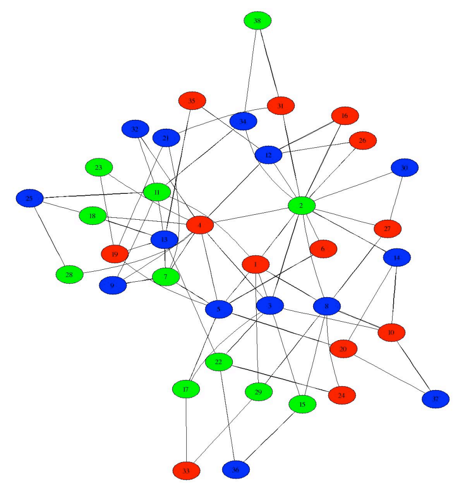
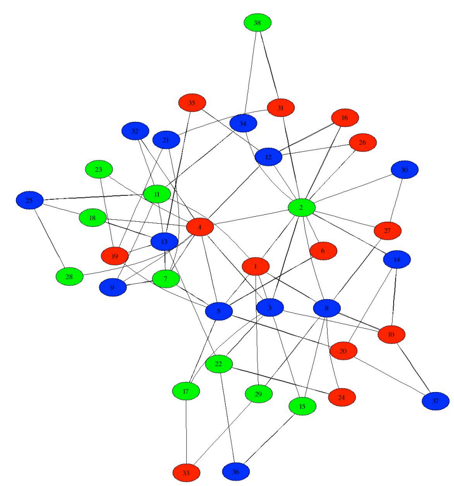
# Models of Network Formation

Networked Life
NETS 112
Fall 2017
Prof. Michael Kearns

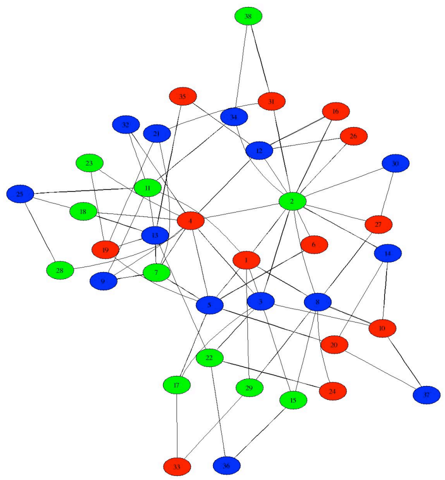


# Roadmap

- Recently: typical large-scale social and other networks exhibit:
  - giant component with small diameter
  - sparsity
  - heavy-tailed degree distributions
  - high clustering coefficient
- These are empirical phenomena
- What could “explain” them?
- One form of explanation: simple models for network formation or growth that give rise to these structural properties
- Next several lectures:
  - Erdös-Renyi (random graph) model
  - “Small Worlds” models
  - Preferential Attachment
- Discussion of structure exhibited (or not) by each

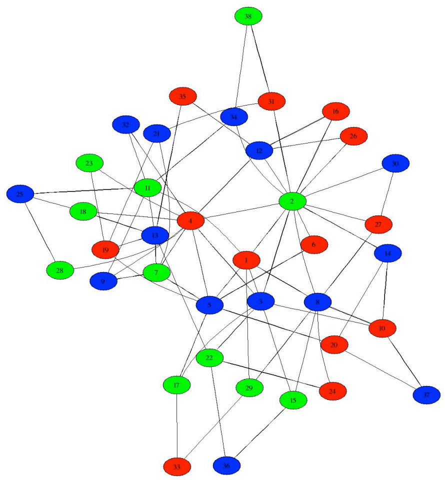


# Models of Network Formation
# I. The Erdös-Renyi (Random Graph) Model



# The Erdös-Renyi (Random Graph) Model

- Really a randomized algorithm for generating networks
- Begin with N isolated vertices, no edges
- Add edges gradually, one at a time
- Randomly select two vertices not already neighbors, add edge
- So edges are added in a random, unbiased fashion
- About the simplest (dumbest?) formation model possible
- But what can it already explain?

