# **Polar Transformer Networks**

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# Abstract

Convolutional neural networks (CNNs) are equivariant with respect to translation; a translation in the input causes a translation in the output. Attempts to generalize equivariance have concentrated on rotations. In this paper, we combine the idea of the spatial transformer, and the canonical coordinate representations of groups (polar transform) to realize a network that is invariant to translation, and equivariant to rotation and scale. A conventional CNN is used to predict the origin of a polar transform. The polar transform is performed in a differentiable way, similar to the Spatial Transformer Networks, and the resulting polar representation is fed into a second CNN. The model is trained endto-end with a classification loss. We apply the method on variations of MNIST, obtained by perturbing it with clutter, translation, rotation, and scaling. We achieve state of the art performance in the rotated MNIST, with fewer parameters and faster training time than previous methods, and we outperform all tested methods in the SIM2MNIST dataset, which we introduce.

#### 1. Introduction

The quest for invariance/equivariance has been investigated since the beginnings of computer vision and pattern recognition whether at the global/pattern level or the local feature level [8]. The state of the art in the "hand crafted" world could be summarized through feature descriptors like SIFT [18] where the intrinsic scale or rotation of a region is selected [17, 2] and the region is normalized: the selection is possible because a descriptor is equivariant and the region after the normalization is invariant to scale and/or rotation. An equivariant descriptor whether a LoG response over scales or a histogram over rotations has to be computed by sampling the corresponding transformations. This transformation sampling was the motivation for the search of steerable representations where the feature response to any transformation instance could be interpolated by a finite basis of filters. Such steerability was proven for rotations of Gaussian derivatives [6] and was extended to combine scale with translations in the shiftable pyramid [24]. Exhaustive sampling for the creation of such a basis was proposed by [21] using the SVD. In parallel, Segman et al. [22] realized invariance can be achieved by the construction of appropriate kernels acting as Fourier basis in a new system of canonical coordinates where a transformation can be expressed as a translation if it satisfies certain conditions. Following this work, Nordlund [20] and Teo and Hel-Or [9, 26] have proposed a methodology for the computation of bases for equivariance spaces given the Lie generators of a transformation. Last, Mallat [23] proposed the scattering transform that produces invariant representations in translation, scaling, and rotations.



Figure 1. In the log-polar representation, rotations around the origin become vertical shifts, and dilations around the origin become horizontal shifts. Top two rows: sequence of  $45^{\circ}$  rotations, and the correspondent polar images. The distance between the yellow and green lines is proportional to the angle of rotation. Bottom two rows: sequence of  $\sqrt{2}$  dilations, and the correspondent polar images. The distance between the yellow and green lines is proportional to the scale factor.

Today there is a consensus that the most powerful fea-

ture representations can be learnt instead of being handcrafted. Equivariance to translations through convolutions and invariance to local deformations through pooling are now textbook material in deep learning ([15], p.335). But what about more general transformations like affine transformations. There are two approaches to this problem: the Spatial Transformer [10] reproduces the canonization similar to scale selection by learning the canonical pose of the input image and using it to warp the image producing an invariant representation. The other line of research is based on enforcing an equivariant structure in the convolutional filters [29] or producing transformed copies of the filters on which a group-convolution produces an equivariant representations [3].

In this paper, we propose the Polar Transformer Network (PTN), which combines the ideas of the Spatial Transformer and group-convolutions to achieve equivariance in the 2D-similarity group of translations, rotations, and dilations. In the first step our network learns the translation of the image to be recognized and shifts the original image. In the second step it transforms the original image around the new origin into a log-polar coordinate system where translations represent global rotations and scalings. In this coordinate system, planar convolutions are nothing more than group-convolutions in angle and scale. This approach allows us to avoid the use of the computationally expensive fully-connected layers for learning pose in the Spatial Transformer and produce an equivariant representation in rotations and dilations.

Our approach overcomes the weakness of the harmonic networks [29] and group convolutions [3] in not capturing dilations and the weakness of group convolution in being able to deal only with finite rotations. However, unlike those approaches, we deal with only global rotations and dilations like the Spatial Transformer does.

