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Geometric Methods and Applications for Computer Science and Engineering, Second Edition

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*To my wife, Anne, my children, Mia, Philippe,
and Sylvie, and my grandchildren, Bahari
and Demetrius*

Preface

This book is an introduction to fundamental geometric concepts and tools needed for solving problems of a geometric nature with a computer. Our main goal is to present a collection of tools that can be used to solve problems in computer vision, robotics, machine learning, computer graphics, and geometric modeling.

During the ten years following the publication of the first edition of this book, optimization techniques have made a huge comeback, especially in the fields of computer vision and machine learning. In particular, *convex optimization* and its special incarnation, *semidefinite programming (SDP)*, are now widely used techniques in computer vision and machine learning, as one may verify by looking at the proceedings of any conference in these fields. Therefore, we felt that it would be useful to include some material (especially on convex geometry) to prepare the reader for more comprehensive expositions of convex optimization, such as Boyd and Vandenberghe [2], a masterly and encyclopedic account of the subject. In particular, we added Chapter 7, which covers separating and supporting hyperplanes.

We also realized that the importance of the SVD (singular value decomposition) and of the pseudo-inverse had not been sufficiently stressed in the first edition of this book, and we rectified this situation in the second edition. In particular, we added sections on PCA (principal component analysis) and on best affine approximations and showed how they are efficienlty computed using SVD. We also added a section on quadratic optimization and a section on the Schur complement, showing the usefulness of the pseudo-inverse.

In this second edition, many typos and small mistakes have been corrected, some proofs have been shortened, some problems have been added, and some references have been added. Here is a list containing brief descriptions of the chapters that have been modified or added.

- Chapter 3, on the basic properties of convex sets, has been expanded. In particular, we state a version of Carathéodory's theorem for convex cones (Theorem 3.2), a version of Radon's theorem for pointed cones (Theorem 3.6), and Tverberg's theorem (Theorem 3.7), and we define centerpoints and prove their existence (Theorem 3.9).

- Chapter 7 is new. This chapter deals with separating hyperplanes, versions of Farkas's lemma, and supporting hyperplanes. Following Berger [1], various versions of the separation of open or closed convex subsets by hyperplanes are proved as consequences of a geometric version of the Hahn–Banach theorem (Theorem 7.1). We also show how various versions of Farkas's lemma (Lemmas 7.3, 7.4, and 7.5) can be easily deduced from separation results (Corollary 7.4 and Proposition 7.3). Farkas's lemma plays an important result in linear programming. Indeed, it can be used to give a quick proof of so-called strong duality in linear programming. We also prove the existence of supporting hyperplanes for boundary points of closed convex sets (Minkowski's lemma, Proposition 7.4). Unfortunately, lack of space prevents us from discussing polytopes and polyhedra. The reader will find a masterly exposition of these topics in Ziegler [3].
- Chapter 14 is a major revision of Chapter 13 (Applications of Euclidean Geometry to Various Optimization Problems) from the first edition of this book and has been renamed "Applications of SVD and Pseudo-Inverses." Section 14.1, about least squares problems, and the pseudo-inverse has not changed much, but we have added the fact that AA^+ is the orthogonal projection onto the range of A and that A^+A is the orthogonal projection onto $\mathrm{Ker}(A)^\perp$, the orthogonal complement of $\mathrm{Ker}(A)$. We have also added Proposition 14.1, which shows how the pseudo-inverse of a normal matrix A can be obtained from a block diagonalization of A (see Theorem 12.7). Sections 14.2, 14.3, and 14.4 are new.
  In Section 14.2, we define various matrix norms, including operator norms, and we prove Proposition 14.4, showing how a matrix can be best approximated by a rank-k matrix (in the $\| \|_2$ norm).
  Section 14.3 is devoted to principal component analysis (PCA). PCA is a very important statistical tool, yet in our experience, most presentations of this concept lack a crisp definition. Most presentations identify the notion of principal components with the result of applying SVD and do not prove why SVD does in fact yield the principal components and directions. To rectify this situation, we give a precise definition of PCAs (Definition 14.3), and we prove rigorously how SVD yields PCA (Theorem 14.3), using the Rayleigh–Ritz ratio (Lemma 14.2).
  In Section 14.4, it is shown how to best approximate a set of data with an affine subspace in the least squares sense. Again, SVD can used to find solutions.
- Chapter 15 is new, except for Section 15.1, which reproduces Section 13.2 from the first edition of this book. We added the definition of the positive semidefinite cone ordering, $\succeq$, on symmetric matrices, since it is extensively used in convex optimization.
  In Section 15.2, we find a necessary and sufficient condition (Proposition 15.2) for the quadratic function $f(x) = \frac{1}{2}x^\top Ax + x^\top b$ to have a minimum in terms of the pseudo-inverse of A (where A is a symmetric matrix). We also show how to accommodate linear constraints of the form $C^\top x = 0$ or affine constraints of the form $C^\top x = t$ (where $t \neq 0$).
  In Section 15.3, we consider the problem of maximizing $f(x) = x^\top Ax$ on the unit sphere $x^\top x = 1$ or, more generally, on the ellipsoid $x^\top Bx = 1$, where A is a symmetric matrix and B is symmetric, positive definite. We show that these

