Spring, 2010 CIS 511

Introduction to the Theory of Computation
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Homework 5

March 25, 2010; Due April 8, 2010

"A problems" are for practice only, and should not be turned in.

Problem A1. Given any two context-free languages L_1 and L_2 over the same alphabet Σ, prove that $L_1 \cup L_2$ and $L_1 L_2$ are also context-free.

Problem A2. Let Σ and Δ be some alphabets, and let $h\colon \Sigma^* \to \Delta^*$ be a homomorphism. Given any language $L \subseteq \Sigma^*$, recall that

$$h(L) = \{h(w) \in \Delta^* \mid w \in L\}.$$

Prove that if L is context-free, then $h(L)$ is also context-free.

Problem A3. Given any language $L \subseteq \Sigma^*$, let

$$L^R = \{w^R \mid w \in L\},$$

the *reversal language of* L (where w^R denotes the reversal of the string w). Prove that if L is context-free, then L^R is also context-free.

"B problems" must be turned in.

Problem B1 (60 pts). Give context-free grammars for the following languages:

(a) $L_5 = \{wcw^R \mid w \in \{a,b\}^*\}$ (w^R denotes the reversal of w)

(b) $L_6 = \{a^m b^n \mid 1 \leq m \leq n \leq 2m\}$

For any fixed integer $K \geq 2$,

$L_7 = \{a^m b^n \mid 1 \leq m \leq n \leq Km\}$

(c) $L_8 = \{a^n b^n \mid n \geq 1\} \cup \{a^n b^{2n} \mid n \geq 1\}$

(d) $L_9 = \{a^m b^n a^m b^p \mid m, n, p \geq 1\} \cup \{a^m b^{4n} a^p b^{4n} \mid m, n, p \geq 1\}$

(e) $L_{10} = \{xcy \mid |x| = 2|y|,\ x, y \in \{a,b\}^*\}$

In each case, give a brief justification of the fact that your grammar generates the desired language.

Problem B2 (40 pts). Given a context-free language L and a regular language R, prove that $L \cap R$ is context-free.

Do not use PDA's to solve this problem!

Hint. Without loss of generality, assume that $L = L(G)$, where $G = (V, \Sigma, P, S)$ is in Chomsky normal form, and let $R = L(D)$, for some DFA $D = (Q, \Sigma, \delta, q_0, F)$. Use a kind of cross-product construction as sketched below. Construct a CFG G_2 whose set of nonterminals is $Q \times N \times Q \ \cup \{S_0\}$, where S_0 is a new nonterminal, and whose productions are of the form:

$$S_0 \to (q_0, S, f),$$

for every $f \in F$;

$$(p, A, \delta(p, a)) \to a \quad \text{iff} \quad (A \to a) \in P,$$

for all $a \in \Sigma$, all $A \in N$, and all $p \in Q$;

$$(p, A, s) \to (p, B, q)(q, C, s) \quad \text{iff} \quad (A \to BC) \in P,$$

for all $p, q, s \in Q$ and all $A, B, C \in N$;

$$S_0 \to \epsilon \quad \text{iff} \quad (S \to \epsilon) \in P \text{ and } q_0 \in F.$$

Prove that for all $p, q \in Q$, all $A \in N$, all $w \in \Sigma^+$, and all $n \geq 1$,

$$(p, A, q) \underset{lm}{\overset{n}{\Longrightarrow}}_{G_2} w \quad \text{iff} \quad A \underset{lm}{\overset{n}{\Longrightarrow}}_{G} w \quad \text{and} \quad \delta^*(p, w) = q.$$

Conclude that $L(G_2) = L \cap R$.

Problem B3 (40 pts). Give context-free grammars for the languages

$$\begin{aligned} L_1 &= \{xcy \mid x \neq y,\, x, y \in \{a, b\}^*\} \\ L_2 &= \{xcy \mid \ x \neq y^R,\, x, y \in \{a, b\}^*\}. \end{aligned}$$

Problem B4 (10 pts). Prove that the function, $f \colon \Sigma^* \to \Sigma^*$, given by

$$f(w) = a_1^{|w|}$$

is primitive recursive ($\Sigma = \{a_1, \ldots, a_N\}$).