We present competitive results in the rotated MNIST as well as in a new dataset we introduce here the SIM2MNIST.

To summarize our contributions:

- We develop a neural network architecture capable of learning an image representation invariant to shifts, and equivariant to rotations and dilations.
- We propose the polar transformer module, which performs a log-polar transform in a differentiable way, amenable to backpropagation training with the transform origin being a latent variable.
- We show how the polar transform origin can be learned effectively as the centroid of a single channel heatmap predicted by a fully convolutional network.

# 2. Related Work

One of the first equivariant feature extraction schemes was proposed by Granlund [20] who suggested the discrete sampling of 2D-rotations of a complex angle modulated filter. About the same time, the image and optical processing community discovered the Mellin transform as a modification of the Fourier transform [31, 1]. The Fourier-Mellin's modulus is invariant and its phase is shift-equivariant to rotation and scale.

During the 80's and 90's, invariances of integral transforms were developed using a systematic approach based on the Lie generators of the underlying group transformations, starting from one-parameter transforms [5] and generalizing to subgroups of the affine group [22]. Lie group transformations could be converted to translations in canonical coordinates if the corresponding Lie generators were linearly independent. This is true, for example, for rotation and scalings but not for translations and rotations. Moreover, this constrains the maximum dimension of the subgroup to be the dimension of the space where the group is acting (two in the case of images).

Closely related to the equivariance/invariance work is the framework of steerability, namely, the ability to interpolate the response to any group action from the response of a finite-dimensional filter basis. An exact steerability framework was started in [6] where rotational steerability for Gaussian derivatives was explicitly computed, and was extended to the shiftable pyramid [24] handling rotations and scale. Approximate steerability was constructed via the SVD by Perona [21] using multiple group action copies for learning a lower dimensional representation of the image deformation.

A unification of the "Lie generator school" and approximate steerability was achieved by Teo and Hel-Or [26] where an SVD is used to reduce the number of basis functions for a given transformation group. Teo and Hel-Or developed the most extensive framework for steerability [26, 9] and proposed the first approach for non-Abelian groups by starting with the largest abelian group for which an exact steerable function space can be constructed, and incrementally steering for the remaining subgroups.

The most recent "handcrafted" framework for equivariance has been the scattering transform [23] which compiles rotated and dilated wavelet copies. It starts with the basic idea inherent in SIFT [18] about equivariance of anchor points such as the maxima of filtered responses in (translation) space. However, the response of such a filter bank is not translation invariant. Translation invariance is enforced by taking the modulus of the filter response and further convolving the modulus with the next filter. The final scattering coefficient is invariant to translations and continuous to local rotations and scalings.

Laptev and Savinov et al. [13] incorporate a pooling

operator over feature maps obtained from several rotated versions of the input in order to achieve transformationinvariance, with the overhead of running each forward and backward pass as many times as the number of input transformations.

In the era of learned features, methods of enforcing group equivariance fall to two main directions. The first is to enforce equivariance by constraining the structure of the filters as in the original Lie-generator derived approaches [22, 9]. Harmonic networks [29] use complex harmonic functions which by themselves are equivariant to rotations. Applied locally they produce a simultaneous rotation and translation equivariance.

The second direction is taken up by Cohen and Welling [3] who create rotational copies of existing learnt filters enabling convolution in the rotation group. Inherently, such a convolution is rotationally equivariant and again, since it is applied locally, translation equivariance is well approximated. They also prove that the rotation equivariance flows through the rectifier and pooling elements. In a similar vein, Dielemann et al. [4] make rotated and flipped copies of the input and build a separate CNN of each copy. The stack of all outputs is then equivariant and is processed for classification. The same idea underlies the work in [7] which produces maps of multidimensional groups sampled at a finite set of anchor points. More recently, Zhou et al. [30] applies rotating filters that produce explicitly oriented feature maps to achieve rotation invariant features. To realize equivariance Lenc and Vedaldi [16] propose a transformation layer consisting first of a permutation/indexing and then a linear filter that act as a group-convolution.