problems are completely solved by diagonalizing A with respect to an orthogonal matrix. We also briefly consider the effect of adding linear constraints of the form $C^\top x = 0$ or affine constraints of the form $C^\top x = t$ (where $t \neq 0$).

- Chapter 16 is new. In this chapter, we define the notion of *Schur complement*, and we use it to characterize when a symmetric 2×2 block matrix is either positive semidefinite or positive definite (Proposition 16.1, Proposition 16.2, and Theorem 16.1).
- Chapter 17 is also brand new. In this chapter, we show how a computer vision problem, contour grouping, can be formulated as a quadratic optimization problem involving a Hermitian matrix. Because of the extra dependency on an angle, this optimization problem leads to finding the derivative of eigenvalues and eigenvectors of a normal matrix X. We derive explicit formulas for these derivatives (in the case of eigenvectors, the formula involves the pseudo-inverse of X) and we prove their correctness. It appears to be difficult to find these formulas together with a clean and correct proof in the literature. Our optimization problem leads naturally to the consideration of the *field of values* (or *numerical range*) $F(A)$ of a complex matrix A. A remarkable property of the field of values is that it is a convex subset of the plane, a theorem due to Toeplitz and Hausdorff, for which we give a short proof using a deformation argument (Theorem 17.1). Properties of the fields of values can be exploited to solve our optimization problem. This chapter describes current and exciting research in computer vision.
- Chapter 18 (which used to be Chapter 14 in the first edition) has been slightly expanded and improved. Our experience in teaching the material of this chapter, an introduction to manifolds and Lie groups, is that it is helpful to review carefully the notion of the derivative of a function $f\colon E \to F$ where E and F are normed vector spaces. Thus we added Section 18.7, which provides such a review. We also state the inverse function theorem and define immersions and submersions. Section 18.8 has also been slightly expanded. We added Proposition 18.6 and Theorem 18.7, which are often useful in proving that various spaces are manifolds; we defined critical and regular values and defined Morse functions; and we made a few cosmetic improvements in the paragraphs following Definition 18.20. A number of new problems on manifolds have been added.
- The only change to Chapter 19 (Chapter 15 in the first edition) is the inclusion of a more complete treatment of the *Frenet frame* for nD curves in Section 19.10.
- Similarly, the only change to Chapter 20 (Chapter 16 in the first edition) is the addition of Section 20.12, on *covariant derivatives* and the *parallel transport*.

Besides adding problems to all the chapters listed above we added one more problem to Chapter 2.