Problem B5 (30 pts). *Ackermann's function* A is defined recursively as follows:

$$\begin{aligned} A(0,\, y) &= y + 1, \\ A(x+1,\, 0) &= A(x,\, 1), \\ A(x+1,\, y+1) &= A(x,\, A(x+1, y)). \end{aligned}$$

Prove that

$$\begin{aligned}A(0,\,x) &= x+1,\\ A(1,\,x) &= x+2,\\ A(2,\,x) &= 2x+3,\\ A(3,\,x) &= 2^{x+3}-3,\end{aligned}$$

and

$$A(4,x) = \left.2^{2^{\cdot^{\cdot^{\cdot^{2^{16}}}}}}\right\}^{x} - 3,$$

with $A(4,0) = 16-3 = 13$. Equivalently (and perhaps less confusing)

$$A(4,x) = \left.2^{2^{\cdot^{\cdot^{\cdot^{2^{2}}}}}}\right\}^{x+3} - 3.$$

Problem B6 (20 pts). Let $\Sigma = \{a_1, \ldots, a_k\}$ be some alphabet and suppose $g, h_1, \ldots, h_k$ are some total functions, with $g\colon (\Sigma^*)^{n-1} \to \Sigma^*$, and $h_i\colon (\Sigma^*)^{n+1} \to \Sigma^*$, for $i = 1, \ldots, k$. If we write $\overline{x}$ for $(x_2, \ldots, x_n)$, for any $y \in \Sigma^*$, where $y = a_{i_1} \cdots a_{i_m}$ (with $a_{i_j} \in \Sigma$), define the following sequences, u_j and v_j, for $j = 0, \ldots, m+1$:

$$\begin{aligned}u_0 &= \epsilon\\ u_1 &= u_0 a_{i_1}\\ &\vdots\\ u_j &= u_{j-1} a_{i_j}\\ &\vdots\\ u_m &= u_{m-1} a_{i_m}\\ u_{m+1} &= u_m a_i\end{aligned}$$

and

$$\begin{aligned}v_0 &= g(\overline{x})\\ v_1 &= h_{i_1}(u_0, v_0, \overline{x})\\ &\vdots\\ v_j &= h_{i_j}(u_{j-1}, v_{j-1}, \overline{x})\\ &\vdots\\ v_m &= h_{i_m}(u_{m-1}, v_{m-1}, \overline{x})\\ v_{m+1} &= h_i(y, v_m, \overline{x}).\end{aligned}$$

(i) Prove that

$$v_j = f(u_j, \overline{x}),$$

for $j = 0, \ldots, m+1$, where f is defined by primitive recursion from g and the h_i's, that is

$$\begin{aligned}
f(\epsilon, \overline{x}) &= g(\overline{x}) \\
f(ya_1, \overline{x}) &= h_1(y, f(y, \overline{x}), \overline{x}) \\
&\vdots \\
f(ya_i, \overline{x}) &= h_i(y, f(y, \overline{x}), \overline{x}) \\
&\vdots \\
f(ya_k, \overline{x}) &= h_k(y, f(y, \overline{x}), \overline{x}),
\end{aligned}$$

for all $y \in \Sigma^*$ and all $\overline{x} \in (\Sigma^*)^{n-1}$. Conclude that f is a total function.

(ii) Use (i) to prove that if g and the h_i's are RAM computable, then the function, f, defined by primitive recursion from g and the h_i's is also RAM computable.

Problem B7 (50 pts). Write a computer program implementing a Turing machine interpreter. Your program should output

1. The list of intructions of the input Turing machine, M (as quintuples of the form (p, a, b, m, q), as in described in class).
2. The sequence of Instantaneous Descriptions that the TM goes through while it executes M. The first ID contains the input string(s) as explained in class.
3. Each ID should display the current state that the machine is in, the current contents of the tape and the position of the reading head.

Test your interpreter on several Turing machines (and input strings).

TOTAL: 250 points.