Our approach is close to both approaches above: it cannot cover local rotation equivariance but covers scaling in addition to translation and rotation. It borrows the well known log-polar coordinates (identified as canonical coordinates in [22]) to implement the convolution in the rotation-dilation group while translation is fixed like an anchor similar to the Spatial Transformer.

### 3. Theoretical Background

This section is divided into two parts, the first offering a review of equivariance and group convolutions which can be reduced to the special and familiar case of translational convolution. The second offers an explicit example of group equivariance using the group of 2D similarity transformations (SIM(2)) comprised of translations, dilations and rotations. Insights from this example enable the application of the SIM(2) group convolution as a traditional translational convolution by reparameterization.

#### 3.1. Group Equivariance

Equivariant representations are highly sought after in filter representations since they enable the prediction of the filter response given an input transformation. Let G be a transformation group and  $L_g I$  be the result of the group action on an image I. We say that a mapping  $\Phi$  is equivariant to actions of the group G if

$$\Phi(L_g I) = L'_g(\Phi(I)). \tag{1}$$

Operators  $L_g$  and  $L'_g$  do not need to be the same but they must satisfy  $L_{gh} = L_g L_h$ . We use a primed  $L'_g$  [3] to allow for operators to apply on a different space on the left and right hand side of the above equation. Equivariance becomes invariance when  $L'_g$  is the identity.

In the context of image classification,  $g \in G$  may be thought of as an image deformation, and  $\Phi$ , a mapping from the image to an alternate representation like the output of a CNN layer. CNNs inherit the property of translation equivariance from the convolution operation. This is observed as translations in the image resulting in translations of the feature representation. This equivariance property is independent of the convolutional kernel.

How would this inherent property of convolutional networks apply to other groups? The key is in the generalization of the translational convolution to group convolution. Let f(h) and  $\phi(h)$  be real valued functions on G, for example,  $L_h f = f(h^{-1})$ . Then group convolution is defined as [12]

$$(f \star_G \phi)(g) = \int_{h \in G} f(h)\phi(h^{-1}g) \, dh.$$
 (2)

The group convolution reduces to the traditional translational convolution when G is the translation group with addition as the group operator,

$$(f \star \phi)(x) = \int_{h} f(h)\phi(h^{-1}x) dh$$
  
= 
$$\int_{h} f(h)\phi(x-h) dh.$$
 (3)

The group convolution definition is quite esoteric since it requires the definition of integrability over a group and the appropriate measure dg. We will continue using this notation without defining the corresponding measure because at the end we will work with a discretized implementation (sum instead of integral) of the group convolution.

It can be proven that group convolution is always group equivariant:

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$$L_{a}f \star_{G} \phi)(g) = \int_{h \in G} f(a^{-1}h)\phi(h^{-1}g) dh$$
  
=  $\int_{b \in G} f(b)\phi((ab)^{-1}g) db$   
=  $\int_{b \in G} f(b)\phi(b^{-1}a^{-1}g) db$   
=  $(f \star_{G} \phi)(a^{-1}g)$   
=  $L_{a}((f \star_{G} \phi)(g)).$  (4)

The definition of group convolution looks different for the very first layer where the group is acting on the image. We show in Figure 2 how the convolution on the original image is also equivariant to the group action (in that case rotation). We illustrate the filter response as a function of the rotation angle (parameter of  $g \in SO(2)$ ) and how it is shifted when the input image is rotated.



Figure 2. Group-convolutions for SO(2). The images in the left most column differ by 90° rotation. It is shown that application of the group convolution with any filter results in an equivariant representation. The filters are shown in the top row. The innerproduct each of filters (rotated from  $0 - 360^\circ$ ) and the image is plotted in blue for the top image and red for the bottom image. Observe how the filter response is shifted by 90°.