As in the first edition, there is some additional material on the web site `http://www.cis.upenn.edu/~jean/gbooks/geom2.html`

This material has not changed, and the chapter and section numbers are those of the first edition. A graph showing the dependencies of chapters is shown in Figure 0.1.

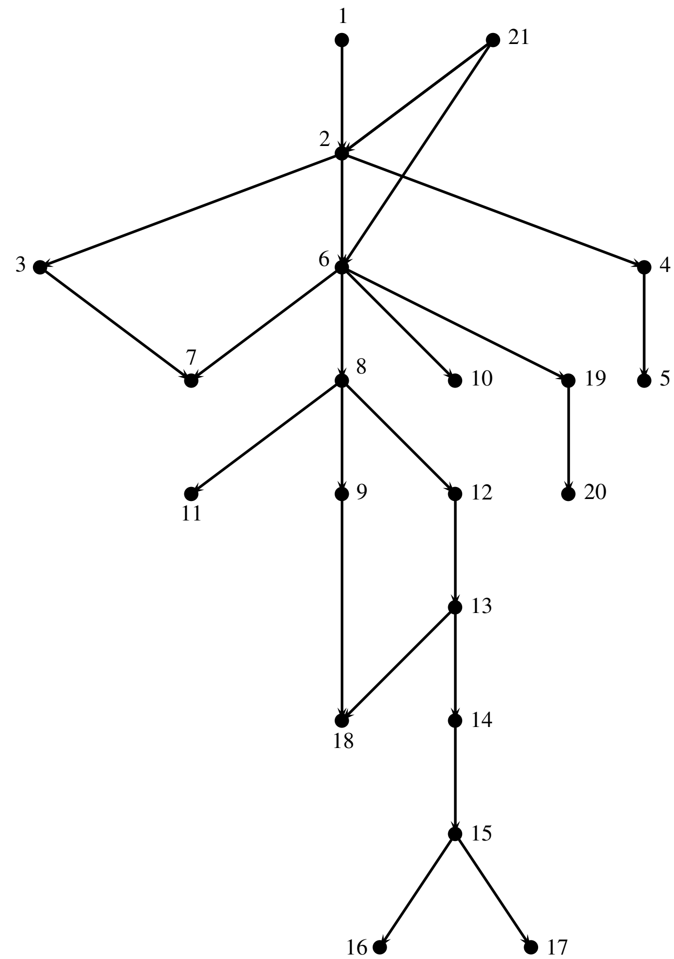

Fig. 0.1 Dependency of chapters.

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Philadelphia, March 2011 *Jean Gallier*

Preface to the First Edition

Many problems arising in engineering, and notably in computer science and mechanical engineering, require geometric tools and concepts. This is especially true of problems arising in computer graphics, geometric modeling, computer vision, and motion planning, just to mention some key areas. This book is an introduction to fundamental geometric concepts and tools needed for solving problems of a geometric nature with a computer. In a previous text, Gallier [24], we focused mostly on affine geometry and on its applications to the design and representation of polynomial curves and surfaces (and *B*-splines). The main goal of this book is to provide an introduction to more sophisticated geometric concepts needed in tackling engineering problems of a geometric nature. Many problems in the above areas require some nontrivial geometric knowledge, but in our opinion, books dealing with the relevant geometric material are either too theoretical, or else rather specialized. For example, there are beautiful texts entirely devoted to projective geometry, Euclidean geometry, and differential geometry, but reading each one represents a considerable effort (certainly from a nonmathematician!). Furthermore, these topics are usually treated for their own sake (and glory), with little attention paid to applications.

