A Linear/Non-Linear Model for Classical Linear Logic

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Background

Linear Logic (LL)

Introduced by Girard (1987), LL is a constructive logic that keeps track of resources. That is, every resource is used exactly once...

$$\frac{}{\cdot \vdash 1} \qquad \frac{\Gamma_1 \vdash \Delta_1, A_1 \quad \Gamma_2 \vdash \Delta_2, A_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, A_1 \otimes A_2}$$

$$\frac{\Gamma_1, A_1 \vdash \Delta_1 \quad \Gamma_2, A_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2, A_1 ⅋ A_2 \vdash \Delta_1, \Delta_2} \qquad \frac{}{\bot \vdash \cdot}$$

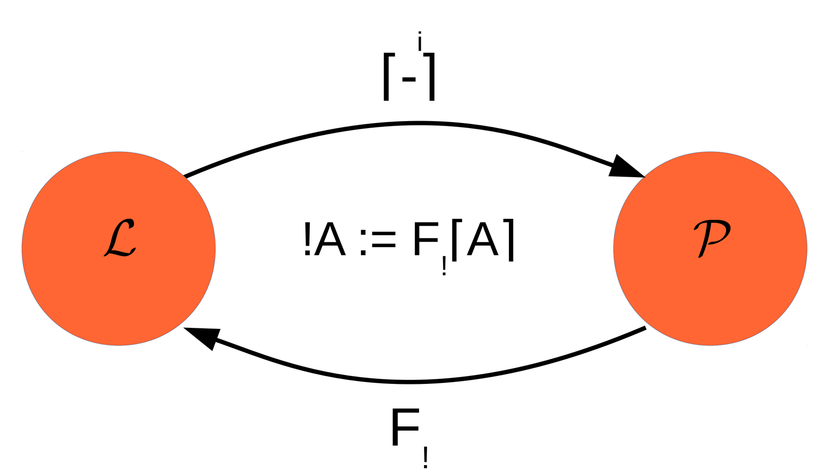
...except for exponential (non-linear) resources.

$$!A \to 1 \qquad !A \to !A,\ !A \qquad \bot \to ?A \qquad ?A,\ ?A \to ?A$$

Linear/Non-Linear Models

Benton (1995) introduced a model of intuitionistic LL (ILL) that syntactically separates the linear propositions A from non-linear propositions !A. The model partitions ! into two components:

Linear A ⇒ Persistent $\lceil A \rceil$
Persistent P ⇒ Linear $F_! P$



As a type theory, this linear/non-linear partition can be used to separate regular λ-calculus terms from linear terms.

