Trimester Program on
Computational Manifolds and Applications

Introduction to Computational Manifolds and Applications

Differential Operators on Manifolds

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Summary

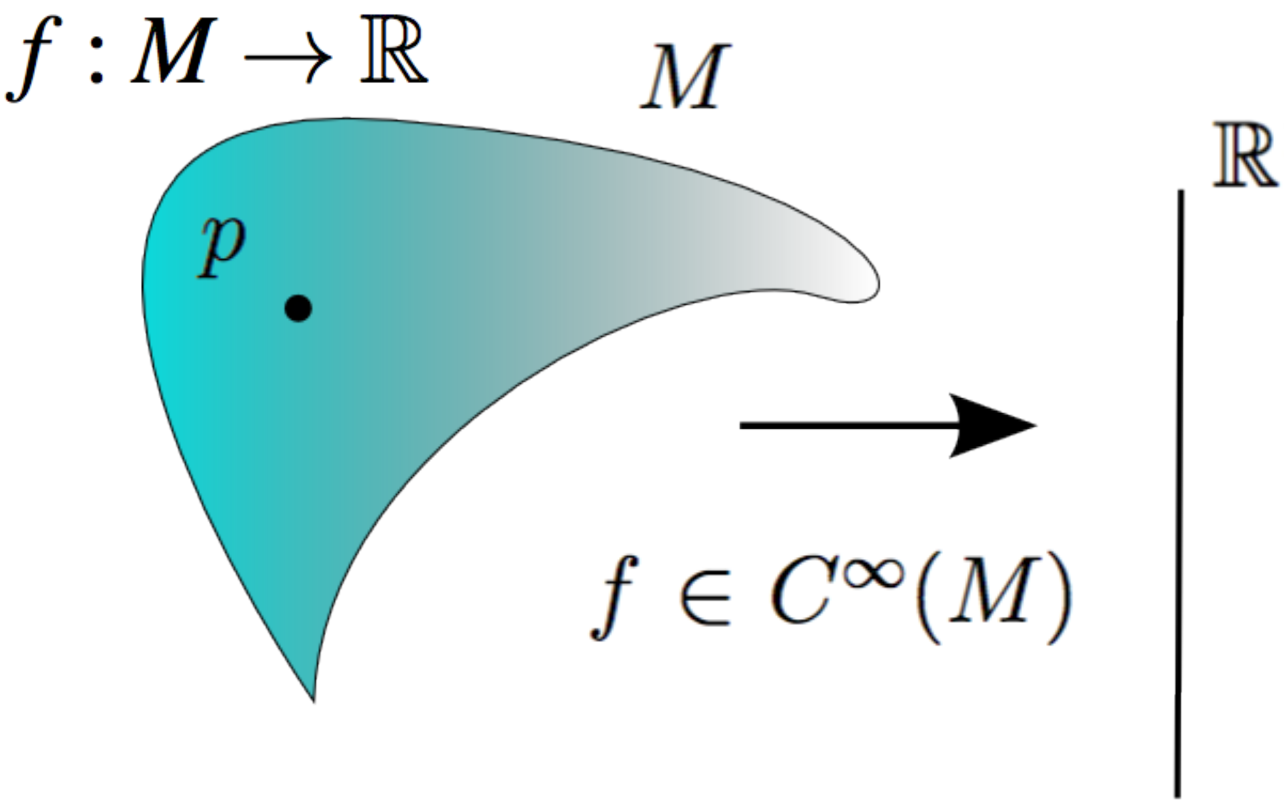
Today (Tuesday): Differential Operators on Surfaces

- Differential operators in the parametric domain
- Cotangent formula
- Belkin's approach
- SPH-based scheme

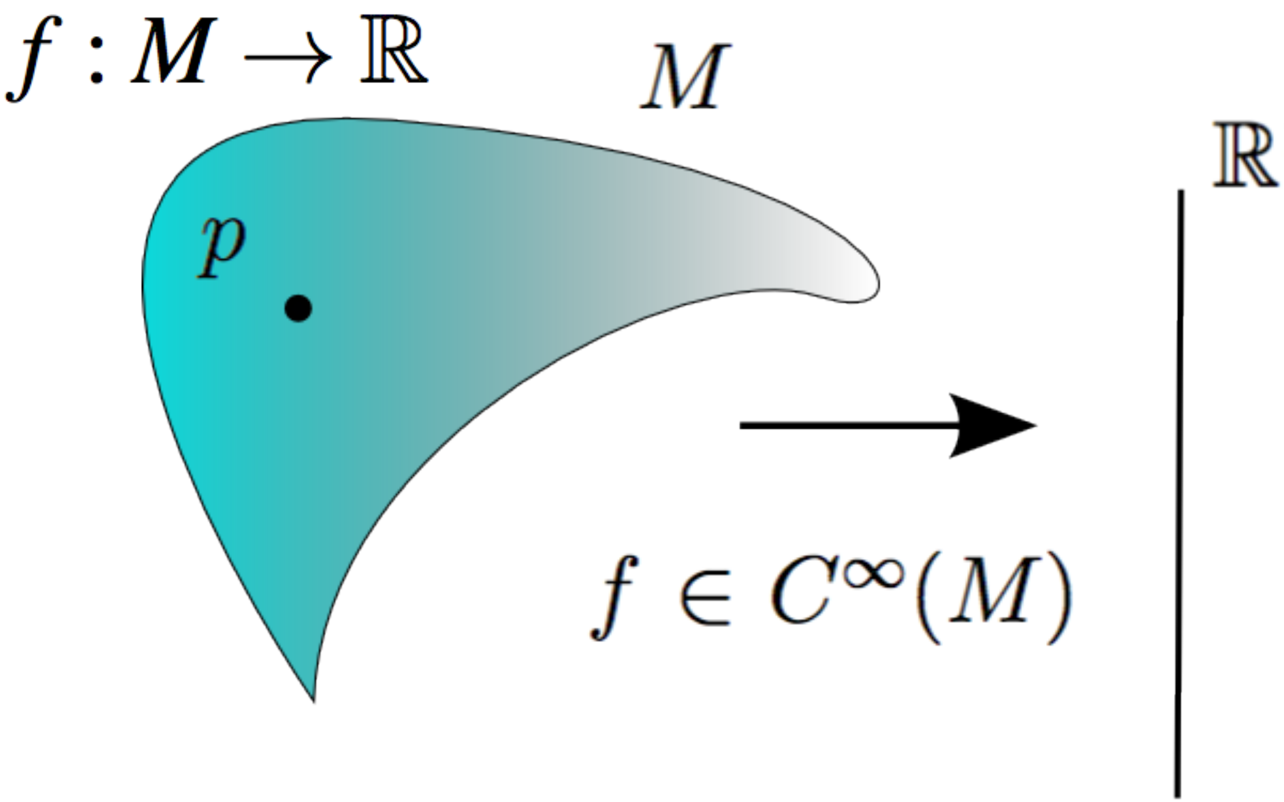
Thursday: Manifold Harmonics and Applications