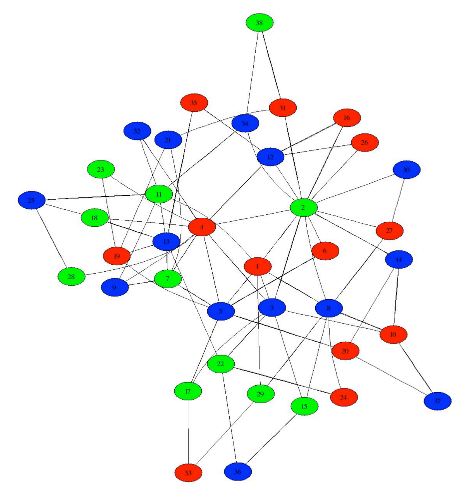


# The Erdös-Renyi (Random Graph) Model

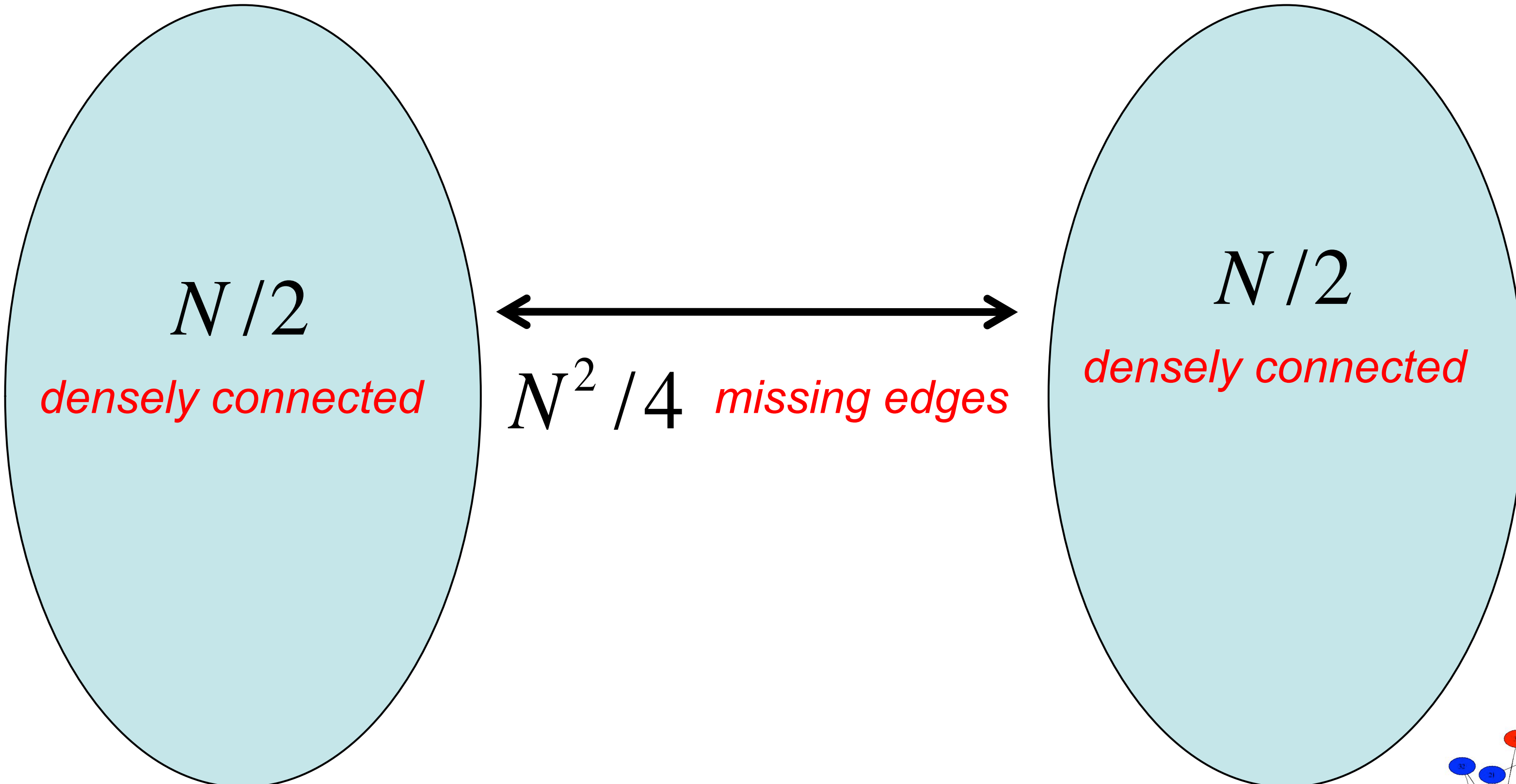
- After adding E edges, edge density is

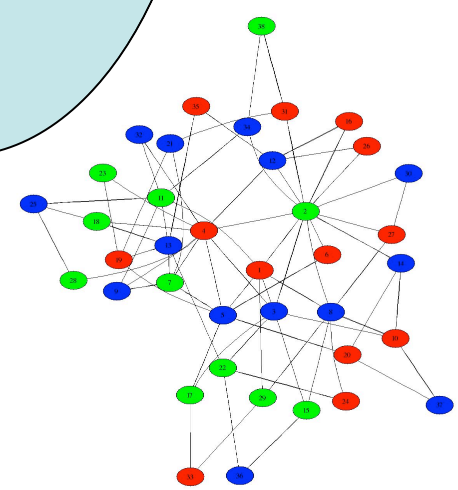
$$p = E/(N(N-1)/2)$$

- As E increases, p goes from 0 to 1
- Q: What are the likely structural properties at density p?
  - e.g. as p = 0 → 1, small diameter occurs; single connected component
- At what values of p do “natural” structures emerge?
- We will see:
  - many natural and interesting properties arise at rather “small” p
  - furthermore, they arise very suddenly (tipping/threshold)
- Let’s examine the Erdös-Renyi simulator



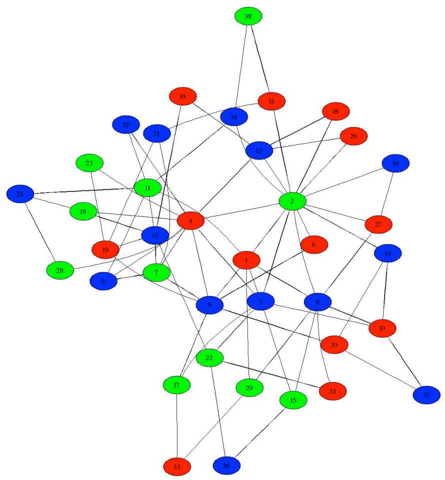
# Why Can't There Be Two Large Components?



# Threshold Phenomena in Erdös-Renyi

- Theorem: In Erdös-Renyi, as N becomes large:
  - If p < 1/N, probability of a giant component (e.g. 50% of vertices) goes to 0
  - If p > 1/N, probability of a giant component goes to 1, and all other components will have size at most log(N)