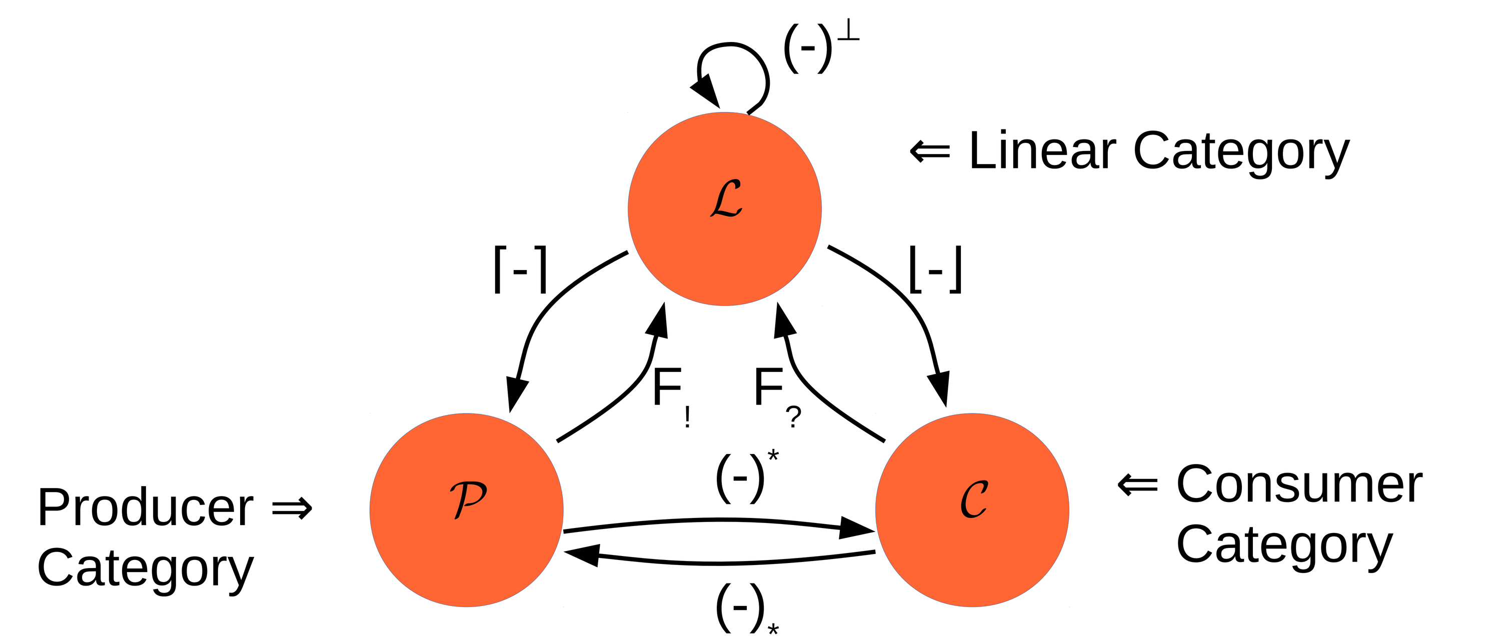
Generalizing to Classical Linear Logic

Models of LL, like Benton's, tend to focus on ILL. But classical LL adds nontrivial structure in the form of the duality operator.

A	$A^\perp$
A⊗B	$A^\perp ⅋ B^\perp$
1	⊥
A&B	$A^\perp \oplus B^\perp$
⊤	0
!A	$?A^\perp$

Classical linear logic is constructive, unlike classical logic, so it has a well-defined categorical interpretation. *Our goal is to present this model under the linear/non-linear paradigm, while giving duality, not linear implication, the prominent role.*

LPC Categorical Model



Producing and Consuming

The producer category $\mathcal{P}$ has natural transformations to interpret weakening and contraction, which form a comonoid for every object.

$$e_P : P \to 1 \qquad d_P : P \to P \otimes P$$

Similarly, every object in $\mathcal{C}$ must form a monoid with components

$$e_C : \bot \to C \qquad d_C : C ⅋ C \to C$$

The categories are in duality with each other, which means

$$(P^*)_* \simeq P \qquad (C_*)^* \simeq C$$

Linearity

The linear category $\mathcal{L}$ is linearly distributive:

$$\delta : A \otimes (B ⅋ C) \to (A \otimes B) ⅋ C$$

It also has dualities:

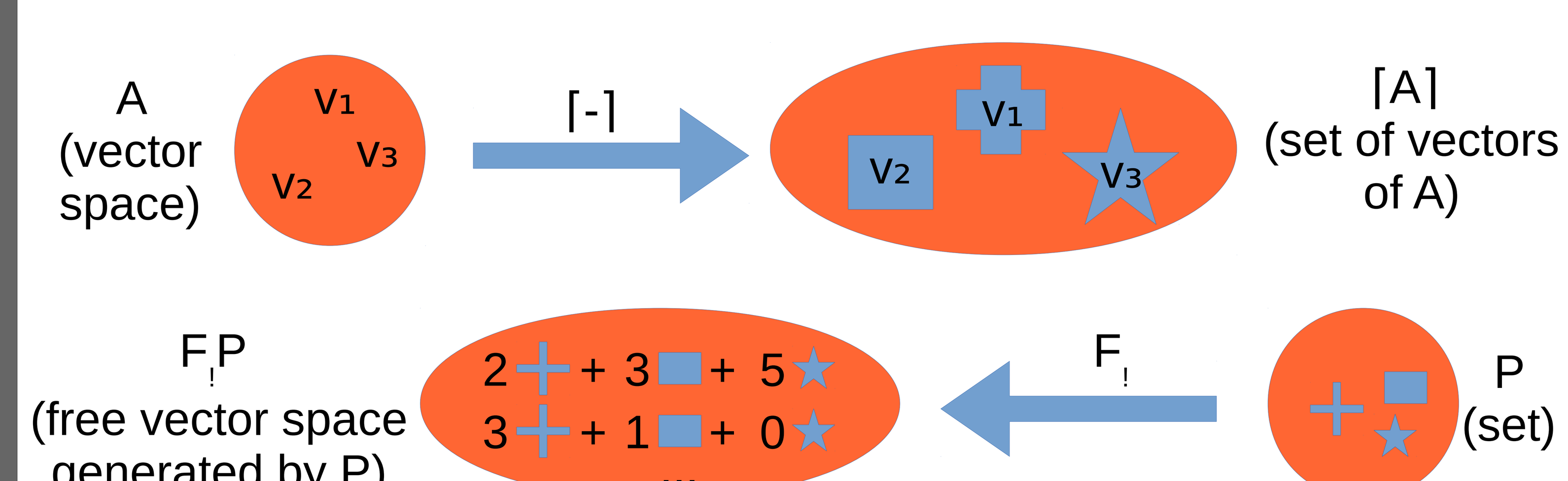
$$\gamma_A^\perp : A^\perp \otimes A \to \bot \qquad \gamma_A^1 : 1 \to A ⅋ A^\perp$$