This book is an attempt to fill this gap. We present a coherent view of geometric methods applicable to many engineering problems at a level that can be understood by a senior undergraduate with a good math background. Thus, this book should be of interest to a wide audience including computer scientists (both students and professionals), mathematicians, and engineers interested in geometric methods (for example, mechanical engineers). In particular, we provide an introduction to affine geometry, projective geometry, Euclidean geometry, basics of differential geometry and Lie groups, and a glimpse of computational geometry (convex sets, Voronoi diagrams, and Delaunay triangulations). This material provides the foundations for the algorithmic treatment of curves and surfaces, some basic tools of geometric modeling. The right dose of projective geometry also leads to a rigorous and yet smooth presentation of rational curves and surfaces. However, to keep the size of this book reasonable, a number of topics could not be included. Nevertheless, they can be found in the additional material on the web site: see `http://www.cis.`

upenn.edu/~jean/gbooks/geom2.html. This is the case of the material on rational curves and surfaces.

This book consists of sixteen chapters and an appendix. The additional material on the web site consists of eight chapters and an appendix: see http://www.cis.upenn.edu/~jean/gbooks/geom2.html.

- The book starts with a brief introduction (Chapter 1).
- Chapter 2 provides an introduction to affine geometry. This ensures that readers are on firm ground to proceed with the rest of the book, in particular, projective geometry. This is also useful to establish the notation and terminology. Readers proficient in geometry may omit this section, or use it *as needed*. On the other hand, readers totally unfamiliar with this material will probably have a hard time with the rest of the book. These readers are advised do some extra reading in order to assimilate some basic knowledge of geometry. For example, we highly recommend Pedoe [42], Coxeter [9], Snapper and Troyer [52], Berger [2, 3], Fresnel [22], Samuel [51], Hilbert and Cohn–Vossen [31], Boehm and Prautzsch [5], and Tisseron [54].
- Basic properties of convex sets and convex hulls are discussed in Chapter 3. Three major theorems are proved: Carthéodory's theorem, Radon's theorem, and Helly's theorem.
- Chapter 4 presents a construction (the "hat construction") for embedding an affine space into a vector space. An important application of this construction is the projective completion of an affine space, presented in the next chapter. Other applications are treated in Chapter 20 on the web site, see http://www.cis.upenn.edu/~jean/gbooks/geom2.html.
- Chapter 5 provides an introduction to projective geometry. Since we are not writing a treatise on projective geometry, we cover only the most fundamental concepts, including projective spaces and subspaces, frames, projective maps, multiprojective maps, the projective completion of an affine space, cross-ratios, duality, and the complexification of a real projective space. This material also provides the foundations for our algorithmic treatment of rational curves and surfaces, to be found on the web site (Chapters 18, 19, 21, 22, 23, 24); see http://www.cis.upenn.edu/~jean/gbooks/geom2.html.
- Chapters 6, 8, and 9, provide an introduction to Euclidean geometry, to the groups of isometries $\mathbf{O}(n)$ and $\mathbf{SO}(n)$, the groups of affine rigid motions $\mathbf{Is}(n)$ and $\mathbf{SE}(n)$, and to the quaternions. Several versions of the Cartan–Dieudonné theorem are proved in Chapter 8. The *QR*-decomposition of matrices is explained geometrically, both in terms of the Gram–Schmidt procedure and in terms of Householder transformations. These chapters are crucial to a firm understanding of the differential geometry of curves and surfaces, and computational geometry.
- Chapter 10 gives a short introduction to some fundamental topics in computational geometry: Voronoi diagrams and Delaunay triangulations.
- Chapter 11 provides an introduction to Hermitian geometry, to the groups of isometries $\mathbf{U}(n)$ and $\mathbf{SU}(n)$, and the groups of affine rigid motions $\mathbf{Is}(n,\mathbb{C})$ and $\mathbf{SE}(n,\mathbb{C})$. The generalization of the Cartan–Dieudonné theorem to Hermitian spaces can be found on the web site, Chapter 25; see http://www.

