 2^{N \setminus \{i\}}$), and a payoff demand $d_i \in \mathbb{R}$. The outcome of the game is then as follows. The actions $a = (a_1, \dots, a_n)$ determine a network $g(a)$ by requiring that a link ij is in $g(a)$ if and only if $j \in a_i$ and $i \in a_j$. However, the network that

[25]The difference between this definition of strong stability from Jackson and van den Nouweland (2000) and that of Dutta and Mutuswami (1997) is as follows. The above definition allows for a deviation to be valid if some members are strictly better off and others are weakly better off, while the definition in Dutta and Mutuswami (1997) considers a deviation valid only if all members of a coalition are strictly better off. While the difference is fairly minor, this stronger notion implies pairwise stability while Dutta and Mutuswami's (1997) definition does not.

[26]See also Slikker and van den Nouweland (2001), Mutuswami and Winter (2000), and Nieva (2002), for similar approaches, and van den Nouweland (2003) for a detailed description.

is eventually formed is determined by checking which components of $g(a)$ are actually feasible in terms of the demands submitted. That is, if $h \in C(g(a))$, then h is actually formed if and only if $\sum_{i \in N(h)} d_i \leq v(h)$.[27] In cases where $\sum_{i \in N(h)} d_i > v(h)$, the links in h are all deleted and the players in $N(h)$ are left without any links.

As I discuss below, the simultaneous bargaining over allocations and network formation can make an important difference in conclusions about the efficiency of the networks that are formed. This means that it is an idea which must be carefully accounted for. The main difficulty with this approach is the specification of the bargaining game, whose fine details (such as how the game ends) can be very important in determining what networks form and how value is distributed.

Dynamic Models

Beyond the one-time models of network formation, one can also take a dynamic perspective where networks are formed over time. The first such approach was taken by Watts (1997) in the context of the symmetric connections model (Example 1). She modeled this as follows. First let me introduce some terminology that will be useful later and is helpful in discussing Watts' ideas.

A network g' is *adjacent* to a network g if $g' = g + ij$ or $g' = g - ij$ for some ij.

A network g' *defeats* another network g if either $g' = g - ij$ and $Y_i(g', v) > Y_i(g', v)$, or if $g' = g + ij$ with $Y_i(g', v) \geq Y_i(g', v)$ and $Y_j(g', v) \geq Y_j(g', v)$ with at least one inequality holding strictly.

Note that under this terminology, a network is pairwise stable if and only if it is not defeated by an (adjacent) network.

Watts' process can then be described as follows. The network begins as an empty network. At each time $t \in \{1, 2, \ldots\}$ a link is randomly identified. The current network is altered if and only if the addition or deletion of the link would defeat the current network. Thus, players add or delete links through myopic considerations of whether this would increase their payoffs.

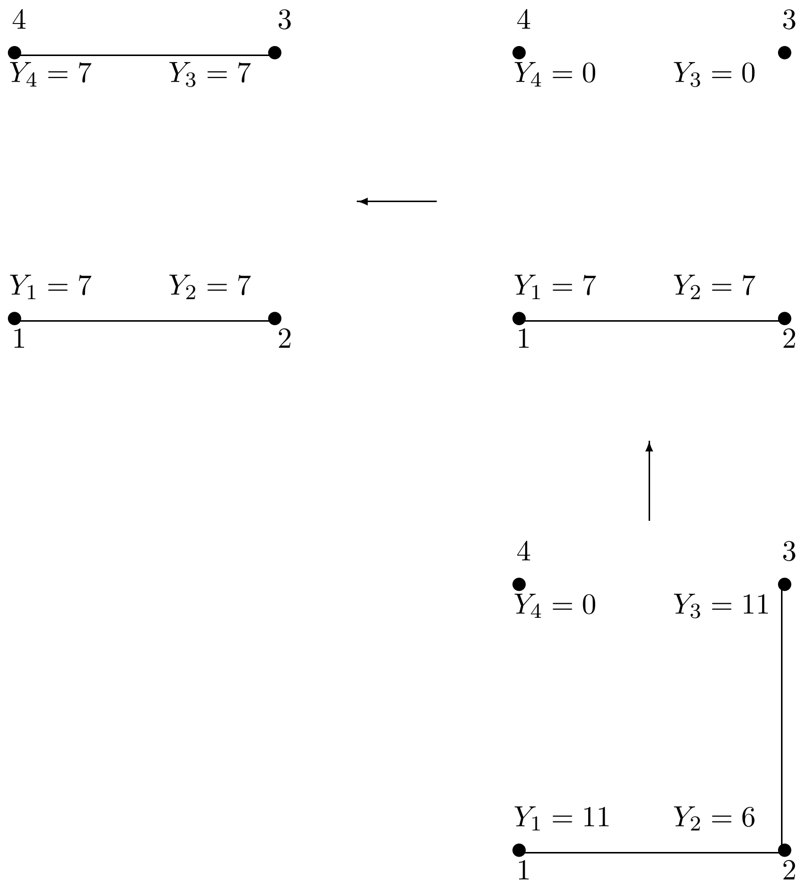
Watts says that a network has reached a *stable state* if there is some time t after which no links would ever be added or deleted.

The set of stable states is clearly a subset of the pairwise stable networks.

The following notion from Jackson and Watts (2002a) captures this notion of sequences of networks where each network defeats the previous one.

[27]To understand these definitions, it is useful to think of v being component additive, so that $\sum_{h \in C(g)} v(h) = v(g)$.

An *improving path*[28] is a sequence of networks $\{g_1, g_2, \ldots, g_K\}$ where each network g_k is defeated by the subsequent (adjacent) network g_{k+1}.



An improving path in this example is the sequence of networks $\{12, 23\}$, $\{12\}$, $\{12, 34\}$. Here $\{12, 23\}$ is defeated by $\{12\}$, as 2 benefits by severing the link 23, and this in turn is defeated by $\{12, 34\}$ as 3 and 4 both benefit by adding the link 34.

A network is pairwise stable if and only if it has no improving paths emanating from it. Note that a stable state is any pairwise stable network that can be reached by an improving path from the empty network.

[28]The term path here refers to a sequence of networks and should not be confused with a path inside a network.

A difficulty with the idea of a stable state is that in some situations one can get stuck at the empty network because any single link results in a negative value, even though it might be that larger networks are valuable. If one can start at any network, then any pairwise stable network could be reached by an improving path. But without specifying the process more fully, it is not clear what the right starting conditions are. Introducing some stochastics into the picture solves this quite naturally.

Stochastic Dynamic Models

There are two dynamic approaches which can overcome this difficulty of getting stuck at a network. One is to do away with the myopic nature of players choices, as discussed below. This makes sense in situations where the networks are relatively small, players know each other and can make good predictions about future plays. The other approach is to introduce some randomness in the network formation process, where links might be added or deleted via some exogenous stimulus or simply by error or experiment on the part of the players.

This second approach of introducing random perturbations to the formation process was first studied by Jackson and Watts (2002a).[29] The setup is described as follows. Start at some network g. At each time $t \in \{1, 2, \ldots\}$ a link ij is randomly identified. Just as in the notion of improving path, we check whether the players in question would like to add the link if it is not in the network or sever the link if it is in the network. What is new is that the intentions of the players are only carried out with probability $1 - \varepsilon$, and with probability $\varepsilon > 0$ the reverse happens. Given these random perturbations in the process, it will go on forever, and has a chance of visiting any network. Some networks are more likely to be visited than others, as some can only be reached through a series of errors, while others are more naturally reached through the intentions of the players. We can then examine this process to see which networks have the highest probability of being reached.[30]

When pairwise stable networks exist, then this analysis of stochastic stability will

[29]This was further studied in the context of the play of non-cooperative games by Jackson and Watts (2002b), Goyal and Vega-Redondo (2000), and Droste, Gilles, and Johnson (2001). See also Skyrms and Pemantle (2000) for a reinforcement based evolutionary analysis of games played on networks. See Goyal (2003) for some discussion of those papers.