Example: Linear Algebra

$\mathcal{L}$:=
Objects: finite-dimensional vector spaces over a finite field
Morphisms: linear maps.

$\mathcal{P}$:=
Objects: Finite sets
Morphisms: Functions

$\mathcal{C}$:=
Objects: Finite sets
Morphisms: Inverse functions



LPC Logic

Propositions

Linear terms
A,B ::= 1 | A⊗B | ⊥ | A⅋B | 0 | A⊕B | ⊤ | A&B | $F_! P$ | $F_? C$

Producers
P,Q ::= 1 | P⊗Q | $\lceil A \rceil$

Consumers
C,D ::= ⊥ | C⅋D | $\lfloor A \rfloor$

Duality is defined as a meta-operation based on De Morgan's laws.

Two Sequent Calculi, One Logic

Linear	Persistent
$\Gamma \vdash \Delta$	$\Gamma \Vdash \Delta$
Contexts range over A,P,C	Contexts range over P,C

Travel between the linear and persistent worlds by means of the adjunctions $F_! \dashv \lceil - \rceil$, $\lfloor - \rfloor \dashv F_?$.

$$\frac{\Gamma^P \Vdash \Delta^C, P}{\Gamma^P \vdash \Delta^C, F_! P} \qquad \frac{\Gamma^P \vdash \Delta^C, A}{\Gamma^P \Vdash \Delta^C, \lceil A \rceil} \qquad \frac{\Gamma^P, A \vdash \Delta^C}{\Gamma^P, \lfloor A \rfloor \Vdash \Delta^C} \qquad \frac{\Gamma^P, C \Vdash \Delta^C}{\Gamma^P, F_? C \vdash \Delta^C}$$

Embedding ! and ?

The regular promotion rules for linear logic are admissible in LPC.

$$\frac{\Gamma^! \vdash \Delta^?, A}{\Gamma^! \vdash \Delta^?, !A} \dashrightarrow \frac{\dfrac{\Gamma^P \vdash \Delta^C, A}{\Gamma^P \Vdash \Delta^C, \lceil A \rceil}}{\Gamma^P \vdash \Delta^C, F_! \lceil A \rceil} \qquad \frac{\Gamma^!, A \vdash \Delta^?}{\Gamma^!, ?A \vdash \Delta^?} \dashrightarrow \frac{\dfrac{\Gamma^P, A \vdash \Delta^C}{\Gamma^P, \lfloor A \rfloor \Vdash \Delta^C}}{\Gamma^P, F_? \lfloor A \rfloor \vdash \Delta^C}$$

Future Work

We have defined a logic and categorical model for the linear/non-linear paradigm extended to classical linear logic. Possible extensions include:

- Extending other models of intuitionistic LL to the classical case
- Exploring connections between linear logic and linear algebra. Possibilities include vector manipulation over finite fields, like cryptography.
- Construct a term language for the LPC logic with a well-defined equational model. Prove soundness of the categorical model with respect to terms.

References

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