`cis.upenn.edu/~jean/gbooks/geom2.html`. A short introduction to Hilbert spaces, including the projection theorem, and the isomorphism of every Hilbert space with some space $l^2(K)$, can also be found on the web site in Chapter 26, see `http://www.cis.upenn.edu/~jean/gbooks/geom2.html`.
- Chapter 12 provides a presentation of the spectral theorems in Euclidean and Hermitian spaces, including normal, self-adjoint, skew self-adjoint, and orthogonal linear maps. Normal form (in terms of block diagonal matrices) for various types of linear maps are presented.
- The singular value decomposition (SVD) and the polar form of linear maps are discussed quite extensively in Chapter 13. The pseudo-inverse of a matrix and its characterization using the Penrose properties are presented.
- Chapter 14 presents some applications of Euclidean geometry to various optimization problems. The method of least squares is presented, as well as the applications of the SVD and QR-decomposition to solve least squares problems. We also describe a method for minimizing positive definite quadratic forms, using Lagrange multipliers.
- Chapter 18 provides an introduction to the linear Lie groups, via a presentation of some of the classical groups and their Lie algebras, using the exponential map. The surjectivity of the exponential map is proved for $\mathbf{SO}(n)$ and $\mathbf{SE}(n)$.
- An introduction to the local differential geometry of curves is given in Chapter 19 (curvature, torsion, the Frenet frame, etc).
- An introduction to the local differential geometry of surfaces based on some lectures by Eugenio Calabi is given in Chapter 20. This chapter is rather unique, as it reflects decades of experience from a very distinguished geometer.
- Chapter 21 is an appendix consisting of short sections consisting of basics of linear algebra and analysis. This chapter has been included to make the material self-contained. Our advice is to use it *as needed*!

A very elegant presentation of rational curves and surfaces can be given using some notions of affine and projective geometry. We push this approach quite far in the material on the web; see `http://www.cis.upenn.edu/~jean/gbooks/geom2.html`. However, we provide only a cursory coverage of CAGD methods. Luckily, there are excellent texts on CAGD, including Bartels, Beatty, and Barsky [1], Farin [17, 18], Fiorot and Jeannin [20, 21], Riesler [50], Hoschek and Lasser [33], and Piegl and Tiller [43]. Although we cover affine, projective, and Euclidean geometry in some detail, we are far from giving a comprehensive treatment of these topics. For such a treatment, we highly recommend Berger [2, 3], Samuel [51], Pedoe [42], Coxeter [11, 10, 8, 9], Snapper and Troyer [52], Fresnel [22], Tisseron [54], Sidler [45], Dieudonné [13], and Veblen and Young [57, 58], a great classic.

Similarly, although we present some basics of differential geometry and Lie groups, we only scratch the surface. For instance, we refrain from discussing manifolds in full generality. We hope that our presentation is a good preparation for more advanced texts, such as Gray [27], do Carmo [14], Berger and Gostiaux [4], and Lafontaine [36]. The above are still fairly elementary. More advanced texts on differential geometry include do Carmo [15, 16], Guillemin and Pollack [29], Warner

[59], Lang [37], Boothby [6], Lehmann and Sacré [38], Stoker [53], Gallot, Hulin, and Lafontaine [25], Milnor [41], Sharpe [44], Malliavin [39], and Godbillon [26].

It is often possible to reduce interpolation problems involving polynomial curves or surfaces to solving systems of linear equations. Thus, it is very helpful to be aware of efficient methods for numerical matrix analysis. For instance, we present the *QR*-decomposition of matrices, both in terms of the (modified) Gram–Schmidt method and in terms of Householder transformations, in a novel geometric fashion. For further information on these topics, readers are referred to the excellent texts by Strang [48], Golub and Van Loan [28], Trefethen and Bau [55], Ciarlet [7], and Kincaid and Cheney [34]. Strang's beautiful book on applied mathematics is also highly recommended as a general reference [46]. There are other interesting applications of geometry to computer vision, computer graphics, and solid modeling. Some good references are Trucco and Verri [56], Koenderink [35], and Faugeras [19] for computer vision; Hoffman [32] for solid modeling; and Metaxas [40] for physics-based deformable models.