[30]More formally, we end up with a finite state aperiodic and irreducible Markov process. Techniques for characterizing the limiting distribution of such processes as $\varepsilon \to 0$ are well developed. In particular, a theorem by Freidlin and Wentzell (1981), as adapted to the study of stochastic stability by Kandori, Mailath, and Rob (1993) and Young (1993), is the key tool. Jackson and Watts (2002a) provide details of the adaptation of this tool to the network setting.

select a subset of them. When pairwise stable networks do not exist, the limit of the above process will involve cycles of networks which are randomly visited over time.

The advantages of these dynamic analyses is that they can select from among the pairwise stable networks. In the case of stochastic stability, essentially the most robust or easy to reach networks are selected. The disadvantage with this approach is that the limit points of the dynamics can be difficult to identify in some applications.[31]

Farsighted Network Formation

Finally, as alluded to earlier, there is another aspect of network formation that deserves attention. The above definitions generally either have some myopic aspect to them, or some artificial stopping point (a finite horizon to the game) which limits the ability of other players to react. For instance, the adding or severing of one link might lead to the subsequent addition or severing of another link. Depending on the context, this might be an important consideration. In large networks it might be that players have very little ability to forecast how the network might change in reaction to the addition or deletion of a link. In such situations the myopic solutions are quite reasonable. However, if players have very good information about how others might react to changes in the network, then these are things that one wants to allow for either in the specification of the game or in the definition of the stability concept. Recent work by Page, Wooders, and Kamat (2002), Watts (2002), and Dutta, Ghosal and Ray (2002), and Deroian (2002) address this issue. This is surveyed in the chapter by Page and Kamat (2003) and so I will not discuss that here.

The Existence of Stable Networks

The existence of stable networks depends on which of the above approaches one takes to modeling stability.

In the case where one uses the sequential game of Aumann and Myerson (1988) or Currarini and Morelli (2000), existence of a (subgame perfect) equilibrium can be established through results in the game theoretic literature.[32]

When looking at the simultaneous move game of Myerson (1991), existence of a variety of types of equilibria are easily established, again via standard theorems.

[31]It does depend on the application. See Jackson and Watts (2002a) for more detail and some examples (such a the bipartite matching problems) where such techniques have sharp predictions.

[32]The game of Aumann and Myerson is a finite extensive form game of perfect information, and so existence follows easily from well-known theorems. Currarini and Morelli have a few more hoops to jump through as they have infinite action spaces.

The main challenges arise in coming to grips with the issue of mutual consent needed to form a link. This occurs either in using a coalition based solution concept to solve Myerson's game, or when one moves to notions such as pairwise stability, strong stability, and stable states. Let me turn to what is known about these issues.

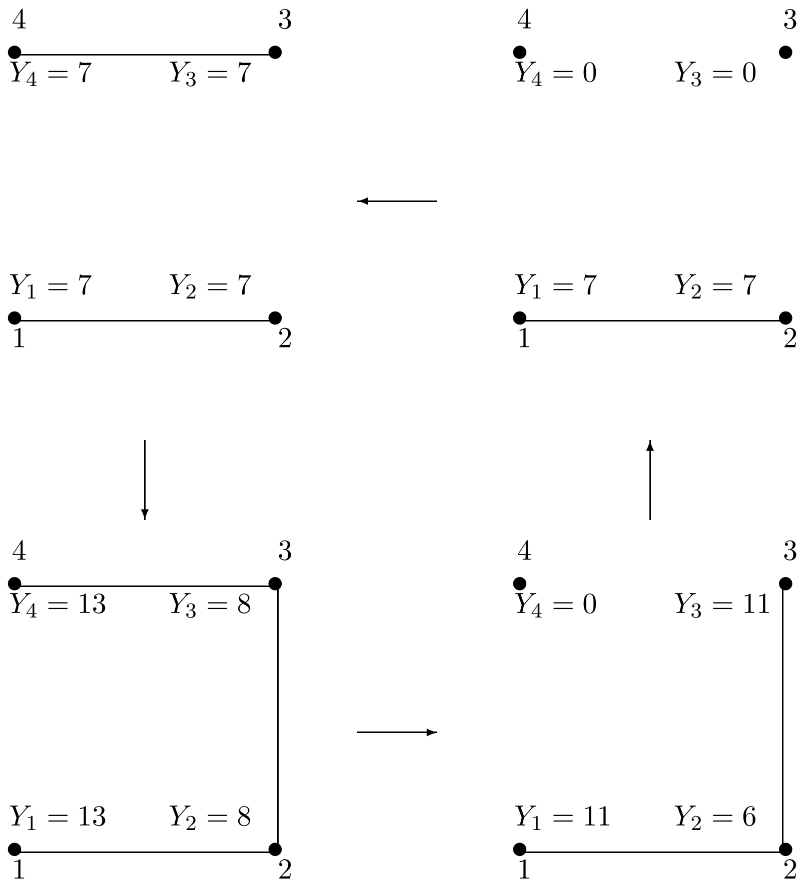
Given the finite number of possible networks, it then follows that if there does not exist any pairwise stable network, then there must exist at least one cycle: an improving path $\{g_1, g_2, \ldots, g_K\}$ where $g_1 = g_K$. Indeed, there are situations where there does not exist any pairwise stable network and following improving paths only lead to cycles with no undefeated networks.

This is demonstrated in the following example from Jackson and Watts (2002a).

Example **12** *Exchange Networks – Non-existence of a Pairwise Stable Network*

The society consists of $n \geq 4$ players who get value from trading goods with each other. Without going into details (see Jackson and Watts (2002a) for those), the idea is that players have random endowments and may gain from trading with others. The more players who are linked, the greater the potential gains from trade, but with a diminishing return to the number of players added. Moreover, there is an externality in that the link is costly to the players who are directly involved, but this may benefit other players through the improved flow of goods through the network. The non-existence of a pairwise stable network is due to these external effects. Players near the end of a "line" network of more than two, wish to sever the link to the end players as that link is more costly to maintain than it directly benefits the players involved. However, once one gets down to separate single links, two such links would like to join up.

This is illustrated in the following figure.



A cycle in this example is $\{12, 34\}$ is defeated by $\{12, 23, 34\}$ which is defeated by $\{12, 23\}$ which is defeated by $\{12\}$ which is defeated by $\{12, 34\}$.

Jackson and Watts (2001) provide a result characterizing when it is that there are no cycles and there exist pairwise stable networks. (Note that if there are no pairwise stable networks, then there must exist a cycle.)

Y and v *exhibit no indifference* if for any two adjacent networks, one defeats the other.

PROPOSITION **1** *Fix v and Y. If there exists a function $w : G \rightarrow I\!R$ such that [g' defeats g] if and only if [$w(g') > w(g)$ and g' and g are adjacent], then there are no* *cycles. Conversely, if Y and v exhibit no indifference, then there are no cycles only if there exists a function $w : G \to I\!R$ such that [g' defeats g] if and only if [$w(g') > w(g)$ and g' and g are adjacent]*

The function w has an intuitive relationship to a potential function as defined in non-cooperative games (see Monderer and Shapley (199?)).