- Note: at edge density p, expected/average degree is p(N-1) ~ pN
- So at p ~ 1/N, average degree is ~ 1: incredibly sparse
- So model "explains" giant components in real networks
- General "tipping point" at edge density q (may depend on N):
  - If p < q, probability of property goes to 0 as N becomes large
  - If p > q, probability of property goes to 1 as N becomes large
- For example, could examine property "diameter 6 or less"

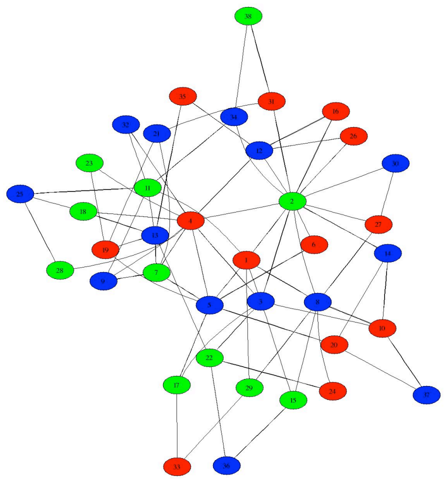


# Threshold Phenomena in Erdös-Renyi

- Theorem: In Erdös-Renyi, as N becomes large:
  - Threshold at

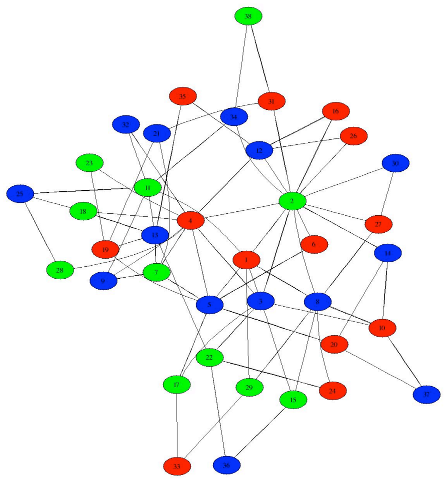
$$p \sim \log(N)/N^{5/6}$$

  - for diameter 6.
  - Note: degrees growing (slightly) with N
  - If N = 300M (U.S. population) then average degree pN ~ 500
  - If N = 7BN (world population) then average degree pN ~ 1000
  - Not unreasonable figures…
- At p not too far from 1/N, get strong connectivity
- Very efficient use of edges