Novelties

As far as we know, there is no fully developed modern exposition integrating the basic concepts of affine geometry, projective geometry, Euclidean geometry, Hermitian geometry, basics of Hilbert spaces with a touch of Fourier series, basics of Lie groups and Lie algebras, as well as a presentation of curves and surfaces both from the standard differential point of view and from the algorithmic point of view in terms of control points (in the polynomial and rational case).

New Treatment, New Results

This books provides an introduction to affine geometry, projective geometry, Euclidean geometry, Hermitian geometry, Hilbert spaces, a glimpse at Lie groups and Lie algebras, and the basics of local differential geometry of curves and surfaces. We also cover some classics of convex geometry, such as Carathéodory's theorem, Radon's theorem, and Helly's theorem. However, in order to help the reader assimilate all these concepts with the least amount of pain, we begin with some basic notions of affine geometry in Chapter 2. Basic notions of Euclidean geometry come later only in Chapters 6, 8, 9. Generally, noncore material is relegated to appendices or to the web site: see `http://www.cis.upenn.edu/~jean/gbooks/geom2.html`.

We cover the standard local differential properties of curves and surfaces at an elementary level, but also provide an in-depth presentation of polynomial and rational curves and surfaces from an algorithmic point of view. The approach (sometimes called *blossoming*) consists in multilinearizing everything in sight (getting *polar forms*), which leads very naturally to a presentation of polynomial and rational curves and surfaces in terms of control points (Bézier curves and surfaces). We present many algorithms for subdividing and drawing curves and surfaces, all implemented in *Mathematica*. A clean and elegant presentation of control points with weights (and control vectors) is obtained by using a construction for embedding an affine space into a vector space (the so-called "hat construction," originating in Berger [2]). We also give several new methods for drawing efficiently closed rational curves and surfaces, and a method for resolving base points of triangular rational surfaces. We give a quick introduction to the concepts of Voronoi diagrams and Delaunay triangulations, two of the most fundamental concepts in computational geometry. As a general rule, we try to be rigorous, but we always keep the algorithmic nature of the mathematical objects under consideration in the forefront.

Many problems and programming projects are proposed (over 230). Some are routine, some are (very) difficult.

Applications

Although it is core mathematics, geometry has many practical applications. Whenever possible, we point out some of these applications, For example, we mention some (perhaps unexpected) applications of projective geometry to computer vision (camera calibration), efficient communication, error correcting codes, and cryptography (see Section 5.13). As applications of Euclidean geometry, we mention motion interpolation, various normal forms of matrices including *QR*-decomposition in terms of Householder transformations and *SVD*, least squares problems (see Section 14.1), and the minimization of quadratic functions using Lagrange multipliers (see Section 15.1). Lie groups and Lie algebras have applications in robot kinematics, motion interpolation, and optimal control. They also have applications in physics. As applications of the differential geometry of curves and surfaces, we mention geometric continuity for splines, and variational curve and surface design (see Section 19.11 and Section 20.13). Finally, as applications of Voronoi diagrams and Delaunay triangulations, we mention the nearest neighbors problem, the largest empty circle problem, the minimum spanning tree problem, and motion planning (see Section 10.5). Of course, rational curves and surfaces have many applications to computer-aided geometric design (CAGD), manufacturing, computer graphics, and robotics.

Many Algorithms and Their Implementation

Although one of our main concerns is to be mathematically rigorous, which implies that we give precise definitions and prove almost all of the results in this book, we are primarily interested in the representation and the implementation of concepts and tools used to solve geometric problems. Thus, we devote a great deal of efforts to the development and implemention of algorithms to manipulate curves, surfaces, triangulations, etc. As a matter of fact, we provide *Mathematica* code for most of the geometric algorithms presented in this book. We also urge the reader to write his own algorithms, and we propose many challenging programming projects.

Open Problems

Not only do we present standard material (although sometimes from a fresh point of view), but whenever possible, we state some open problems, thus taking the reader to the cutting edge of the field. For example, we describe very clearly the problem of resolving base points of rectangular rational surfaces (this material is on the web site, see `http://www.cis.upenn.edu/~jean/gbooks/geom2.html`).