- Some theoretical background
- Mesh Filtering
- Rustamov Embedding
- Fiedler tree
- Other applications

Scalar Functions on Surfaces



Scalar Functions on Surfaces

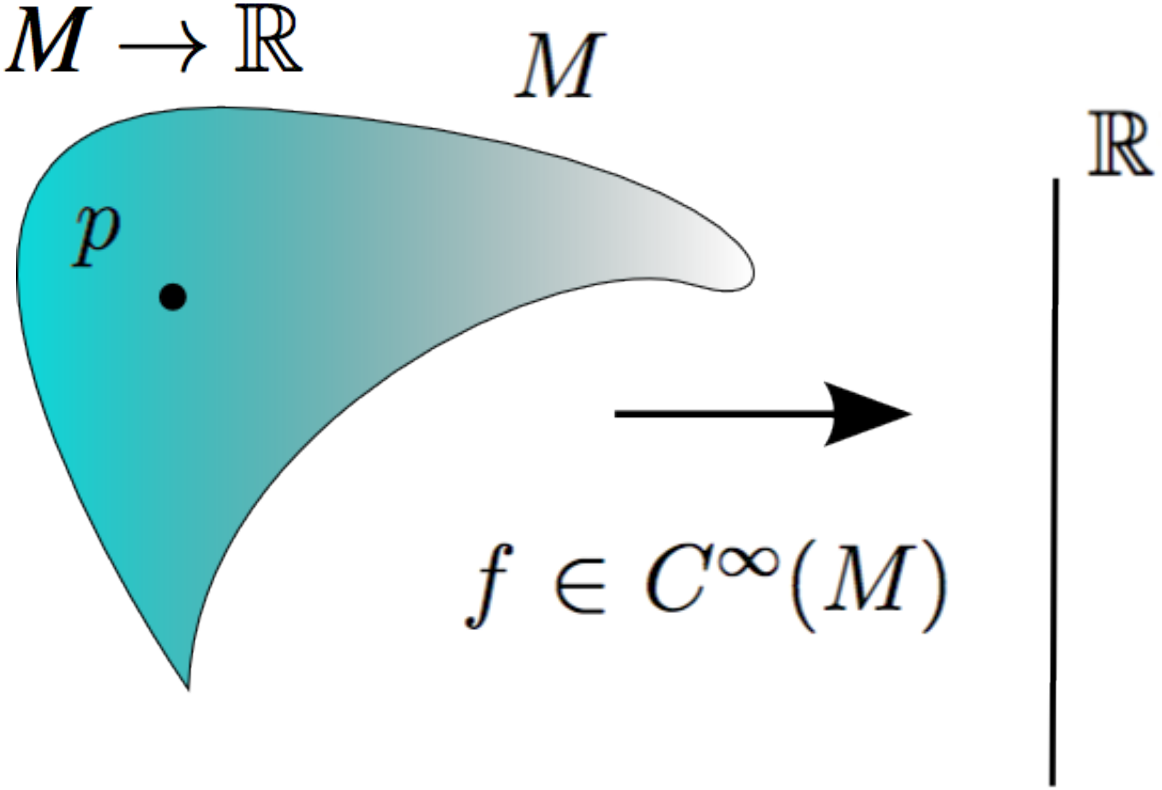


Since f is defined only on M it does not make sense to write:

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}$$

Scalar Functions on Surfaces

$f : M \to \mathbb{R}$



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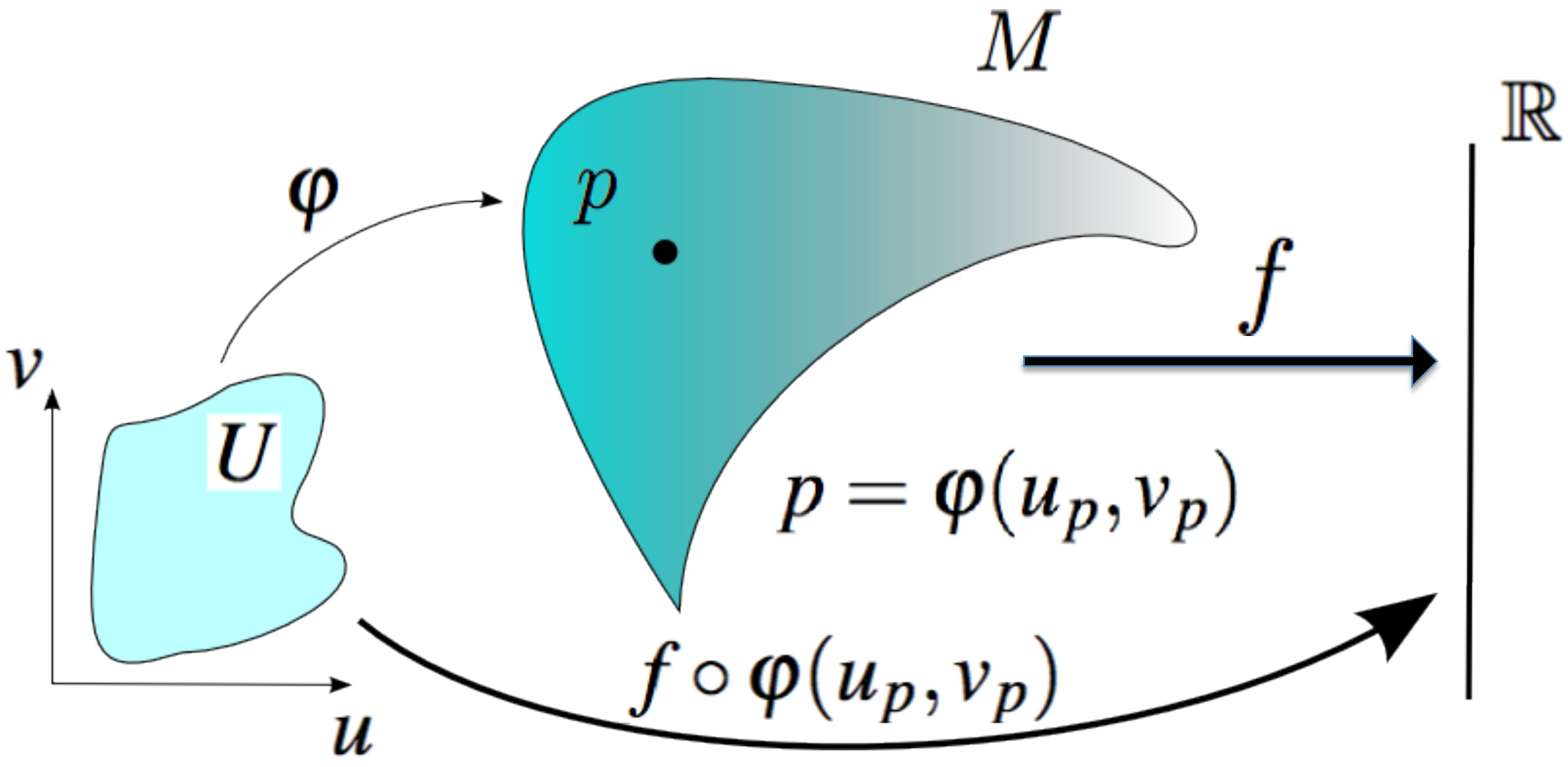
May not be on M