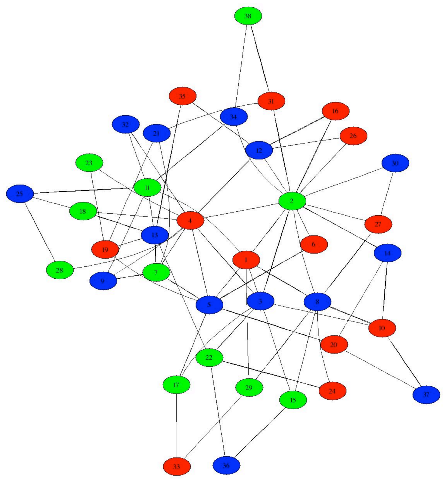
# Threshold Phenomena in Erdös-Renyi

- In fact: Any *monotone property* of networks exhibits a threshold phenomenon in Erdös-Renyi
  - monotone: property continues to hold if you add edges to the networks
  - e.g. network has a group of K vertices with at least 71% neighbors
  - e.g. network has a cycle of at least K vertices
- Tipping is the rule, not the exception

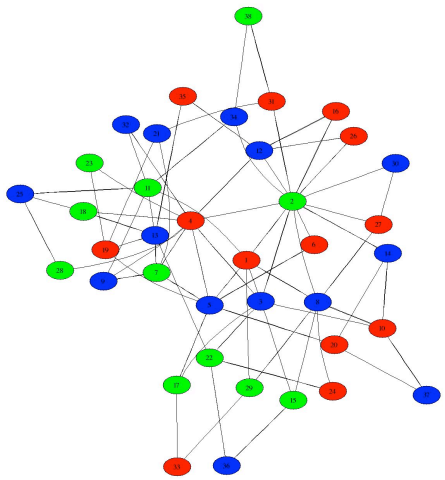


# What Doesn't the Model Explain?

- Erdös-Renyi explains giant component and small diameter
- But:
  - degree distribution not heavy-tailed; exponential decay from mean (Poisson)
  - clustering coefficient is *exactly* p
- To explain these, we'll need richer models with greater realism

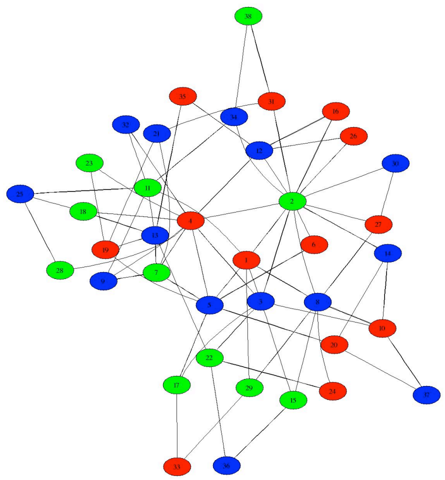


# Models of Network Formation
# II. Clustering Models



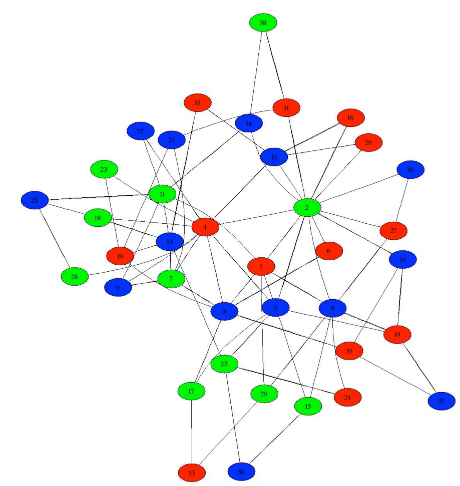
# Roadmap

- So far:
  - Erdös-Renyi exhibits small diameter, giant connected component
  - Does not exhibit high edge clustering or heavy-tailed degree distributions
- Next: network formation models yielding high clustering
  - Will also get small diameter "for free"
- Two different approaches:
  - "program" or "bake" high clustering into the model
  - balance "local" or "geographic" connectivity with long-distance edges