While the above proposition seems difficult to use given that one must find some such w, it has some surprisingly simple applications. One application is to prove existence of pairwise stable networks under the Myerson value which is a prominent allocation rule.

The Myerson Value

The Myerson value is an allocation rule that was defined by Myerson (1977), in the context of cooperative games with communication (aka cooperation) structures, that is a variation on the Shapley value. This rule was subsequently referred to as the Myerson value (see Aumann and Myerson (1988)). The Myerson value also has a corresponding allocation rule in the context of network games as well, as shown by Jackson and Wolinsky (1996). That allocation rule is expressed as follows.

$$Y_i^{MV}(g,v) = \sum_{S \subset N \setminus \{i\}} (v(g|_{S \cup i}) - v(g|_S)) \left(\frac{\#S!(n - \#S - 1)!}{n!} \right).$$

The Myerson value follows Shapley Value style calculations and allocates value based on those calculations. That is, we can think of building up our network by adding players one by one, and then seeing what value is generated through this process. Players are allocated their marginal contributions to generating overall value. In this process there are many different orders in which this could be done and the factor on the right hand side accounts for averaging over all of the different orderings through which we could calculate the marginal contributions of players.

The following Proposition is due to Jackson (2003).

Proposition **2** *There exists a pairwise stable network relative to Y^{MV} for every v. Moreover, all improving paths relative to Y^{MV} and under any v emanating from any network lead to pairwise stable networks. Thus, there are no cycles under the Myerson Value allocation rule.*

This can be proven a corollary to Proposition 1 by noting that

$$Y_i^{MV}(g,v) - Y_i^{MV}(g-ij,v) = w(g) - w(g-ij),$$

where

$$w(g) = \sum_{S\subset N} v(g|_S) \left(\frac{(\#S-1)!(n-\#S)!}{n!} \right).$$

This proposition shows that existence of pairwise stable networks is at least very well behaved for some prominent allocation rules. In fact, existence of pairwise stable networks is also straightforward for two other natural rules: egalitarian and component-wise egalitarian allocation rules.

Egalitarian Rules

The following allocation rules were defined by Jackson and Wolinsky (1996).

The *egalitarian allocation rule* Y^e is defined by $Y_i^e(g,v) = \frac{v(g)}{n}$. Here simply set $w(g) = \frac{v(g)}{n}$ and apply Proposition 1, or alternatively simply note that under the egalitarian rule, any efficient network will be pairwise stable.

The *component-wise egalitarian allocation rule* Y^{ce} is defined as follows. For a component additive v and network, Y^{ce} is such that for any $h \in C(g)$ and each $i \in N(h)$

$$Y_i^{ce}(g,v) = \frac{v(h)}{\#N(h)}.$$

For v that is not component additive, $Y^{ce}(g,v) = Y^e(g,v)$ for all g; so that Y^{ce} splits the value $v(g)$ equally among all players if v is not component additive. The component-wise egalitarian rule is one where the value of each component is split equally among the members of the component; provided this can be done - that is, within the limits of component additivity.

Under the component-wise egalitarian rule, one can also always find a pairwise stable network. However, for this rule one cannot apply Proposition 1. Instead one must follow other lines of proof. As noted by Jackson (2003), an algorithm for finding a pairwise stable network is as follows:[33] find a component h that maximizes the payoff $Y_i^{ce}(h,v)$ over i and h. Next, do the same on the remaining population $N \setminus N(h)$, and so on. The collection of resulting components forms the network.[34]

[33]This is specified for component additive v's. For any other v, Y^e and Y^{ce} coincide.

[34]This follows the same argument as existence of core-stable coalition structures under the weak top coalition property in Banerjee, Konishi and Sönmez (2001). Note, however, that the networks identified by this algorithm are not necessarily strongly stable under the definition used here.

Now that we have seen some of the methods for modeling network formation, let us turn to one of the main foci of the literature: the relationship between the stable networks and the efficient networks.

5 The Relationship Between Stability and Efficiency

Some of the very central questions about network formation concern the conditions under which the networks which are formed by the players turn out to be efficient from an overall societal perspective. In order to discuss these issues we need to define what we mean by efficiency.

An obvious notion of efficiency is simply maximizing the overall total value among all possible networks. This notion was referred to as strong efficiency by Jackson and Wolinsky (1996), but I will simply refer to it as efficiency.

Efficiency

A network g is *efficient* relative to v if $v(g) \geq v(g')$ for all $g' \in G$.

It is clear that there will always exist at least one efficient network, given that there are only a finite set of networks.

Once we begin to define things relative to a fixed allocation rule, then there is another natural notion of efficiency: the standard notion of Pareto efficiency.

Pareto Efficiency

A network g is *Pareto efficient* relative to v and Y if there does not exist any $g' \in G$ such that $Y_i(g', v) \geq Y_i(g, v)$ for all i with strict inequality for some i.

To understand the relationship between the two definitions, note that g is efficient relative to v if it is Pareto efficient relative to v and Y *for all* Y.

Thus, efficiency is the more natural notion in situations where there is some freedom to reallocate value through transfers, while Pareto efficiency might be more reasonable in contexts where the allocation rule is fixed (and we are not able or willing to make further transfers or to make interpersonal comparisons of utility).

Beyond these notions of efficiency, one may want to consider others. For instance it may be that some reallocation of value is possible, but only under the constraints that the allocations are balanced on each component. Such constraints lead to the following definition of constrained efficiency introduced by Jackson (2003).

A network g is *constrained efficient* relative to v if and only if it is Pareto efficient relative to v and Y for every component balanced and anonymous Y.

With definitions of efficiency in hand, we can examine the central question of the relationship between stability and efficiency of networks.

I begin with the simple model of the symmetric connections model. While this is a highly stylized model, it provides a preview of some of the tension between stability and efficiency and gives an idea of why such a conflict might arise.

The following propositions are from Jackson and Wolinsky (1996).

PROPOSITION **3** *The unique efficient network structure in the symmetric connections model (Example 1) is*

(i) the complete graph g^N *if* $c < \delta - \delta^2$,

(ii) a star encompassing everyone if $\delta - \delta^2 < c < \delta + \frac{(N-2)}{2}\delta^2$, *and*

(iii) no links if $\delta + \frac{(N-2)}{2}\delta^2 < c$.

The efficient networks take simple and intuitive forms. If link costs are high, then it does not make sense to form any links, (iii). If link costs are low enough ($c < \delta - \delta^2$), then it makes sense to form all links as the cost of adding a link is less then the gain from shortening any path of length at least two into a path of length one. The more interesting case arises for intermediate costs of links. Here the only efficient network structure is a star. To see why, note that a star has the minimal number of links needed to connect any set of players. Moreover, it is the (unique) network structure that minimizes the average path length given the minimal number of links.

Given that the star is the only efficient network for intermediate costs of links, we might expect to see some conflict between stability of a network and efficiency. In a star network in the connections model, the center player bears a great deal of cost and provides a great deal of externalities for other players, but is not compensated for those externalities. Thus there will be whole ranges of costs of links, where the efficient networks are not pairwise stable. The description of pairwise stable networks in the symmetric connections model, from Jackson and Wolinsky (1996), is as follows.

PROPOSITION **4** *In the symmetric connections model :*

(i) A pairwise stable network has at most one (non-empty) component.