#### **3.2. Equivariance in SIM(2)**

We will show next how group equivariance can be implemented for the group of 2D similarity transformations, comprising translation, dilations, and rotations.

A similarity transformation  $\rho\in \mathrm{SIM}(2)$  acting on a point in  $\mathbb{R}^2$ 

$$\rho x \to s \, R \, x + t \quad s \in \mathbb{R}^+, \, R \in SO(2), \, t \in \mathbb{R}^2, \tag{5}$$

where SO(2) is the rotation group.

To take advantage of the planar convolution implemented in standard CNN we decompose a  $\rho \in SIM(2)$  into a translation, t in  $\mathbb{R}^2$  and a dilated rotation r in  $SO(2) \times \mathbb{R}^+$ .

Equivariance to SIM(2) will be achieved in two steps. First, the center of the dilated rotation will be learned. The original image will be shifted to the center and it will be transformed to canonical coordinates where SO(2)  $\times \mathbb{R}^+$  actions will correspond to translations. Standard convolution will then be equivalent to group (SIM(2)) convolution.

The first step is just an instance of translational spatial transformer. The translation filters are learnt in such a way that the centroid of the output best approximates the global translation of a template [10] which is being classified.

This transformation of the image  $L_t I = I(t - t_0)$  (appearing also as canonization in [25]) reduces the group action on the original image to a dilated rotation if  $t_o$  is the actual translation.

After de-translation of the original image we perform  $SO(2) \times \mathbb{R}^+$  convolutions on the new image we will call  $I_o = I(x - t_o)$ :

$$f(r) = \int_{x \in \mathbb{R}^2} I_o(x)\phi(r^{-1}x) \ dx \tag{6}$$

and in subsequent layers

$$h(r) = \int_{s \in SO(2) \times \mathbb{R}^+} f(s)\phi(s^{-1}r) \, ds \tag{7}$$

where  $r, s \in SO(2) \times \mathbb{R}^+$ . Such a convolution would require an enormous amount of inner products of index locations which might be viable for the case of  $\pi/2$  rotations in [3] but not for general rotation-dilations.

To overcome indexing and inspired by canonical coordinates of Lie-groups [22] we transform the original image  $I(x, y)^1$  into log-polar coordinates:  $(e^{\xi} \cos(\theta), e^{\xi} \sin(\theta))$ and denote the resulting image as  $\lambda(\xi, \theta)$ . The reader will realize that suddenly our image is defined on SO(2) ×  $\mathbb{R}^+$  and could be written as  $\lambda(s)$  where  $(\xi, \theta)$  is just a parameterization of  $s \in SO(2) \times \mathbb{R}^+$ .

We visualize in Figure 3 how equivariance holds when we apply a set of rotated and dilated copies of filters to two images rotated and scaled to each other. The filter response maps are shown on canonical coordinates and the 3rd row is shifted with respect to the 2nd row.

What we gain from canonical coordinates is that the group convolution can be expressed as a plane/translational convolution. Observe that  $s^{-1}r = \xi_r - \xi, \theta_r - \theta$ . Then

$$\int_{s} f(s)\phi(s^{-1}r) \ ds = \int_{s} \lambda(\xi,\theta)\phi(\xi_{r}-\xi,\theta_{r}-\theta) \ d\xi d\theta.$$
(8)

The main advantage of the log-polar coordinates is that it provides an efficient discretization of  $SO(2) \times \mathbb{R}^+$ . Obviously, it is not a group because the dilation  $\xi$  is not compact.

To summarize our approach if we build:

- 1. a network of translational convolutions,
- 2. take the centroid of the last layer,
- 3. shift the original image to this centroid,
- 4. convert image into log-polar coordinates,
- 5. and apply a network of translational convolutions

<sup>&</sup>lt;sup>1</sup>we abuse the notation here and momentarily we use x as the x-coordinate instead of  $x \in \mathbb{R}^2$ .