# “Programming” Clustering

- Erdös-Renyi:
  - global/background edge density p
  - all edges appear independently with probability p
  - no bias towards connecting friends of friends (distance 2) → no high clustering
- But in real networks, such biases often exist:
  - people introduce their friends to each other
  - people with common friends may share interests (homophily)
- So natural to consider a model in which:
  - the more common neighbors two vertices share, the more likely they are to connect
  - still some “background” probability of connecting
  - still selecting edges randomly, but now with a bias towards friends of friends

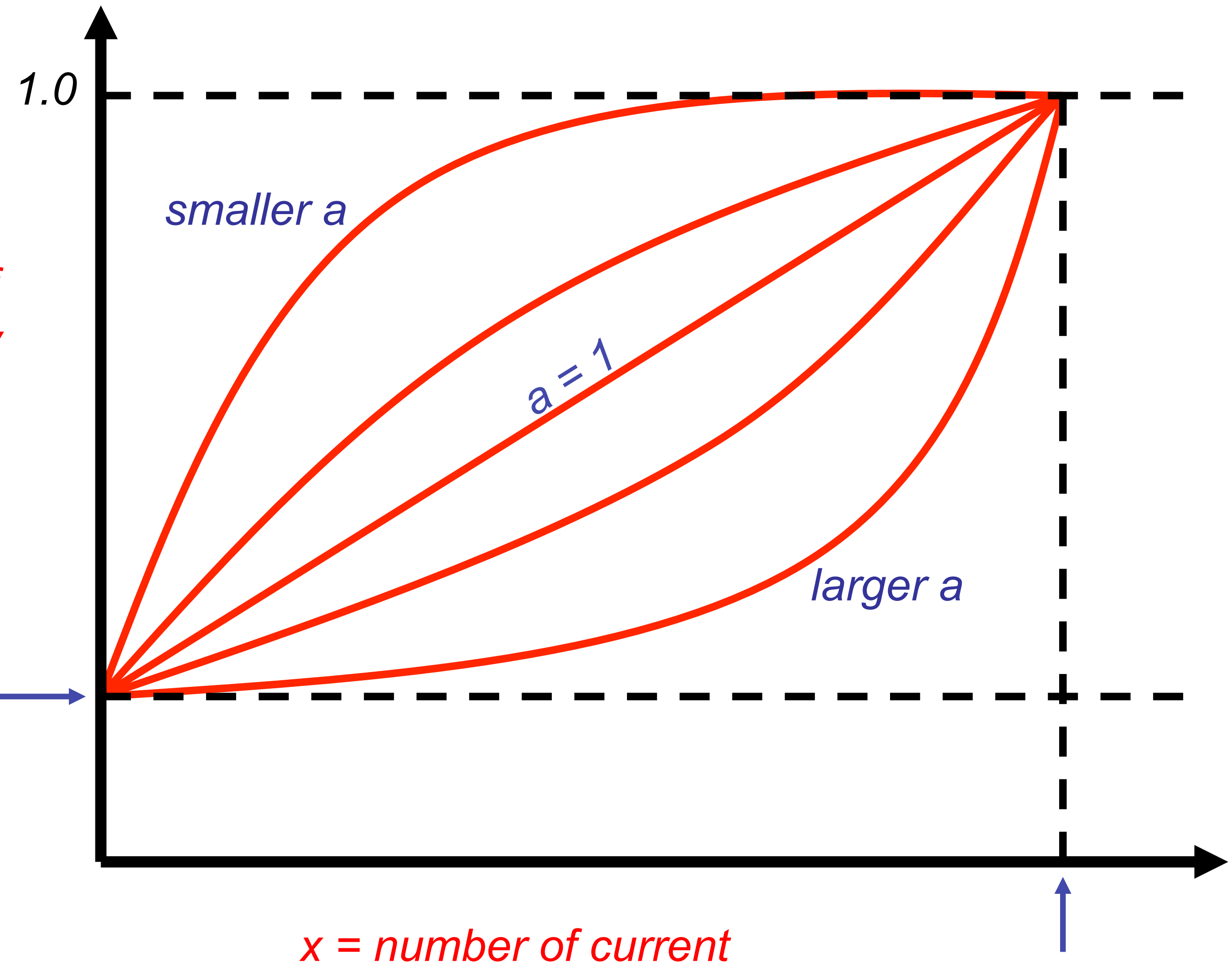


# Making it More Precise: the a-model

y = probability of connecting u & v

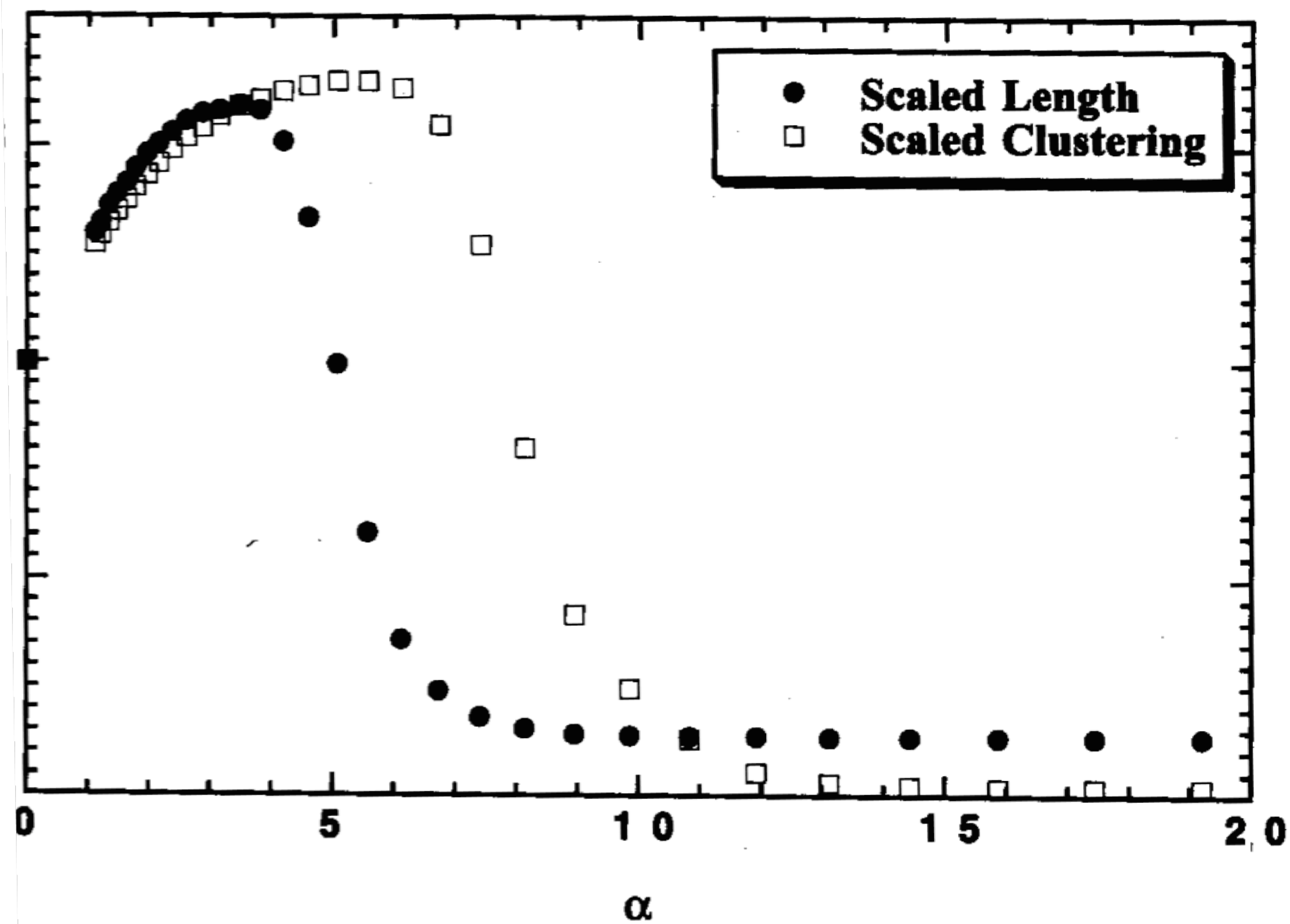
$$y \sim p + (x/N)^{\alpha}$$

“default” probability p




x = number of current common neighbors of u & v

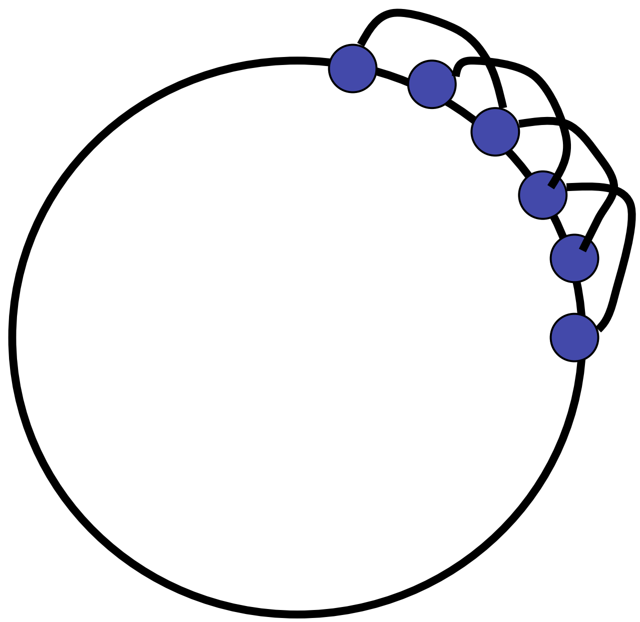
network size N




*From D. Watts, "Small Worlds"*

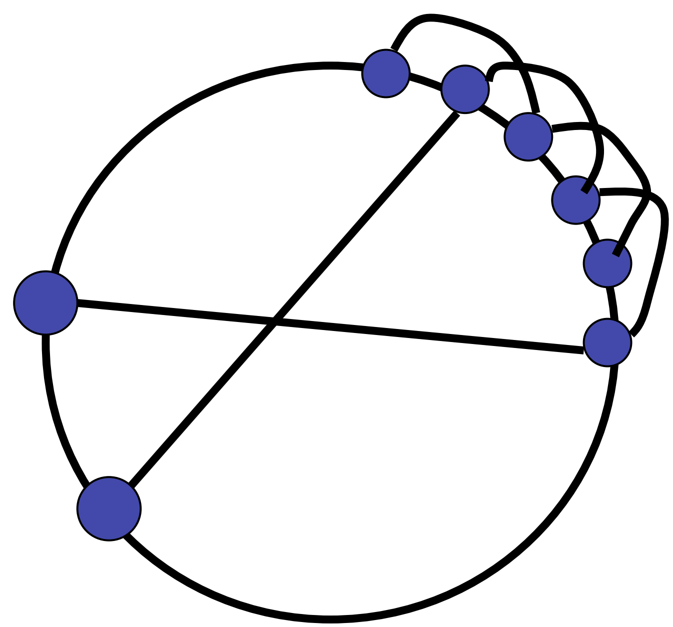
# Clustering Coefficient Example 2

- Network: simple cycle + edges to vertices 2 hops away on cycle
- By symmetry, all vertices have the same clustering coefficient
- Clustering coefficient of a vertex v:
  - Degree of v is 4, so the number of *possible* edges between pairs of neighbors of v is 4 x 3/2 = 6
  - How many pairs of v's neighbors actually *are* connected? 3 --- the two clockwise neighbors, the two counterclockwise, and the immediate cycle neighbors
  - So the c.c. of v is 3/6 = ½
- Compare to overall edge density:
  - Total number of edges = 2N
  - Edge density p = 2N/(N(N-1)/2) ~ 4/N
  - As N becomes large, ½ >> 4/N
  - So this cyclical network is highly clustered