(ii) For $c < \delta - \delta^2$, the unique pairwise stable network is the the complete graph, g^N.

(iii) For $\delta - \delta^2 < c < \delta$, a star encompassing all players is pairwise stable, but not necessarily the unique pairwise stable graph.

(iv) For $\delta < c$, any pairwise stable network which is nonempty is such that each player has at least two links and thus is inefficient.[35]

As we expected, for high and low costs to links, efficient networks coincide with the pairwise stable networks, and the problematic case is for intermediate costs to links.

For instance, consider a situation where $n = 4$ and $\delta < c < \delta + \frac{\delta^2}{2}$. Here a star network is the unique efficient structure. However, the only pairwise stable network is the empty network. To see this, note that since $c > \delta$ a player gets a positive payoff from a link only if it also offers an indirect connection. Thus, clearly the star will not be pairwise stable as the center bears more cost for each link than it gets in benefits. Moreover, this implies that if a network is nonempty and stable then each player must have at least two links, as if i has only one link and it is to j, then j would benefit from severing that link. We can also see in this cost range that a player maintains at most 2 links, since the payoff to a player with three links (given $n = 4$) is less than 0 since $c > \delta$. So, a pairwise stable network would have to be a ring (e.g., $\{12, 23, 34, 14\}$). However, such a network is not pairwise stable since the payoff to any player is increased by severing a link. For instance, 1's payoff in the ring is $2\delta + \delta^2 - 2c$, while severing the link 14 leads to $\delta + \delta^2 + \delta^3 - c$ which is higher since $c > \delta$.

Although the empty network is the unique pairwise stable network, it is not even Pareto efficient. The empty network is Pareto dominated by a line (e.g., $g = \{12, 23, 34\}$). To see this, not that under the line, the payoff to the end players (1 and 4) is $\delta + \delta^2 + \delta^3 - c$ which is greater than 0, and to the middle two players (2 and 3) the payoff is $2\delta + \delta^2 - 2c$ which is also greater than 0 since $c < \delta + \frac{\delta^2}{2}$.

Thus, there exist cost ranges under the symmetric connections model for which all pairwise stable networks are *Pareto* inefficient, and other cost ranges where all pairwise stable networks are efficient. There are also some cost ranges where some pairwise stable networks are efficient and some other pairwise stable networks are not even Pareto efficient.

[35]If $\delta + \frac{(N-2)}{2}\delta^2 > c$, then all pairwise stable networks are inefficient since then the empty graph is also inefficient.

Even when there are some efficient networks that are pairwise stable in the symmetric connections model, they might not be reached. For instance, Watts (1997) shows that as n increases, the probability that the resulting stable state is a star goes to 0. Thus as the population increases the particular ordering which is needed to form a star (the efficient network) becomes less and less likely relative to orderings leading to some other stable states. Watts' (1997) result is stated as follows.

PROPOSITION **5** *Consider the symmetric connections model in the case where* $\delta - \delta^2 < c < \delta$. *As the number of players grows, the probability that a stable state (under the process where each link has an equal probability of being identified) is reached with the efficient network structure of a star goes to 0.*

The above propositions show us that there may be cases where the networks that are pairwise stable (or stable states) are not efficient, nor even Pareto efficient.

At this point there is a series of important questions that come up.

We begin to see from the connections model that some reallocation of value might be natural, and might help reconcile efficiency and stability. For instance, the center of the star could negotiate with the other players to receive some payments or favors for maintaining her links with the other players.[36] If we start to account for such reallocations can we reconcile efficiency and stability?

As there are many different ways in which we might think of these issues, let us list some of the questions along these lines come to mind.

(1) If we can control the allocation rule, can we always design an allocation rule such that at least one efficient network is pairwise stable?

(2) Can we always design an allocation rule such that at least one efficient network is pairwise stable if we impose some minimal conditions on the allocation rule such as anonymity and component balance?

(3) If the answer to (2) is no, what if we weaken the demands on efficiency, or on anonymity or on component balance?

(4) If the answer to (2) is no, is there some nice class of situations where we can design an anonymous and component balanced allocation rule such that at least one efficient network is pairwise stable?

[36] In fact, intuition from the sociology literature would suggest that a player in such a central position should receive a high payoff (e.g., see Burt (1992)).

(5) For given allocation rules, what are the classes of value functions for when it is that efficient and stable networks coincide?

(6) What can we say about these questions for alternative stability notions?

(7) Will efficient networks be formed if bargaining over the allocation and the network formation are tied together?

The answer to question (1) (whether we can design an allocation rule that reconciles efficiency and stability) is yes, and it is easy to see. Consider the egalitarian allocation rule, Y^e. This completely aligns player incentives and overall efficiency as players' payoffs are directly proportional to overall network value. Thus, under the egalitarian allocation rule every efficient network is pairwise stable, and in fact strongly stable.

While this is partly reassuring, it turns out that this answer is really dependent on such a full reallocation of value. A fully egalitarian rule has nice incentive properties, but it is an extreme rule and in particular requires that value be allocated across different components. That is, the egalitarian rule fails to satisfy component balance. In the long run this might be problematic, as some components will be receiving less than their value and might benefit from seceding.

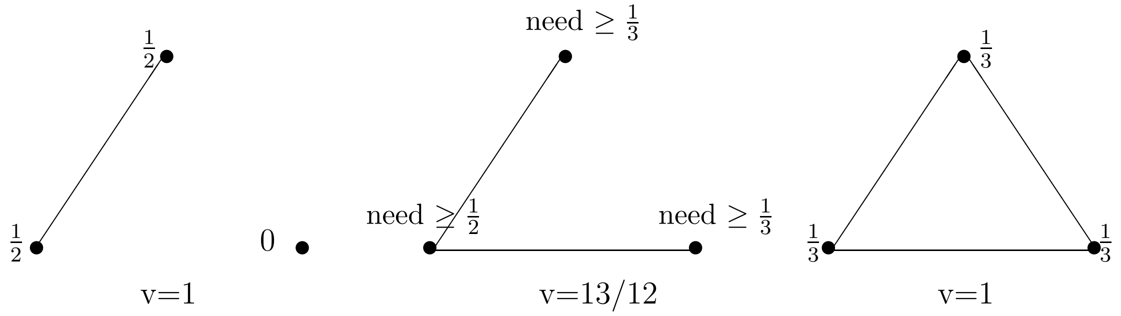
This takes us to question (2) - as to whether or not we can find an allocation rule for which efficiency and stability are reconciled, while at the same time satisfying some simple conditions such as component balance and anonymity. The following proposition shows that there is no component balanced and anonymous allocation rule for which it is always the case that some efficient network is pairwise stable. Thus, the answer to (2) is no. This proposition is due to Jackson and Wolinsky (1996).[37]

Proposition **6** *There does not exist any component balanced and anonymous allocation rule such that for every v there exists an efficient network that is pairwise stable.*

The proof of Proposition 6 shows that there is a particular v such that for every component balanced and anonymous allocation rule none of the constrained efficient networks are pairwise stable.

[37] Jackson (2003) shows that the proposition can be strengthened to only require equal treatment of equals rather than anonymity, and that it also holds if efficiency of the network is replaced by constrained efficiency. Arguably, if one is requiring component balance of the allocation rule, then the efficiency notion should be similarly defined and so constrained efficiency is the appropriate notion. See Jackson (2003) for details.

To see the proof, simply consider the following example with $n = 3$ players. Here v is such that any one-link network has a value of 1, any two-link network has a value of 13/12, and the complete network has a value of 1. This is as pictured in the following figure.