Figure 3. Result of applying the group-convolution for SO(2)  $\times \mathbb{R}^+$ . Images in the left most column differ by a rotation and scaling of  $\pi/4$  and 1.2 respectively. Careful consideration of the resulting heatmaps reveals a shift corresponding to the transformation of the input images.

then the result will be an equivariant feature map with respect to dilated rotations around the origin. A network does also include rectifier and pooling elements which have been proven to preserve equivariance [3].

# 4. Architecture

Our network is composed of two parts, connected by the polar transformer module. The first part is the polar origin predictor, and the second is the classifier. The building block of the network is a  $3 \times 3$  convolutional layer, with some number of filters, followed by batch normalization and a ReLU. We will refer to this building block simply as *block*. Subsampling is performed using strided convolutions in some layers. Figure 4 shows the architecture.

#### 4.1. Polar Origin Predictor

The polar origin predictor operates on the original image and outputs the origin of the log-polar transform coordinates. There are some difficulties in training a neural network to predict coordinates in images. Some approaches [28, 10] attempt to use fully connected layers to directly regress the coordinates, with limited success.

A better option is to predict heatmaps [27, 19], and take their argmax. The problem of using the argmax gradients during backpropagation is that the gradients are zero in all but one point, which impedes learning. The usual way to predict heatmaps is by applying a loss against a ground truth heatmap. In this case, the supervision comes before the argmax, so the gradients of the argmax with respect to the previous layer are not necessary. In our case, the polar origin is not known a priori, it must be learned implicitly by the network. This means we must take the gradient of the output coordinates with respect to the heatmap. We do this, avoiding the argmax, by taking the centroid of the heatmap as our polar origin. The gradient of the centroid with respect to the heatmap is constant and nonzero for all points, which makes the learning possible.

Our polar origin predictor comprises a sequence of blocks, some with subsampling, culminating in a  $1 \times 1$  convolution to generate a single feature map. The polar transform origin is taken as the position of the centroid of this feature map.

#### 4.2. Polar transformer module

The polar transformer module takes in the origin coordinates, the image input, and produces a log-polar representation of the input.

The module is inspired by the Spatial Transformer [11]. In fact, we use the same differentiable image sampling, which allows us to write each output coordinate  $V_i$  in terms of the input U and the coordinates of the source sample points  $(x_i^s, y_i^s)$ . Therefore, to apply the log-polar transform, we need to find the source sample points in terms of the target regular grid  $(x_i^t, y_i^t)$ , which is done as follows,

$$x_i^s = x_0 + r^{x_i^t/W} \cos \frac{2\pi y_i^t}{H}$$
 (9)

$$y_i^s = y_0 + r^{x_i^t/W} \sin \frac{2\pi y_i^t}{H}$$
(10)

Where  $(x_0, y_0)$  is the origin, W, H are the output width and height, and r is the maximum distance from the origin, which is set to  $0.5\sqrt{H^2 + W^2}$  in our experiments.

# 4.3. Wrap-around padding

The polar representation is periodic in the angular axis. When rotating the input by increasing angles, we should see the representation gradually shifting vertically, wrapping around when the first or last row is reached. Hence, we should see the polar image as a cylinder instead of a rectangle.

Most CNN implementations use zero padding before every convolutional layer, in order to maintain the feature map resolutions. We note this is not ideal for polar representations, since the row exactly above the first row should be the last row, and not a row of zeros.

We implement a wrap-around padding in the vertical dimension to handle this. The top of the feature map is padded using the bottom rows, and the bottom is padded using the top rows. Zero-padding is still used for the horizontal dimension.

We observe significant performance improvements when using the wrap-around padding instead of zero-padding (see table 3)



Figure 4. Network architecture. The input image passes through a fully convolutional network, the polar origin predictor, which outputs a heatmap. The centroid of the heatmap (two coordinates), together with the input image, goes into the polar transformer module, which performs a polar transform with origin at the input coordinates. The obtained polar representation is invariant with respect to the original object location; and rotations and dilations are now shifts, which are handled equivariantly by a conventional classifier CNN.