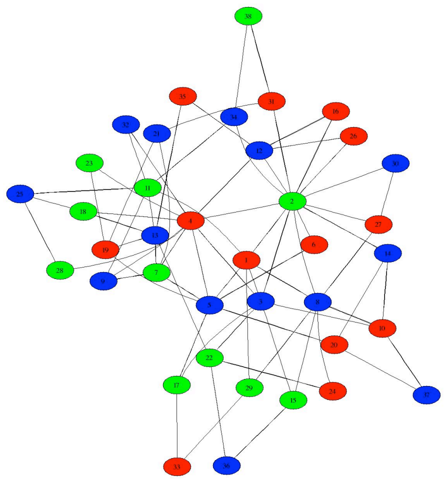
# An Alternative Model

- A different model:
  - start with all vertices arranged on a ring or cycle (or a grid)
  - connect each vertex to all others that are within k cycle steps
  - with probability q, rewire each local connection to a random vertex
- Initial cyclical structure models “local” or “geographic” connectivity
- Long-distance rewiring models “long-distance” connectivity
- q=0: high clustering, high diameter
- q=1: low clustering, low diameter (~ Erdös-Renyi)
- Again is a “magic range” of q where we get both high clustering and low diameter
- Let’s look at this demo

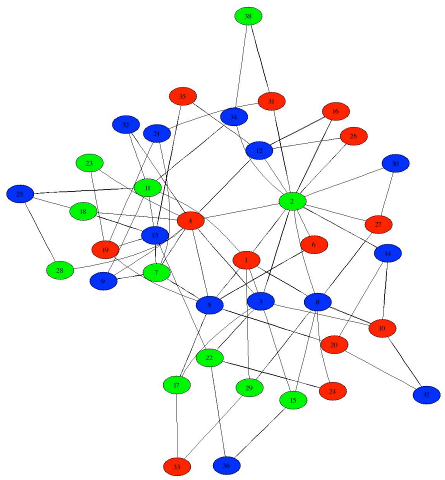


# Summary

- Two rather different ways of getting high clustering, low diameter:
  - bias connectivity towards shared friendships
  - mix local and long-distance connectivity
- Both models require proper “tuning” to achieve simultaneously
- Both a bit more realistic than Erdös-Renyi
- Neither model exhibits heavy-tailed degree distributions

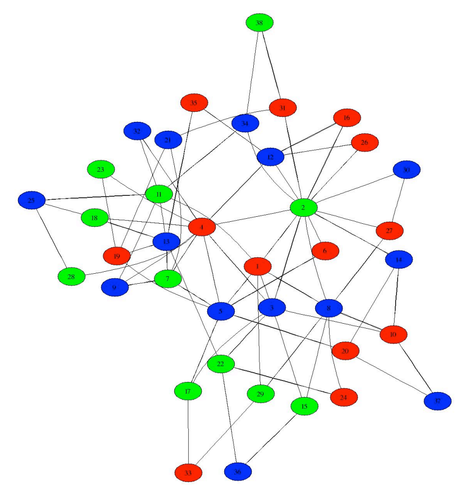


# Models of Network Formation
# III. Preferential Attachment



# Rich-Get-Richer Processes

- Processes in which the more someone has of something, the more likely they are to get more of it
- Examples:
  - the more friends you have, the easier it is to make more
  - the more business a firm has, the easier it is to win more
  - the more people there are at a nightclub, the more who want to go
- Such processes will amplify inequality
- One simple and general model: if you have amount x of something, the probability you get more is proportional to x
  - so if you have twice as much as me, you're twice as likely to get more
- Generally leads to heavy-tailed distributions (power laws)
- Let's look at a simple "nightclub" demo…



# Preferential Attachment

- Start with two vertices connected by an edge
- At each step, add one *new* vertex v with one edge back to *previous* vertices
- Probability a previously added vertex u receives the new edge from v is *proportional to the (current) degree of u*
  - more precisely, probability u gets the edge = (current degree of u)/(sum of all current degrees)
- Vertices with high degree are likely to get *even more* links!
  - …just like the crowded nightclub
- *Generates a power law distribution of degrees*
- Variation: each new vertex initially gets k edges
- Here's another demo

# Summary

- Now have provided network formation models exhibiting each of the universal structure arising in real-world networks
- Often got more than one property at a time:
  - Erdös-Renyi: giant component, small diameter
  - α model, local+long-distance: high clustering, small diameter
  - Preferential Attachment: heavy-tailed degree distribution, small diameter
- Can we achieve all of them simultaneously?
- Probably: mix together aspects of all the models
- Won't be as simple and appealing, though