So, in this example the efficient network structure is a two-link network. Now let us consider what allocations have to be. By anonymity and component balance, the allocations in one-link and three-link networks are completely determined. Each player connected in a one-link network gets an allocation of $\frac{1}{2}$. Each player in the complete network gets an allocation of $\frac{1}{3}$. Let us consider what the possibilities are for a two-link network, with the idea that we would like to make the two-link network pairwise stable. In order for a two-link network to be pairwise stable it must be that the middle player (who has two links) gets an allocation of at least $\frac{1}{2}$ or else he would benefit from severing one of the links. Also, for a two-link network to be pairwise stable the other two players must each get an allocation of at least $\frac{1}{3}$ or else they would benefit from adding a link between them. Unfortunately $\frac{1}{2} + \frac{1}{3} + \frac{1}{3} > \frac{13}{12}$, and so this is not feasible. Thus, there is no possible allocation rule satisfying anonymity and component balance such that the efficient network is pairwise stable here.

The answer to question (3) is that the conditions of anonymity and component balance do play important roles in the incompatibility of stability and efficiency. If we drop either of the conditions then we can reconcile efficiency and stability. That is, Proposition 6 is tight.

Let us examine each aspect of the proposition. If we drop component balance, then as mentioned before the egalitarian allocation rule will always have all efficient networks being (strongly) stable.

If we drop anonymity (or equal treatment of equals), then a careful and clever construction of Y by Dutta and Mutuswami (1997) ensures that some efficient network is strongly stable for a class of v. This is stated in the following proposition.

Let $\mathcal{V}^* = \{v \in \mathcal{V} \mid g \neq \emptyset \Rightarrow v(g) > 0\}$. This is a class of value functions where any network generates a positive value.

The following proposition is due to Dutta and Mutuswami (1997).

PROPOSITION **7** *There exists a component balanced Y such that for any $v \in \mathcal{V}^*$, some efficient network is pairwise stable. Moreover, while Y is not anonymous, it is still anonymous on some networks that are both efficient and pairwise stable.* [38]

This proposition shows that if one can design an allocation rule, and only wishes to satisfy anonymity on stable networks, then efficiency and stability are compatible.

Next, let us consider the question of requiring the allocation rule to be component balanced and anonymous, but weakening efficiency to only require Pareto efficiency. If we do this, then the component-wise egalitarian rule ensures that for any value function at least one pairwise stable network is Pareto efficient, as stated in the following proposition.

Let $g(v, S) = \text{argmax}_{g \in g^S} \frac{v(g)}{\#N(g)}$ denote the network with the highest per capita value out of those that can be formed by players in $S \subset N$.

Given a component additive v, find a network g^v through the following algorithm. Pick some $h_1 \in g(v, N)$ with a maximal number of links out of those in the set. Next, pick some $h_2 \in g(v, N \setminus N(h_1))$ with a maximal number of links out of those in the set. Iteratively, at stage k pick a new component $h_k \in g(v, N \setminus N(\cup_{i \leq k-1} h_i))$ with a maximal number of links. Once there are only empty networks left stop. The union of the components picked in this way defines a network g^v.

The following proposition is a variation on one due to Banerjee (1999).[39]

[38] Dutta and Mutuswami work with a variation of strong stability. As mentioned before, their version of strong stability is not quite a strengthening of pairwise stability, as it only considers one network to defeat another if there is a deviation by a coalitions that makes all of its members strictly better off; while pairwise stability allows one of the two players adding a link to be indifferent. However, one can check that the construction of Dutta and Mutuswami extends to pairwise stability as well.

[39] Banerjee (1999) actually works with a weighted version of the component-wise egalitarian rule, which is a straightforward generalization of this result. Also, he works with a notion of strong stability, but one that only accounts for deviations that make all players strictly better off. Note that g^v will not always be strongly stable under the definition here. Finally, the algorithm here is a bit different from his, as I require the maximal number of links in the definition of each h_k, and this is critical

Proposition 8 *Under a component additive v, a g^v defined by the above algorithm is a pairwise stable and Pareto efficient network under the component-wise egalitarian rule.*

While this proposition is of some interest, given that we are allowing reallocation of value, it's not clear that Pareto efficiency is the right notion of efficiency. In particular, constrained efficiency seems to be more appropriate, and then the proposition no longer is true, as we have seen already in Proposition 6 (and its footnote).

So, reconciling the tension between stability and efficiency will require giving something up in terms of our desired conditions of anonymity, component balance, and efficiency; and so to some extent this tension is a characteristic of network games.

This leads us to another one of our questions (4): is there some nice class of situations where we can design an anonymous and component balanced allocation rule such that at least one efficient network is pairwise stable? That is, the tension arises for some value functions, but not all. What do we know about the structure of value functions for which there is (or is not) a tension?

The following proposition provides a partial answer to (4) by identifying a very particular feature of the tension between efficiency and stability. It shows that in situations where efficient networks are such that each player has at least two links, then there is no tension. So, problems arise only in situations where efficient networks involve players who may be thought of as "loose ends."

A network g has *no loose ends* if for any $i \in N(g)$, $\#\{j|ij \in g\} \geq 2$.

The following proposition is due to Jackson (2003).

Proposition 9 *There exists an anonymous and component balanced allocation rule such that if v is anonymous and has an efficient network with no loose ends, then there is at least one efficient network (with no loose ends) that is pairwise stable.*

The proof of proposition 9 is constructive, showing that a variation on the component-wise egalitarian allocation rule works. This tells us that the tension between efficiency and stability has some natural limits, and must involve situations where efficient networks has some players who have single links.

to guaranteeing pairwise stability. Banerjee does not have to worry about this since his definition of stability only considers deviations where all deviating players are strictly better off.

The analysis in the last few propositions took a "design" perspective, in that the question was asked as to whether there existed any allocation rule that would reconcile efficiency and stability. More generally, however, the allocation rule might be determined naturally by the environment. To the extent that we cannot intervene (or prefer not to unless needed), it is important to know when there will be a tension between efficiency and stability for a given allocation rule.

The difficulty in addressing this issue is that the space of allocation rules is quite large and so providing a characterization of when there are tensions and when not, is an overwhelming task. What we might do, is instead simply look at some natural allocation rules and natural settings. Many of the examples from Section 3 are ones for which this is the approach that has been taken. There a setting, value function, and allocation rule are given by the model, and then one analyzes which networks are stable and can address the issue of whether they are efficient. This is a valuable exercise, and provides some insights. The results, however, are particular to the models in question. Given the limits on the length of this survey, I will not go over those results here.[40]

In order to get a bit broader view, we can look at question (5): For given allocation rules, what are the classes of value functions for when it is that efficient and stable networks coincide?

A natural starting point for this question is to work with the most obvious of component balanced and anonymous allocation rules, the component-wise egalitarian rule. A strong reason for doing this is that we know the egalitarian rule has very nice incentive properties, and so the component-wise version would seem to be a nice one to work with under the constraint of component balance. The following proposition provides a characterization of when the component-wise egalitarian rule works well.

A link ij is *critical* to the graph g if $g - ij$ has more components than g or if i is only linked to j under g.

A critical link is one such that if it is severed, then the component that it was a part of will become two components (or one of the nodes will become disconnected). Let h denote a component which contains a critical link and let h_1 and h_2 denote the components obtained from h by severing that link (where it may be that $h_1 = \emptyset$ or $h_2 = \emptyset$).