#### 4.4. Polar origin augmentation

The polar transform may change significantly between two different origins. In order to improve the robustness of our method, we augment the polar origin during training time. To achieve that, we just add a random shift to the regressed polar origin coordinates. Note that this comes with almost no extra computational cost, as opposed to conventional augmentation methods such as rotating the input image. Table 3 quantifies the performance gains of this kind of augmentation.

# 4.5. Classifier

The classifier is a conventional fully convolutional network; the only differences are that it operates on polar images and uses the wrap-around padding. It comprises a sequence of blocks, some with subsampling. The final layer has the same number of channels as there are classes, from where we take the global average pooling and pass through a softmax classifier.

# 5. Experiments

We train our model for a classification task on variations of MNIST, and compare with several different architectures. All models are trained with the Adam optimizer, with a learning rate of 0.01.

# 5.1. Architectures

We implement the following architectures for comparison,

- **Conventional CNN (CCNN)**, a fully convolutional network, composed of a sequence of convolutional layers and some rounds of subsampling.
- Polar CNN (PCNN), same architecture as CCNN, operating on polar images. The log-polar transform is pre-computed at the image center before training, as in

[10]. The fundamental difference between our method and this is that we learn the polar origin implicitly, instead of fixing it.

- Spatial Transformer Network (STN), our implementation of [11], replacing the localization network by four blocks of 20 filters and stride 2, followed by a 20 unit fully connected layer, which we found to perform better. The transformation regressed is in SIM(2), and a CCNN comes after the transform.
- Polar Transformer Network (PTN), our proposed method. The polar origin predictor comprises three blocks of 20 filters each, with stride 2 on the first block (or the first two blocks, when input is  $96 \times 96$ ). The classification network is the CCNN.
- **PTN-CNN**, we classify based on the sum of the per class scores of instances of PTN and CCNN trained independently.

The following suffixes qualify the architectures described above:

- **S**, "small" network, with seven blocks of 20 filters and one round of subsampling (equivalent to the Z2CNN in [3]).
- **B**, "big" network, with 8 blocks with the following number of filters: 16, 16, 32, 32, 32, 64, 64, 64. Subsampling by strided convolution is used whenever the number of filters increase. We add up to two 2 extra blocks of 16 filters with stride 2 at the beginning to handle larger input resolutions (one for 42 × 42 and two for 96 × 96).
- +, training time rotation augmentation by continuous angles.
- ++, training and test time rotation augmentation. We input 8 rotated versions the the query image and classify using the sum of the per class scores.

Regarding rotation augmentation for polar-based methods; in theory, the effect of input rotation is just a shift in the corresponding polar image, which should not affect the classifier CNN. In practice, however, interpolation and angle discretization effects result in slightly different polar images for rotated inputs, so even the polar-based methods benefit from this kind of augmentation.

# 5.2. Rotated MNIST

The rotated MNIST dataset [14] is composed of  $28 \times 28$ ,  $360^{\circ}$  rotated images of handwritten digits. The training, validation and test sets are of sizes 10k, 2k, and 50k, respectively.

Table 1 shows the results. We divide the analysis in two parts; on the top, we show approaches with smaller networks and no rotation augmentation, on the bottom there are no restrictions.

Between the restricted approaches, the Harmonic Network [29] outperforms the PTN by a small margin, but with almost 4x more training time. This can be explained by the use of convolutions on complex variables, which are more costly. Also worth mentioning is the poor performance of the STN with no augmentation, which shows that learning the transformation parameters is much harder than learning the polar origin coordinates.

Analyzing the unrestricted approaches, we see that all variants of PTN-B outperform the current state of the art, with significant improvements when combined with CCNN and/or test time augmentation. We note that the methods based on TI-Pooling [13] employ rotated versions of the inputs in both training and test time, which roughly corresponds to our ++ variants, and the PTN outperform those methods even with no test time augmentation.

Finally, we note that the PCNN achieves a relatively high accuracy in this dataset because the digits are mostly centered, so using the polar transform origin as the image center is reasonable. Our method, however, outperforms it by a high margin, showing that even with the digit roughly centered, it is possible to find an origin away from the image center that results in a more distinctive representation.