Scalar Functions on Surfaces



$$f \circ \varphi : U \subset \mathbb{R}^2 \to \mathbb{R}$$

Differential Operators on Surfaces

What is the gradient of $f \circ \varphi(u_p, v_p)$?

$$\varphi(u,v) = (x(u,v), y(u,v), z(u,v))$$

$$g_1 = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \quad g_2 = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$$

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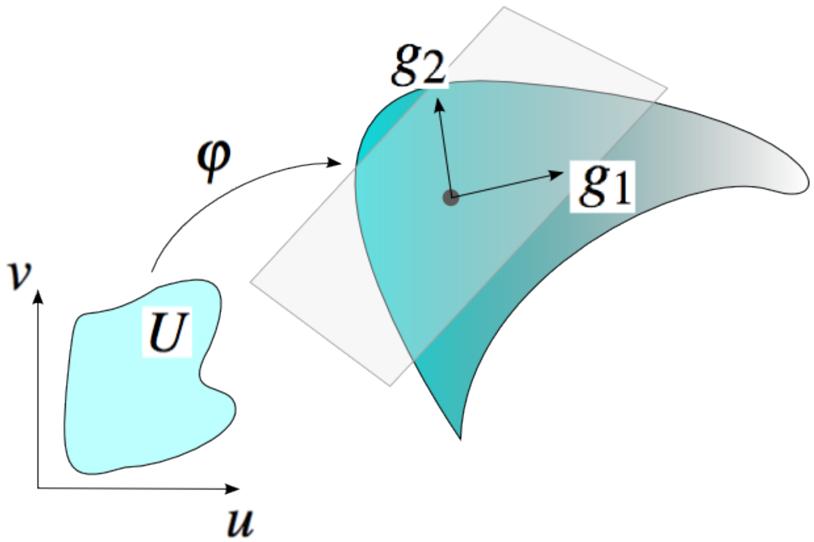

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$$g_{11} =< g_1, g_1 > \quad g_{22} =< g_2, g_2 >$$

$$g_{12} = g_{21} =< g_1, g_2 >$$

$$g = det \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$

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Metric tensor

Differential Operators on Surfaces

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$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Differential Operators on Surfaces

What is the gradient of $f \circ \varphi(u_p, v_p)$?

From the properties of the metric tensor and some algebraic manipulation we get:

$$\nabla f = (g^{11} f_u + g^{12} f_v, g^{22} f_v + g^{21} f_u)$$

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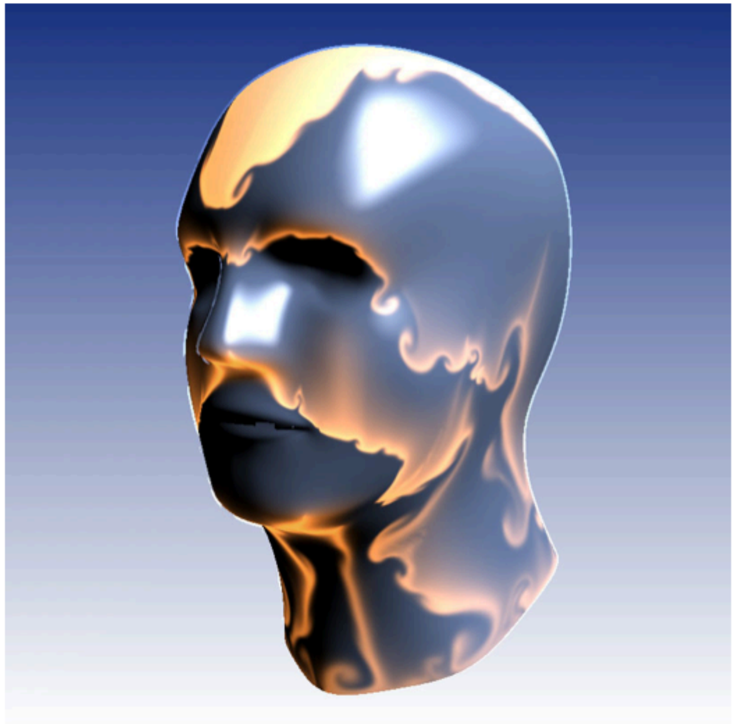
The Laplacian:

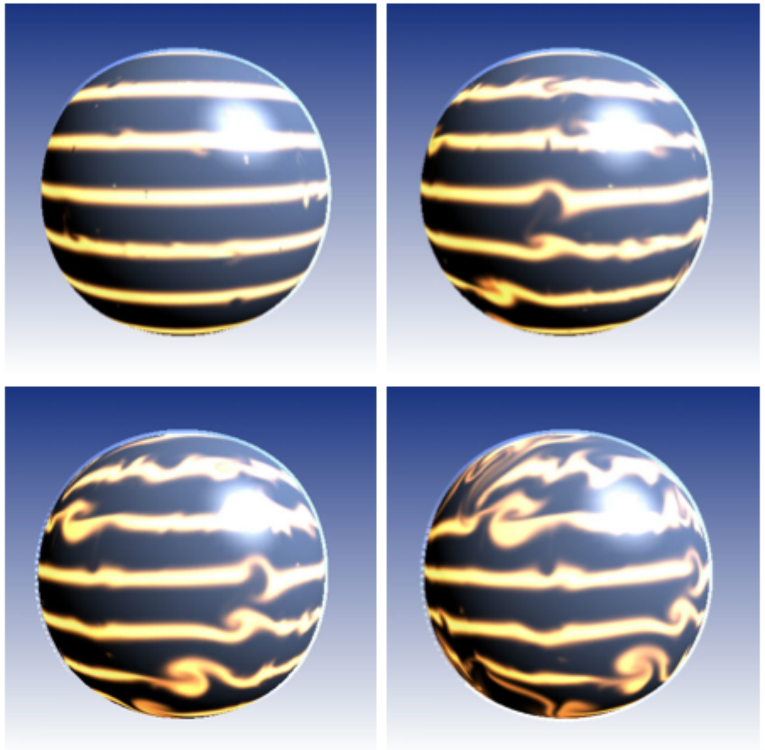
$$\nabla^2 f = \frac{1}{\sqrt{g}} \left(\frac{\partial}{\partial u} \left(\sqrt{g} (g^{11} f_u + g^{12} f_v) \right) + \frac{\partial}{\partial v} \left(\sqrt{g} (g^{21} f_u + g^{22} f_v) \right) \right)$$

Differential Operators on Surfaces

Jos Stam made use of those operators defined on the parametric domain to simulate flows on surfaces.

[Flows on Surfaces of Arbitrary Topology, ACM TOG 2003]





Differential Operators on Surfaces

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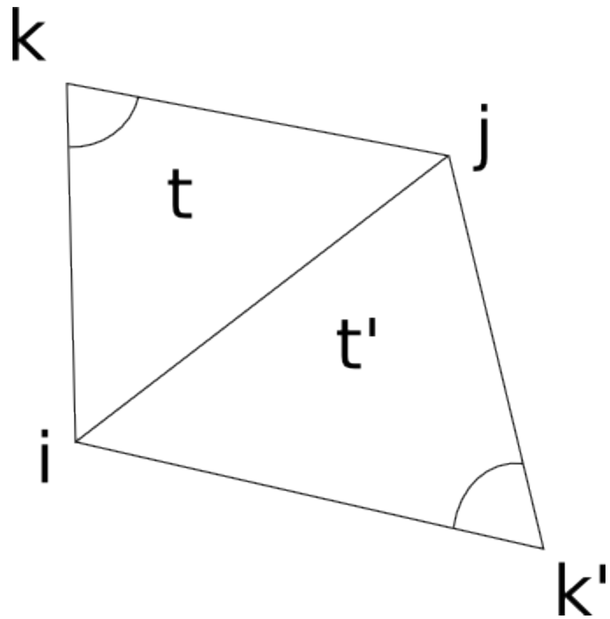

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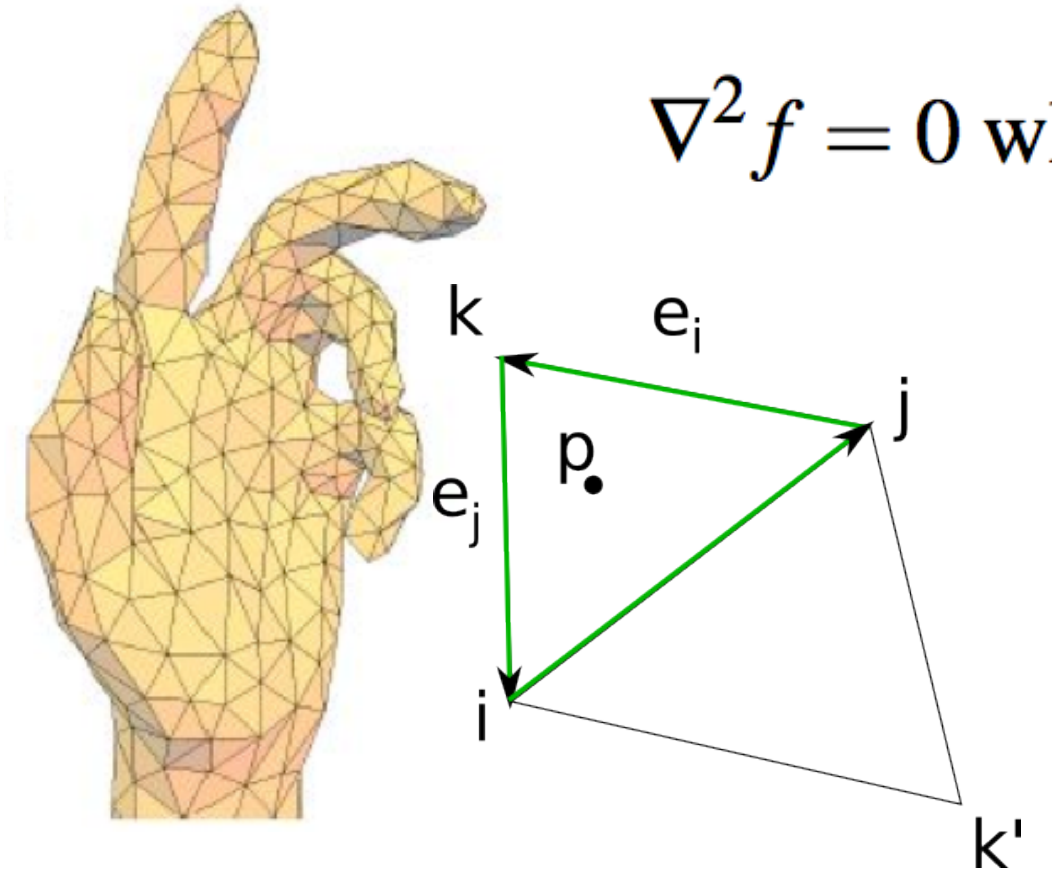