The pair (g, v) satisfies *critical link monotonicity* if, for any critical link in g and its

[40] The reader is referred to the papers themselves. The reader can also find some comparison across some of the bargaining models in Jackson (2003).

associated components h, h_1, and h_2, we have that $v(h) \geq v(h_1) + v(h_2)$ implies that $v(h)/\#N(h) \geq \max[v(h_1)/\#N(h_1), v(h_2)/\#N(h_2)]$.

The following proposition is due to Jackson and Wolinsky (1996)

PROPOSITION **10** *If g is efficient relative to a component additive v, then g is pairwise stable for Y^{ce} relative to v if and only if (g, v) satisfies critical link monotonicity.*

Question (6) asks about the relationship between efficiency and stability for stability notions other than pairwise stability. Short of the consideration of bargaining together with network formation (question (7)), the only analysis has been in terms of strong stability.[41] Note that strong stability goes a long way towards guaranteeing at least Pareto efficiency simply by definition, and so it will not be too surprising that the issue will largely boil down to existence of strongly stable networks. It also turns out that an interesting implication of strong stability is that if we have an efficient network being strongly stable, then the allocation we are working with must be the component-wise egalitarian rule, at least on the given network. This makes the component-wise egalitarian rule a natural one to focus on. These ideas are formalized as follows.

An allocation rule Y is *component decomposable* if $Y_i(g, v) = Y_i(g|_S, v)$ for each component additive v, $g \in G$, $S \in \Pi(g)$, and $i \in S$.

Component decomposability requires that in situations where v is component additive, the way in which value is allocated within a component does not depend on the structure of other components. So, in situations where there are no externalities across components, the allocation within a component is independent of the rest of the network. For instance, the players within a component need not pay attention to, and might not even be aware of, the organization of other components.

The following proposition is due to Jackson and van den Nouweland (2000).

PROPOSITION **11** *Consider any anonymous and component additive value function $v \in \mathcal{V}$. If Y is an anonymous, component decomposable, and component balanced allocation rule and $g \in G$ with $\Pi(g) \neq \{N\}$ is a network that is strongly stable with respect to Y and v, then $Y(g, v) = Y^{ce}(g, v)$ and $Y_i(g, v) = \frac{v(g)}{n}$ for each $i \in N$.*

While Proposition 11 only ties down the allocation rule on strongly stable networks, it still strongly suggests the component-wise egalitarian rule as a focal one. So let us examine when efficient networks are strongly stable under that allocation rule.

[41] There is analysis of other formation models in the context of directed networks. One can find some discussion of that in the chapter by Goyal (2002).

A value function v is *top convex* if some efficient network also maximizes the *per-capita* value among players.[42] That is, the value function v is *top convex* if there exists an efficient g such that $\frac{v(g)}{n} \geq \frac{v(g')}{\#N(g')}$ for all g'.

Top convexity implies that all components of an efficient network must lead to the same per-capita value. If some component led to a lower per capita value than the average, then another component would have to lead to a higher per capita value than the average which would contradict top convexity.

Jackson and van den Nouweland (2001) prove the following proposition.

PROPOSITION **12** *Consider any anonymous and component additive value function v. The set of efficient networks coincides with the set of strongly stable networks under the component-wise egalitarian rule if and only if v is top convex. Moreover, the set of strongly stable networks is nonempty under the component-wise egalitarian rule if and only if v is top convex.*

To get some feeling for the top-convexity condition, note that in the symmetric connections model v is top convex for all values of $\delta \in [0, 1)$ and $c \geq 0$, so that all networks that are strongly stable with respect to Y^{ce} and v are efficient with respect to v. This means that top-convexity is a condition that is satisfied in some natural situations. However, it is still a demanding condition that has strong implications.

Simultaneous Network Formation and Allocation of Value

Finally, let us turn to question (7) regarding what happens when the allocation of value and the formation of the network occur as part of the same bargaining process.

Currarini and Morelli (2000) show that for a wide class of value functions, all subgame perfect equilibria of their formation game are efficient. As it applies for a fairly broad class of value functions, it shows that under some assumptions the tension between stability and efficiency may be overcome if bargaining over value is tied to link formation.

A value function v satisfies size monotonicity if $v(g) > v(g - ij)$ for every g and critical link $ij \in g$.

While this also a demanding condition, it is one that is often satisfied in situations where large networks are efficient. The demanding aspect is the requirement that it hold for all g, and it is not clear the extent to which that is vital to the following result.

The following proposition is due to Currarini and Morelli (2000).

[42] A related condition is called "domination by the grand coalition," as defined in the context of a cooperative game by Chatterjee, Dutta, Ray, and Sengupta (1993).

Proposition 13 *If v satisfies size monotonicity, then every (subgame perfect) equilibrium of the Currarini and Morelli bargaining and network formation game leads to an efficient network.*

Mutuswami and Winter (2000) also discuss a similar network formation game and show that such positive results hold for even more general value functions, under a slightly different formulation. In their analysis, players receive increasing benefits in the size of the network and incur increasing costs. The game is such that instead of making demands on how much value they desire, players indicate how much they will contribute towards the cost of links. Moreover, Mutuswami and Winter show that a variation on the game results in payoffs that mirror the Shapley value calculations of a related cooperative game.

One aspect of these results deserves discussion. The finite ending point of the game provides for strong bargaining power for early players in the sequence. They essentially demand what can maximally be extracted given what the other players will end up getting from the subsequent game. The maximum value comes from the efficient network, as the game really boils down to a bargaining one. It is not so clear what would happen if the bargaining protocol went on in some way, either with discounting or with players being able to revise demands and actions. Proposition 13 is still very important in pointing out the potential role of simultaneous formation of links and bargaining over value, but whether this will turn out to be robust to variations in the protocol is not yet clear.

6 The Myerson Value and alternative Allocation Rules

In addition to the literature that has concentrated on questions of whether or not efficient networks are formed, there is also a literature that has looked in detail at the axiomatic foundations of some allocation rules. To some extent, the axiomatic treatment of allocation rules is the "cooperative" counter-part to the "non-cooperative" analysis of the stability of networks. The axiomatic literature largely grew out of the cooperative game theory literature and mostly followed cooperative games with communication or cooperation structures. However, almost all of the studies there have fairly easy extensions to the more general network game setting.

Much of the literature on cooperative games with communication structures is discussed in the chapter by van den Nouweland (2003), and so here I only briefly discuss here a small part of that axiomatic literature. In particular I discuss a part that is closely related to the idea that networks are not fixed, but something that is subject to the discretion of the players involved; which is the part that is most closely linked to ideas of network formation.

In order to discuss these issues, let us first observe a characterization of the Myerson Value allocation rule, which is the most prominent allocation rule.

Equal Bargaining Power and the Myerson Value

An allocation rule satisfies equal bargaining power if for any component additive v and $g \in G$

$$Y_i(g) - Y_i(g - ij) = Y_j(g) - Y_j(g - ij).$$

Note that equal bargaining power does *not* require that players split the marginal value of a link. It just requires that they equally benefit or suffer from its addition. It is possible (and generally the case) that $Y_i(g) - Y_i(g - ij) + Y_j(g) - Y_j(g - ij) \neq v(g) - v(g - ij)$.

The following proposition from Jackson and Wolinsky (1996) is a fairly direct extension of Myerson's (1977) result from the setting of cooperative games with communication structures to the network game setting.