### 5.3. Other MNIST variants

We also perform experiments in other MNIST variants.

- **MNIST R**, we replicate it from [11]. It has 60k training and 10k testing samples, where the digits of the original MNIST are rotated between  $[-90^{\circ}, 90^{\circ}]$ . It is also know as half-rotated MNIST [13].
- **MNIST RTS**, we replicate it from [11]. It has 60k training and 10k testing samples, where the digits of the original MNIST are rotated between  $[-45^\circ, 45^\circ]$ , scaled between 0.7 and 1.2, and shifted within a  $42 \times 42$  black canvas.

Model	error [%]	params	time [s]
PTN-S	1.83 (0.04)	27k	3.64 (0.04)
PCNN-S	2.6 (0.08)	22k	2.61 (0.04)
CCNN-S	5.76 (0.35)	22k	2.43 (0.02)
STN-S	7.87 (0.18)	43k	3.90 (0.05)
HNet [29]	<b>1.69</b> <sup>1</sup>	33k	13.29 (0.19)
P4CNN [3]	$2.28^{1}$	22k	-
PTN-B+	1.14 (0.08)	129k	4.38 (0.02)
PTN-B++	0.95 (0.09)	129k	$4.38^{2}$
PTN-CNN-B+	1.01 (0.06)	254k	7.36
PTN-CNN-B++	0.89 (0.06)	254k	7.36 <sup>2</sup>
PCNN-B+	1.37 (0.00)	124k	3.30 (0.04)
CCNN-B+	1.53 (0.07)	124k	2.98 (0.02)
STN-B+	1.31 (0.05)	146k	4.57 (0.04)
ORN-8 [30]	2.25 <sup>1</sup>	$\approx 1 \mathrm{M}$	3.17 <sup>3</sup>
OR-TIPooling [30]	1.54 <sup>1</sup>	$\approx 1 \mathrm{M}$	-
TI-Pooling [13]	1.2 <sup>1</sup>	$\approx 1 \mathrm{M}$	42.90

<sup>1</sup> As reported in the original paper

<sup>2</sup> Test time performance is 8x slower when using test time augmentation

<sup>3</sup> This runs in Torch, all the others in Tensorflow; comparison may not be fair

Table 1. Performance on rotated MNIST. Errors are averages of several runs, with standard deviations within parenthesis. Times are average training time for one epoch. All variants of PTN-B outperform the current state of the art, with significant improvements when combined with CCNN and/or test time augmentation.

• **SMI2MNIST**, we introduce a more challenging dataset, based on MNIST, perturbed by random transformations from SIM(2). The images are 96 × 96, with 360° rotations; the scale factors range from 1 to 2.4, and the digits can appear anywhere in the image. The training, validation and test set have size 10k, 5k, and 50k, respectively.

Table 2 shows the results.

We can see that the PTN performance mostly matches the STN on both MNIST R and RTS. The deformations on these datasets are mild and data is plenty, so the performance may be saturated.

On SIM2MNIST, however, the deformations are more challenging and the training set 5x smaller. The PCNN performance is significantly lower, which reiterates the importance of predicting the best polar origin. The HNet outperforms the other methods (except for the PTN), thanks to its translation and rotation equivariance properties. Our method is more efficient both in number of parameters and training time, and is also inherently equivariant to dilations; hence, it achieves the best performance for the SIM2MNIST dataset by a large margin.