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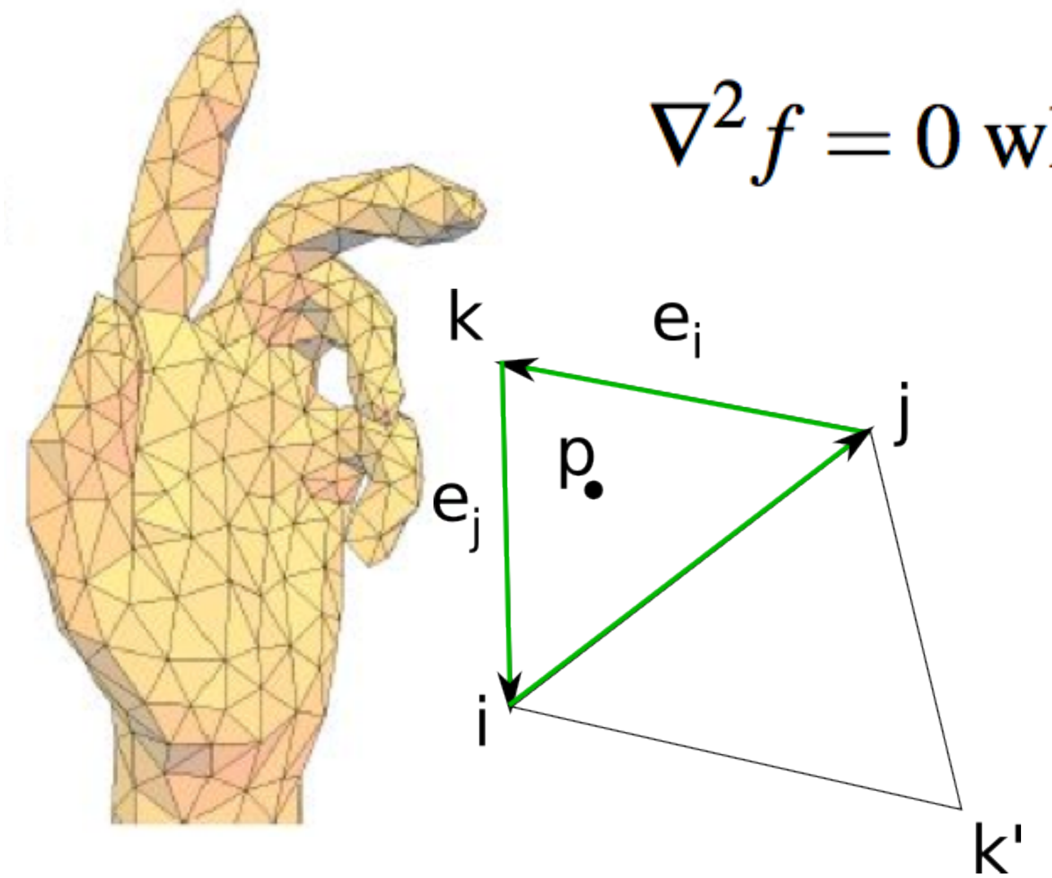

$$\varphi(u_i, u_j) = k + u_i e_j - u_j e_i$$

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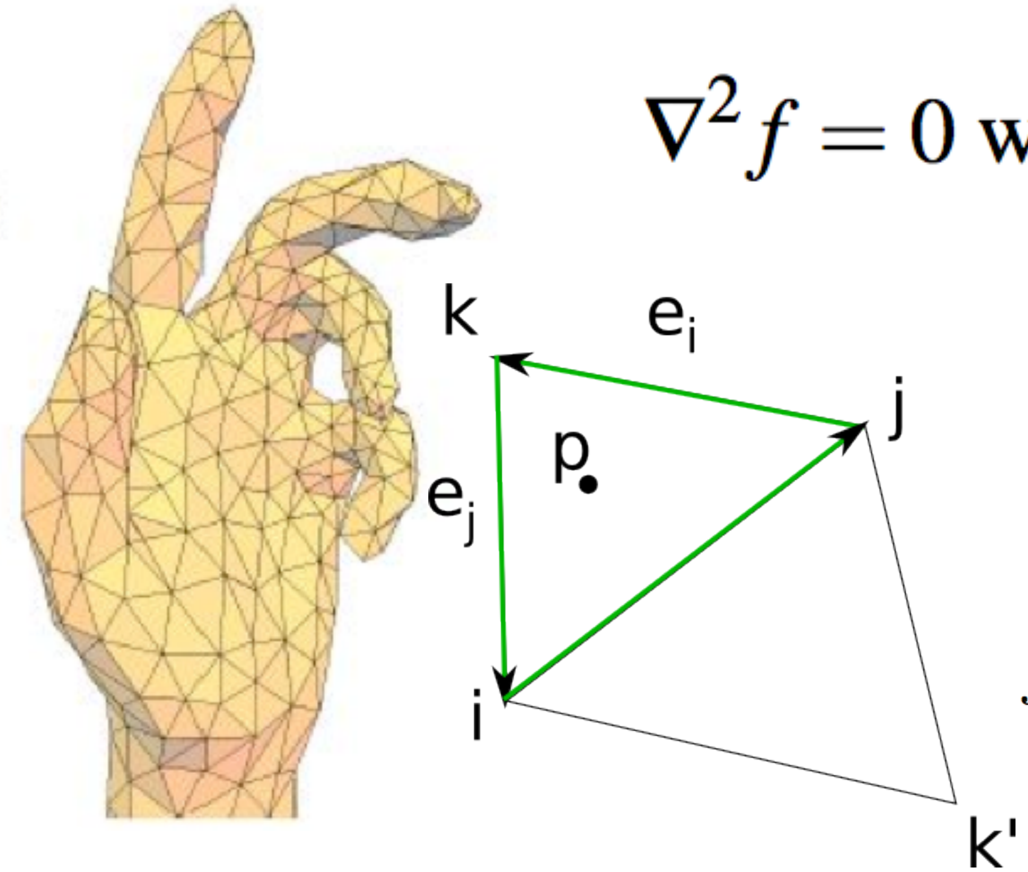