Proposition **14** *Y satisfies component balance and equal bargaining power if and only if $Y(g, v) = Y^{MV}(g, v)$ for all $g \in G$ and any component additive v.*

Dutta and Mutuswami (1997) extend the characterization to allow for weighted bargaining power, and show that one obtains a version of a weighted Shapley (Myerson) value.

While the Myerson value is an interesting allocation rule, the perspective it takes is problematic from a network formation perspective.[43] The basic problem with it is that the value of other possible networks is not properly accounted for in its calculations. This is especially bothersome in situations where the network is something that can be changed or is being formed. The basic idea is as follows. If the network is something that can be changed, or is such that alternative possible network structures are taken into account when bargaining over how to allocate value, then values of alternative networks, *and not just sub-networks*, should be important in determining the allocation.

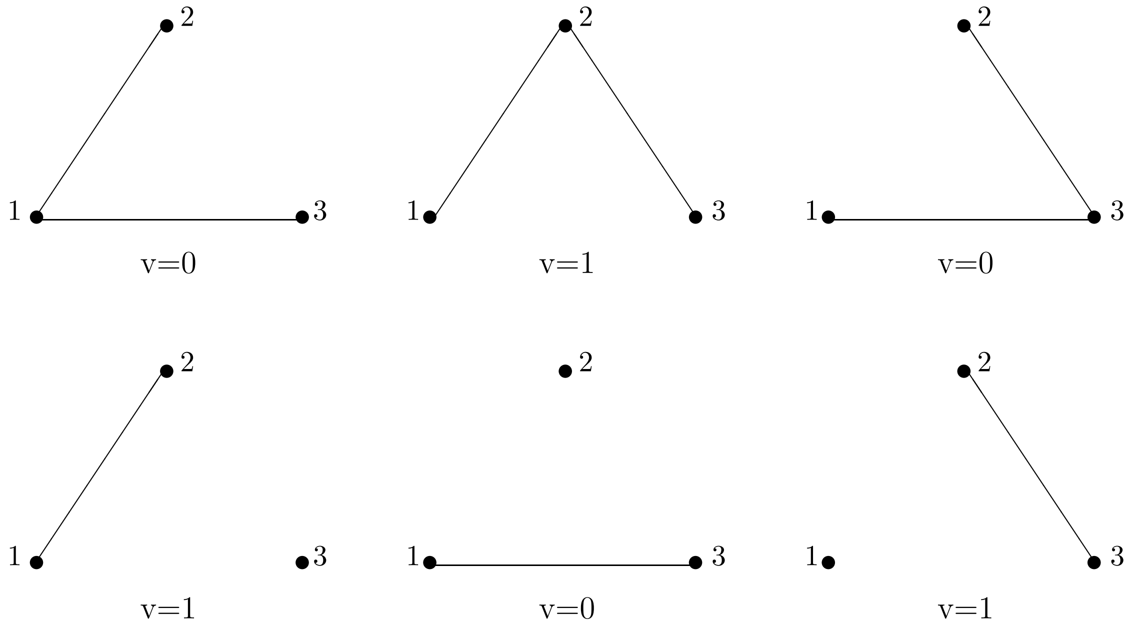
[43] See Jackson (2003b) for a detailed discussion.

If the network is completely fixed and cannot be changed, then it is not clear why the value of sub-networks (and only sub-networks) should enter allocation calculations.

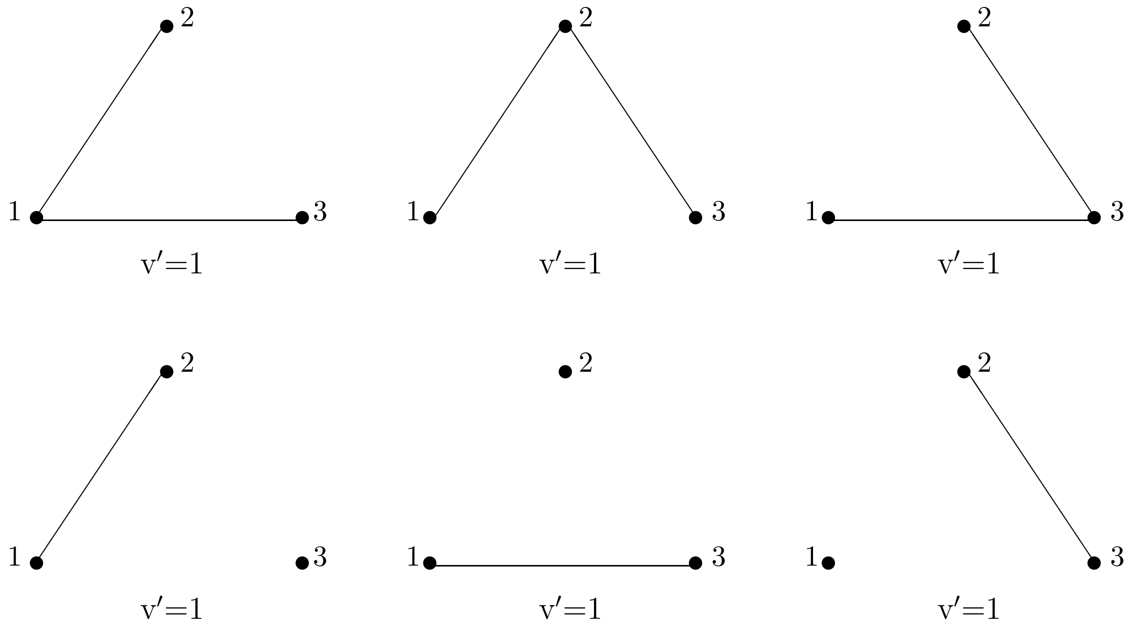
These criticisms can be made more precise by looking at some very simple examples.

Example **13** *A Criticism of the Myerson Value*

Consider a value function v where $v(\{12\}) = v(\{23\}) = 1$, $v(\{12, 23\}) = 1$, and $v(g) = 0$ for all other networks. In this case only the network $\{12, 23\}$ and its nonempty subnetworks generate value.



Consider another value function v' defined by $v'(g) = 1$ for all $g \neq \emptyset$. That is, under v' the value every non-empty network is 1.



Note that

$$Y^{MV}(\{12,23\},v) = Y^{MV}(\{12,23\},v') = \left(\frac{1}{6},\frac{2}{3},\frac{1}{6}\right).$$

The Myerson Value allocation rule provides the same allocation on the network $\{12,23\}$, regardless of whether the value function is v or v'. In particular, player 2 gets a bigger allocation in the network $\{12,23\}$ than the other players. This reflects player 2's status in two links in the network, and comes about through the Shapley value style calculations underlying the Myerson value, where we can think of building up the network $\{12,23\}$ by adding players one at a time.

While player 2's position is special in the network $\{12,23\}$, player 2's status is not at all special if the value function is v'. That is, any player could have equally well served that central position. In fact, any non-empty network would provide the same value as the network $\{12,23\}$. To the extent that the network is something that can be altered, there is no reason that player 2 should enjoy special treatment under v', and one might argue that all players should receive equal payments.