#### **5.4.** Ablation study

We quantify the performance boost obtained with wrap around padding, polar origin augmentation, and training

	MNIST R		MNIST RTS		SIM2MNIST <sup>3</sup>				
	error [%]	params	time	error [%]	params	time	error [%]	params	time
PTN-S+	0.88 (0.04)	29k	19.72	0.78 (0.05)	32k	24.48	5.44 (0.03)	35k	11.92
PTN-B+	0.62 (0.04)	129k	20.37	0.57 (0.03)	134k	28.74	5.03 (0.11)	134k	12.02
PCNN-B+	0.81 (0.04)	124k	13.97	0.70 (0.01)	129k	17.19	15.46 (0.22)	129k	5.33
CCNN-B+	0.74 (0.01)	124k	12.79	0.62 (0.07)	129k	15.97	11.73 (0.57)	129k	5.28
STN-B+	0.61 (0.02)	146k	23.12	0.54 (0.02)	150k	27.90	12.35 (1.61)	150k	10.41
STN [11]	$0.7^{1}$	$400k^{1}$	-	<b>0.5</b> <sup>1</sup>	$400k^{1}$	-	-	-	-
HNet <sup>2</sup> [29]	-	-	-	-	-	-	9.28 (0.05)	44k	31.42
TI-Pooling [13]	0.81	$\approx 1 \mathrm{M}$	-	-	-	-	-	-	-

<sup>1</sup> As reported in the original paper

<sup>2</sup> Our modified version, with two extra layers with subsampling to account for larger input

<sup>3</sup> No augmentation is used with SIM2MNIST, despite the + suffixes

Table 2. Performance on MNIST variants. Errors are averages of several runs, with standard deviations within parenthesis. Times are average training time for one epoch. The deformations in MNIST R and RTS are mild, and 5x more training data is available; in those conditions, our STN implementation perform slightly better than the PTN. On SIM2MNIST, however, the PTN outperforms all the other methods by a large margin.

Origin aug.	Rotation aug.	Wrap padding	Error [%]
Yes	Yes	Yes	1.12 (0.03)
No	Yes	Yes	1.33 (0.12)
Yes	No	Yes	1.46 (0.11)
Yes	Yes	No	1.31 (0.06)

Table 3. Ablation study. Rotation and polar origin augmentation during training time, and wrap around padding all contribute to reduce the error. Results are from PTN-B on the rotated MNIST.

time rotation augmentation. Results are based on the PTN-B variant trained on Rotated MNIST. We remove one operation at a time and verify that the performance consistently drops, which indicates that all operations are indeed helpful. Table 3 shows the results.

### 5.5. Visualization

We visualize network activations to confirm our claims about invariance to translation and equivariance to rotations and dilations.

In figure 5, we look at some of the predicted polar origins and the results of the polar transform. We can see that the network learns to reject clutter and to find a suitable origin for the polar transform, and that the representation after the polar transformer module does present the properties claimed.

We proceed to visualize if the properties are preserved in deeper layers. Figure 6 shows the activations of selected channels from the last convolutional layer, for different rotations, dilations, and translations of the input. The reader can verify that the equivariance to rotations and dilations, and the invariance to translations are indeed preserved during the sequence of convolutional layers.



Figure 5. Predicted polar origins and results of the polar transform. The rows alternate between samples from SIM2MNIST, where the predicted origin is shown as a green dot, and their learned polar representation. Note how the polar representation is invariant to the original location of the object, and how rotations and dilations of the object become shifts.



Figure 6. Feature maps on the last convolutional layer. Each row shows a different input and correspondent activations. The first and second rows show that the  $180^{\circ}$  rotation results in a half-width shift of the feature maps. The third and fourth rows show that the  $2.4 \times$  dilation results in a shift right of the feature maps. By comparing the first and third rows, the invariance to translation can be verified.

# 6. Conclusion

We have proposed a novel network whose output is invariant to translations and equivariant to the group of dilations/rotations. We have combined the idea of learning the translation (similar to the spatial transformer) but providing equivariance for the scaling and rotation, avoiding, thus, fully connected layers required for the pose regression in the spatial transformer. Equivariance with respect to dilated rotations is achieved by convolution in this group. Such a convolution would require the production of multiple group copies, however, we avoid this by transforming into canonical coordinates. We improve the state of the art performance on rotated MNIST by a large margin, and outperform all other tested methods on a new dataset we call SIM2MNIST. We expect our approach to be applicable to other problems, where the presence of different orientations and scales hinder the performance of conventional CNNs.

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