$$\varphi(u_i, u_j) = k + u_i e_j - u_j e_i$$

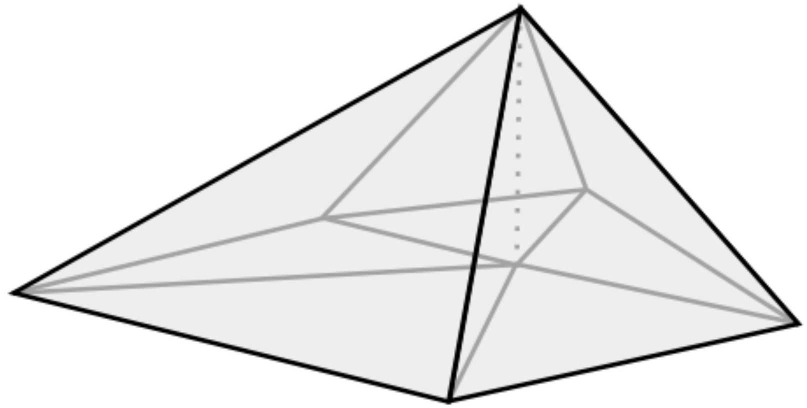
$$\int_t f dS = 2A_t \int_{u_i=0}^{1} \int_{u_j=0}^{1-u_i} f du_j du_i$$

Differential Operators on Surfaces

Using Finite Element Formulation

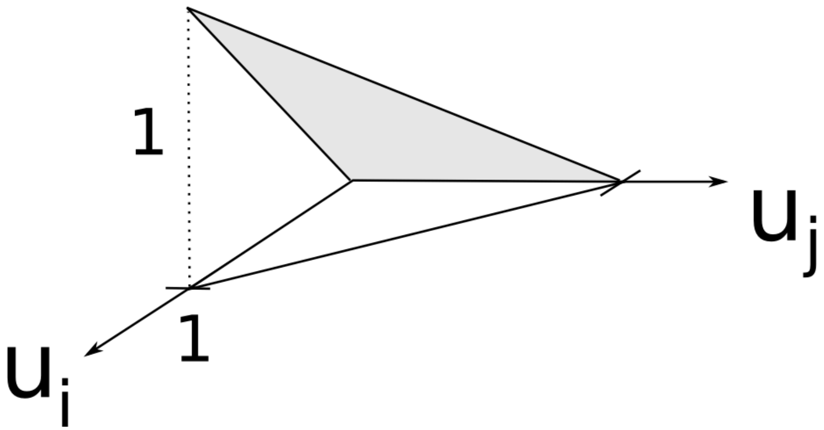
$$\nabla^2 f = 0 \Rightarrow Lf = 0$$

$$l_{ij} = \int_{t \cup t'} < \nabla\phi_i , \nabla\phi_j > dS$$

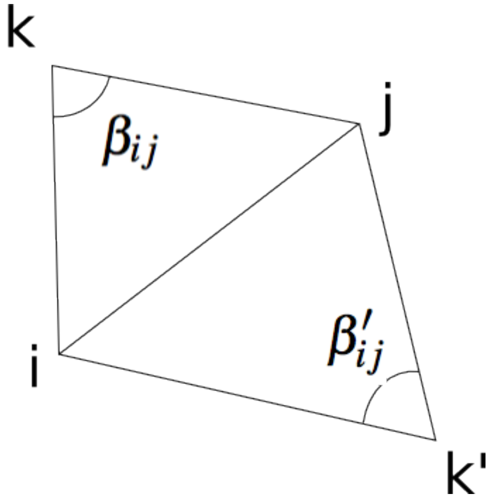


Differential Operators on Surfaces

In the canonical domain



$$< \nabla\phi_i , \nabla\phi_j >= \frac{< e_i , e_j >}{4A_t^2}$$

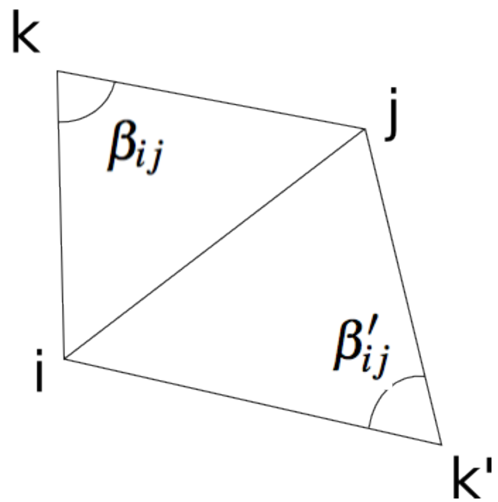


$$2A_t \int_{u_i=0}^{1} \int_{u_j=0}^{1-u_i} < \nabla\phi_i , \nabla\phi_j > du_j du_i =$$

$$\frac{< e_i , e_j >}{4A_t} = \frac{\|e_i\| \|e_j\| \cos(\beta_{ij})}{2\|e_i\| \|e_j\| \sin(\beta_{ij})} = \frac{\cot(\beta_{ij})}{2}$$

Differential Operators on Surfaces

Considering the two triangles sharing the edge ij



$$l_{ij} = \frac{\cot(\beta_{ij}) + \cot(\beta'_{ij})}{2}$$

$$l_{ii} = -\sum_{t \in st(i)} l_{ij}$$

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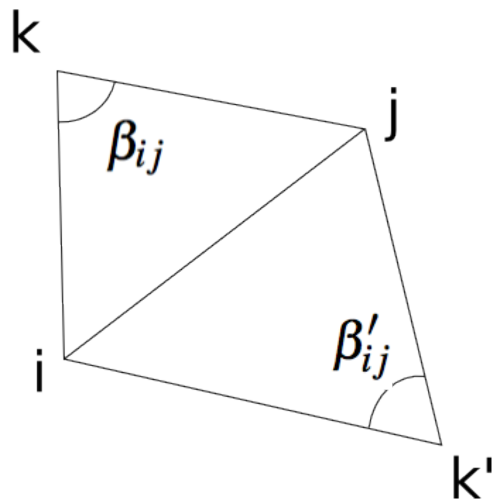