What's Not Covered in This Book

Since this book is already quite long, we have omitted solid modeling techniques, methods for rendering implicit curves and surfaces, the finite elements method, and wavelets. The first two topics are nicely covered in Hoffman [32], and the finite element method is the subject of so many books that we will not attempt to mention any references besides Strang and Fix [47]. As to wavelets, we highly recommend the classics by Daubechies [12], and Strang and Truong [49], among the many texts on this subject. It would also have been nice to include chapters on the algebraic geometry of curves and surfaces. However, this is a very difficult subject that requires a lot of algebraic machinery. Interested readers may consult Fulton [23] or Harris [30].

How to Use This Book for a Course

This books covers three complementary but fairly disjoint topics:

(1) Projective geometry and its applications to rational curves and surfaces (Chapter 5, and on the web page, Chapters 18, 19, 21, 22, 23, 24);
(2) Euclidean geometry, Voronoi diagrams, and Delaunay triangulations, Hermitian geometry, basics of Hilbert spaces, spectral theorems for special kinds of linear maps, SVD, polar form, and basics of Lie groups and Lie algebras (Chapters 6, 8, 9, 10, 11, 12, 13, 14, 18);
(3) Basics of the differential geometry of curves and surfaces (Chapters 19 and 20).

Chapter 21 is an appendix consisting of background material and should be used only *as needed*.

Our experience is that there is too much material to cover in a one–semester course. The ideal situation is to teach the material in the entire book in two semesters. Otherwise, a more algebraically inclined teacher should teach the first or second topic, whereas a more differential-geometrically inclined teacher should teach the third topic. In either case, Chapter 2 on affine geometry should be covered. Chapter 4 is required for the first topic, but not for the second.

Problems are found at the end of each chapter. They range from routine to very difficult. Some programming assignments have been included. They are often quite open-ended, and may require a considerable amount of work. The end of a proof is indicated by a square box (□). The word *iff* is an abbreviation for *if and only if*. References to the web page `http://www.cis.upenn.edu/~jean/gbooks/geom2.html` will be abbreviated as web page.

Hermann Weyl made the following comment in the preface (1938) of his beautiful book [60]:

> The gods have imposed upon my writing the yoke of a foreign tongue that was not sung at my cradle .... Nobody is more aware than myself of the attendant loss in vigor, ease and lucidity of expression.

Being in a similar position, I hope that I was at least successful in conveying my enthusiasm and passion for geometry, and that I have inspired my readers to study some of the books that I respect and admire.

Acknowledgments

This book grew out of lectures notes that I have written as I have been teaching CIS610, *Advanced Geometric Methods in Computer Science*, for the past two years. Many thanks to the copyeditor, David Kramer, who did a superb job. I also wish to thank some students and colleagues for their comments, including Koji Ashida, Doug DeCarlo, Jaydev Desai, Will Dickinson, Charles Erignac, Steve Frye, Edith Haber, Andy Hicks, Paul Hughett, David Jelinek, Marcus Khuri, Hartmut Liefke, Shih-Schon Lin, Ying Liu, Nilesh Mankame, Dimitris Metaxas, Viorel Mihalef, Albert Montillo, Youg-jin Park, Harold Sun, Deepak Tolani, Dianna Xu, and Hui Zhang. Also thanks to Norm Badler for triggering my interest in geometric modeling, and to Marcel Berger, Chris Croke, Ron Donagi, Herman Gluck, David Harbater, Alexandre Kirillov, and Steve Shatz for sharing some of their geometric secrets with me. Finally, many thanks to Eugenio Calabi for teaching me what I know about differential geometry (and much more!). I am very grateful to Professor Calabi for allowing me to write up his lectures on the differential geometry of curves and surfaces given in an undergraduate course in Fall 1994 (as Chapter 20).

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