Algorithms for VWAP and Limit Order Trading

Sham Kakade
TTI (Toyota Technology Institute)

Collaborators: Michael Kearns, Yishay Mansour, Luis Ortiz
Technological Revolutions in Financial Markets

• **Competition**
  - amongst exchanges
  - rise of the ECNs; NASDAQ vs. NYSE

• **Automation**
  - exchanges
  - technical analysis/indicators
  - algorithmic trading

• **Transparency**
  - real-time revelation of low-level transactional data
  - market microstructure
Outline

- formal models for market microstructure
- competitive algorithms for canonical execution problems
- provide a price for VWAP trading
Market Microstructure

Consider a typical exchange for some security
Order books: buy/sell side
- sorted by price; top prices are the bid and ask
Market order:
- give volume, leave price to "the market"
- matched with opposing book
Limit order:
- specify price and volume
- placed in the buy or sell book
Market orders guaranteed transaction but not price; limit orders guaranteed price but not transaction
last price / ticket price
Commercial and Academic Interest in Market Microstructure

- Real-time microstructure revelation enables:
  - optimized execution
  - new automated trading strategies?
    - order books express “market sentiment”

- Early microstructure research:
  - equilibria of limit order games (Parlour et al.)
  - power laws relative to bid/ask (Bouchaud et al.)
  - dynamics of price evolution (Farmer et al.)

- What about the algorithmic issues?
One way trading (OWT)

- The common objective in online analysis
- Sequence of prices:
  - \( p_1, p_2, \ldots, p_t \)
  - \( p_{\text{max}} = \max_i \{p_i\}; p_{\text{min}} = \min_i \{p_i\}; R = \frac{p_{\text{max}}}{p_{\text{min}}} \)
- Q: Compete with the maximum price \( p_{\text{max}} \)?
  - "Yes", assuming infinite liquidity [EFKT]
  - \( O(\log R) \) competitive
The VWAP

- Given a sequence of price-volume trades:
  - \((p_1, v_1), (p_2, v_2), \ldots, (p_T, v_T)\)
- Volume Weighted Average Price (VWAP)
  - \(\text{VWAP} = \frac{\sum p_t v_t}{\sum v_t}\)
- Objective: sell (or buy) tracking VWAP
- A much more modest goal
  - a “trading benchmark“?
  - Why is it important?
  - Can we achieve it?
Typical Trading scenario

• Large mutual fund owns 3% of a company
• Likes to sell 1% of the shares
  - over a month
  - likes to get a “fair price”
• Option 1: Simply sell all the shares
  - huge market impact!
Typical Trading scenario (more)

- Option 2: Sell it to a brokerage
  - What should be the price
  - The future VWAP over the next month
    [minus some commission cost]

- Brokerage: Needs to sell the shares at the VWAP
  (more or less)
  - brokerage takes on risk
VWAP Issues

- Psychological Factors:
  - increased supply
  - market impact
  - less of an issue for the 'brokerage'

- Mechanics:
  - liquidity is the key

- Algorithmic Challenge:
  - get close to the VWAP?
  - what about psychology?
An Online Microstructure Model

- Market places a sequence of price-volume limit orders:
  - \( M = (p_1, v_1), (p_2, v_2), \ldots, (p_T, v_T) \) (+ order types)
  - possibly adversarial
  - ignore market orders!
- Algorithm is allowed to interleave its own limit orders:
  - \( A = (q_1, w_1), (q_2, w_2), \ldots, (q_T, w_T) \)
- Merged sequence determines executions and order books:
  - \( \text{merge}(M, A) = (p_1, v_1), (q_1, w_1), \ldots, (p_T, v_T), (q_T, w_T) \)
  - Now have complex, high-dimensional state
VWAP Results

- **Goal:** Sell K shares at VWAP.
- **How to measure “time”?:**
  - measure time by amount of volume traded
  - assume no order larger than \( \beta \) shares
- **Theorem:** After \( \beta K \) shares traded,
  \[
  \text{AvgRevenue}(S,A) \geq \text{VWAP}(S,A) - \left( \frac{2p_{\text{max}}}{\sqrt{K}} \right)
  \]
- **Worst case commission cost of** \( 2p_{\text{max}} / \sqrt{K} \)
  - relatively mild assumptions
  - don’t address ‘psychology’
- **If time horizon is fixed, “guess” volume**
VWAP Algorithm

- Divide time into equal (executed) volume intervals $I_1, I_2,$...
- Let $VWAP_j$ be the VWAP in volume interval $I_j$
- Consider price levels $(1-\varepsilon)^k$

Algorithm:

After $I_j$, place sell limit order for 1 share at the price $(1-\varepsilon)^k$ nearest $VWAP_j$

- Note if all orders executed, we are within $(1-\varepsilon)$ of overall VWAP
  - since each limit order is $(1-\varepsilon)$ close to $VWAP_j$
The Proof

Algorithm:
After $I_j$, place sell limit order for 1 share at $\sim (1-\varepsilon)^k$ nearest VWAP$_j$

Proof:
• say after interval $I_j$, algo. places order at level $(1-\varepsilon)^m$
• Key Idea: after interval $j$, if price ever rises above the price $(1-\varepsilon)^m$, then our limit order is executed
• Hence, at end of trading, can’t “strand” more than one order at any given price level
• This implies:
  \[ \text{AvgRevenue}(S,A) \geq (1-\varepsilon) \cdot \text{VWAP}(S,A) - \left( \frac{p_{\text{max}}}{\varepsilon K} \right) \]
  • optimize $\varepsilon$!

Implications:
• note that algorithm may not sell any shares?
• Algorithm exploits the power of limit orders!
One Way Trading & Order Books

- **Goal:** sell K shares at highest prices
  - compete with optimal “offline” algorithm

- **Assumptions:**
  - The price is in: \([p_{\text{min}}, p_{\text{max}}]\)
  - define \(R = \frac{p_{\text{max}}}{p_{\text{min}}}\)

- **Theorem:** Algo A has performance that is within a multiplicative factor of \(2\log(R)\log(K)\) of “optimal”
  - worst-case market impact of large trades
    - **proof:**
      - order prices \(p_1 > p_2 > \ldots\) are exec/buy prices
      - want to obtain \(Kp_1\), but cant
      - try to “guess” and obtain \(\max\{kp_k\}\)