$$l_{ij} = \frac{\cot(\beta_{ij}) + \cot(\beta'_{ij})}{2}$$

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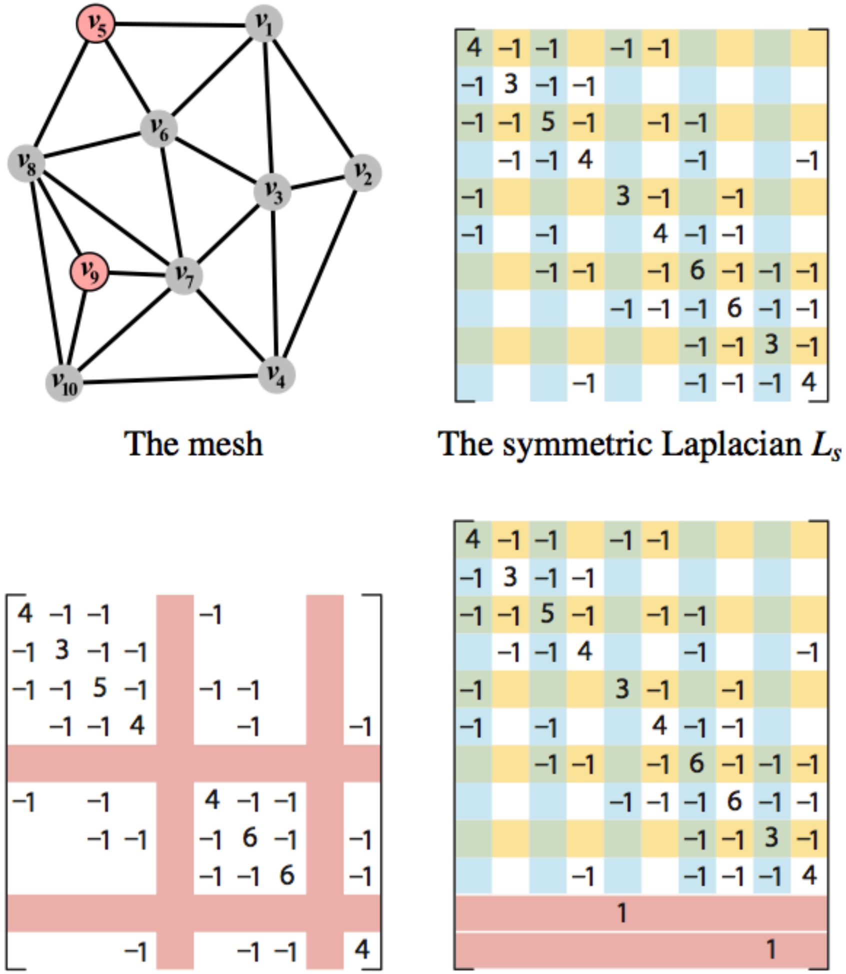
Allows to discretize the Laplace equation directly on the surface.

Boundary Conditions

Least Square-based



The mesh

The symmetric Laplacian L_s

Invertible Laplacian

2-anchor $\tilde{L}$

O. Sorkine, Eurographics 2005.

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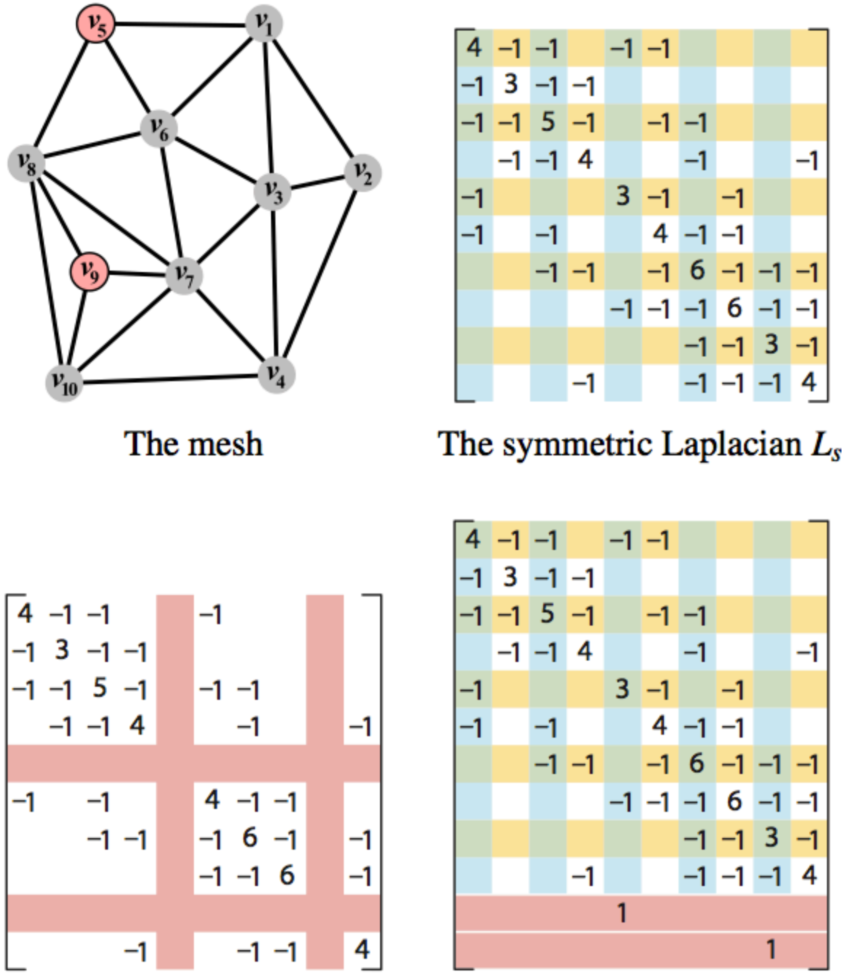

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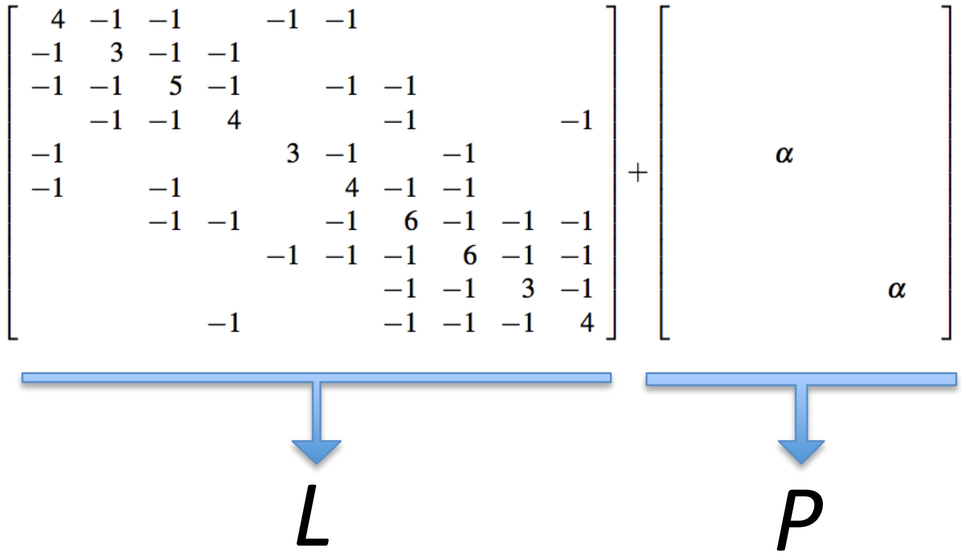
Penalty Method