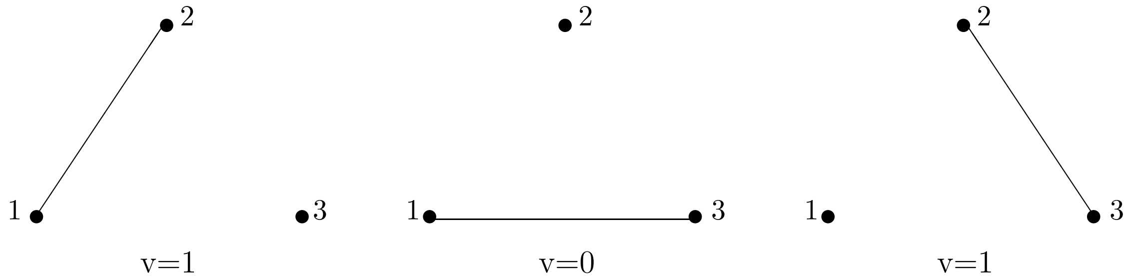
To see some of the issues in more detail, one can look at the conditions that characterize the Myerson value.

EXAMPLE **14** *A Criticism of the Equal Bargaining Power*

Next, let $v(\{12\}) = v(\{23\}) = 1$ and $v(g) = 0$ for all other networks. Here any single link network that involves player 2 will generate a value of 1, while all other networks generate a value of 0.

Any allocation rule, including the Myerson value, that satisfies equal bargaining power (and allocates 0 to the players on the empty network) will have $Y_1(\{12\}, v) = Y_2(\{12\}, v)$.

Other networks have $v = 0$



Here, there is a real asymmetry among the players and player 2 is more a critical player than the others. It is not at all clear why we should require that the allocation to players 1 and 2 be the same in the network $\{12\}$, as player 2 has a viable outside option while player 1 does not.[44]

There are other criticisms that can be made, including pointing out problems with component balance. In response to these criticisms, Jackson (2003b) proposes the following allocation rule. First an auxiliary definition is needed.

Given a value function v, its *monotonic cover* $\widehat{v}$ is defined by

$$\widehat{v}(g) = \max_{g' \subset g} v(g').$$

The monotonic cover of a value function looks at the highest value that can be achieved by building a network out of a given set of links. The monotonic cover captures the perspective that the network is flexible, and so can be reorganized to produce the highest possible value.

Using this perspective leads to a natural adaptation of the Shapley value to network games which results in the following allocation rule, which Jackson called the Player-Based Flexible Network Allocation Rule.[45]

[44]For instance, if one brings in Core-based considerations, then in fact the full value should be given to player 2 in this example.

[45]Jackson (2003b) also proposes other allocation rules. One is a slight variation on this and is

$$Y_i^{PBFN}(g,v) = \frac{v(g)}{\widehat{v}(g^N)} \sum_{S \subset N \setminus \{i\}} (\widehat{v}(g^{S \cup i}) - \widehat{v}(g^S)) \left(\frac{\#S!(n - \#S - 1)!}{n!} \right).$$

This allocation rule looks at the value that any group of players S could generate by forming the best possible network they could. Then through Shapley-style calculations, it looks at the marginal value generated by adding a player to different groups, and then the allocation is in proportion to these marginal contributions.

On a superficial level this rule bears some similarities to the Myerson Value because we see Shapley Value style calculations. However, it is a quite different allocation rule. In fact, it violates both equal bargaining power and component balance, and is characterized by conditions that are violated by the Myerson value. For instance, looking back at Example 13, Y^{PBFN} provides allocations that differ on v and v'. In fact, it agrees with the Myerson Value under v and in contrast is fully egalitarian under v'. In Example 14 it provides higher allocations to player 2 than the others, again in contrast to the Myerson value.

More generally, once one takes this perspective that alternative network structures should matter in determining an allocation, a variety of cooperative game theoretic solutions concepts, including, for example, the Nucleolus, can be called upon in addition to the Shapley value. The simple idea is that the monotonic cover $\widehat{v}$, can be used to define a cooperative game $w(S) = \widehat{v}(g^S)$, which can then be used as a basis for allocating value.[46]

7 Concluding Discussion

Directed Networks

The survey here has focussed on the case of non-directed networks. More specifically, the important aspect here is the treatment of situations where a link between two players requires the consent of both parties. While this non-directed or mutual consent

"link-based". For more on the idea of allocating value based on links rather than players see Meessen (1988) and Borm, Owen, and Tijs (1992). The idea is instead to apply the value to links rather than players, and then to assign the value to players based on the links that they control. Other variations involve using other solution concepts such as the Nucleolus.

[46]See Jackson (2003b) for details.

case covers many (if not most) situations of interest, the case where links are directed and may be formed unilaterally also includes some important settings. For instance, a web site can provide a link or pointer to another web site without the second web site's permission. Likewise if we consider a network of researchers and examine who cites whom, this is another network that is both directed and where links can be formed unilaterally, as one researcher can generally cite another researcher without the second researcher's permission.

While the analysis of directed networks is different from that of non-directed networks, the overall themes end up being similar. The main differences are in modeling network formation, which is simpler due to the unilateral action, and of course in the applications covered. In particular, there still exists a tension between stability and efficiency (see Dutta and Jackson (2000)), and, again, there situations where efficient networks self-organize quite naturally. For instance, Bala and Goyal (2000a) consider a directed version of the connections model, and find that in situations where the decay is not too high (δ is close to 1) efficient networks are the unique strict Nash equilibrium networks in a directed variation of Myerson's (1990) network formation game. This is discussed in more detail in the chapter by Goyal (2002), and so I will not say more here.

Closing Remarks

The literature surveyed here helps us to understand network formation. As networks are pervasive in social and economic interactions, this literature was really inevitable. As we have seen, there are interesting and somewhat unexpected relationships between which networks are efficient from society's perspective and which networks form as the result of player incentives. Another thing that we have learned is that explicit modeling of networks is tractable, and that a valuable theory can be developed.

While the literature has made some progress, there is still good news for researchers. Namely, there are many important and interesting open questions in this area that are manageable and just waiting for attention. Some of these questions involve theoretical modeling, such as developing further understanding of the relationship between stable and efficient networks and how this depends on the setting, and further exploring the simultaneous bargaining over the allocation of value and the formation of networks, and more generally understanding how side payments might affect network formation. But these questions also go well beyond the theoretical, to include the empirical and experimental analysis of models of economic networks. I have not really touched upon these here, partly because those areas are so wide open. Some folks are pioneering into these areas, as we see on the experimental side in Corbae and Duffy (2000), Charness, Corominas-Bosch, and Frechette (2001), and Callander and Plott (2002), and Falk and Kosfeld (2003) (see Kosfeld (2003) for a recent survey).[47] Work on the empirical side has a longer tradition dating back to early studies on contact networks in labor economics (e.g., Rees (1966)), but is enjoying new interest as in in recent work by Topa (2000), Fafchamps and Lund (2000), and Aizer and Currie (2002).

There are also some substantial challenges for the future literature on networks that is coming out of economics and game theory. One challenge is bridging to the sociology ("social networks") literature.[48] That literature is well-established, very large, and full of interesting questions, insights, data sets, and knowledge of network structure and what influences it. The main challenge comes in the differences in terminology, the points of view, and the techniques of analysis. As the literatures continue to grow, the cross fertilization which is just beginning now should become more and more natural.

[47] I should be careful to say that the experimental research on exchange networks from the sociological side is quite extensive (e.g., see Bienenstock and Bonacich (1993)), as is the empirical analysis of various network structures (e.g., see Wasserman and Faust (1994) and the references therein). So, here I am referring more to the questions of network formation and the testing of formal models of formation, as well as analyses of the efficiency of observed networks.

[48] To a more limited extent, the same can be said for bridging to the agent-based computation models, where many network situations have be analyzed. There is more natural overlap there, as the underlying view of players' incentives and the terminology are closer to begin with.

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