$$(L+P)=Pb$$



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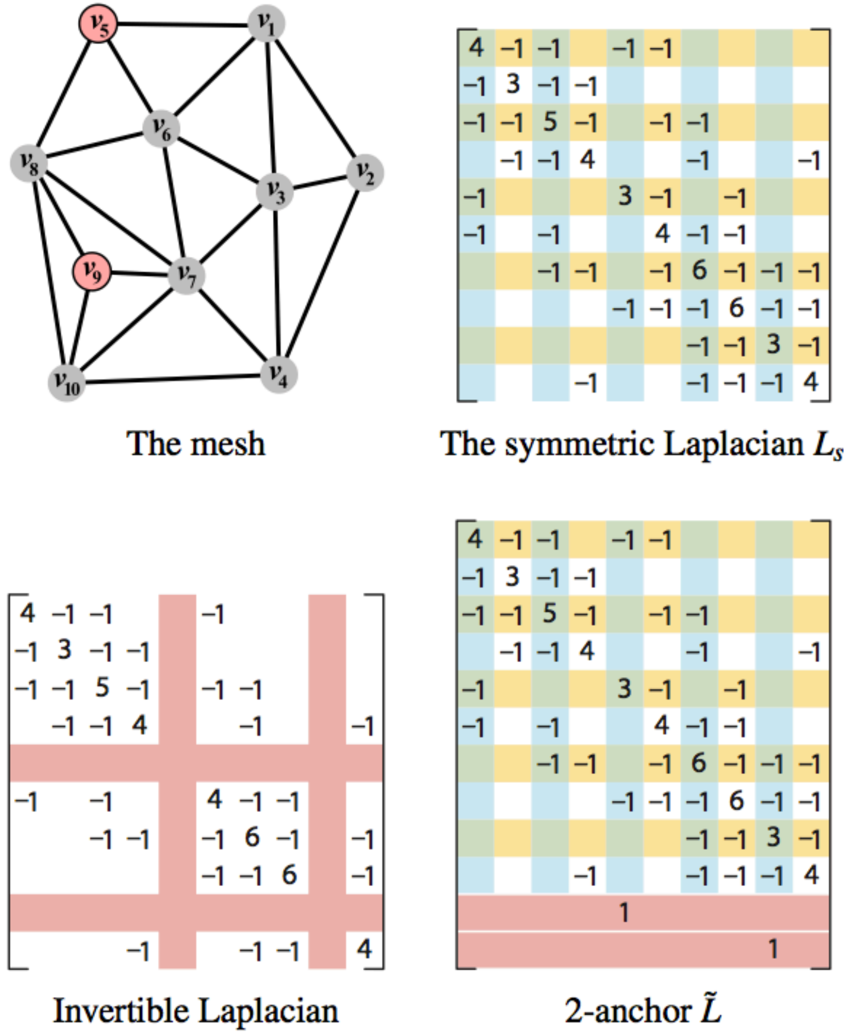

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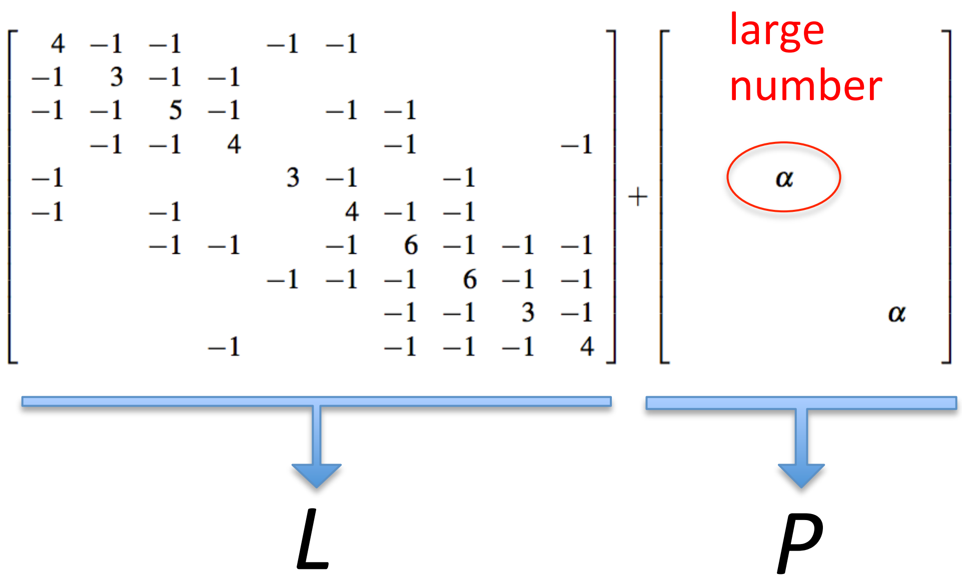

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Differential Operators on Surfaces

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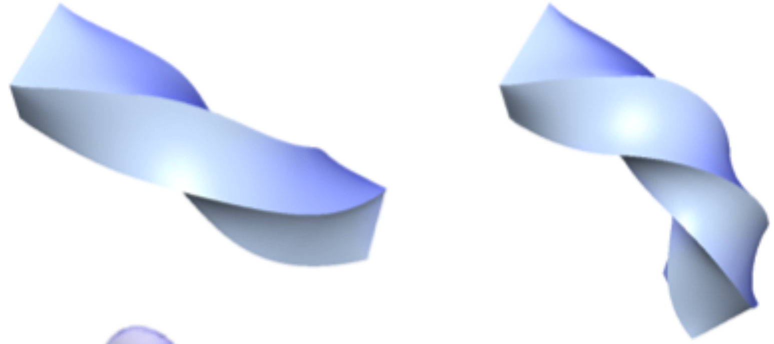
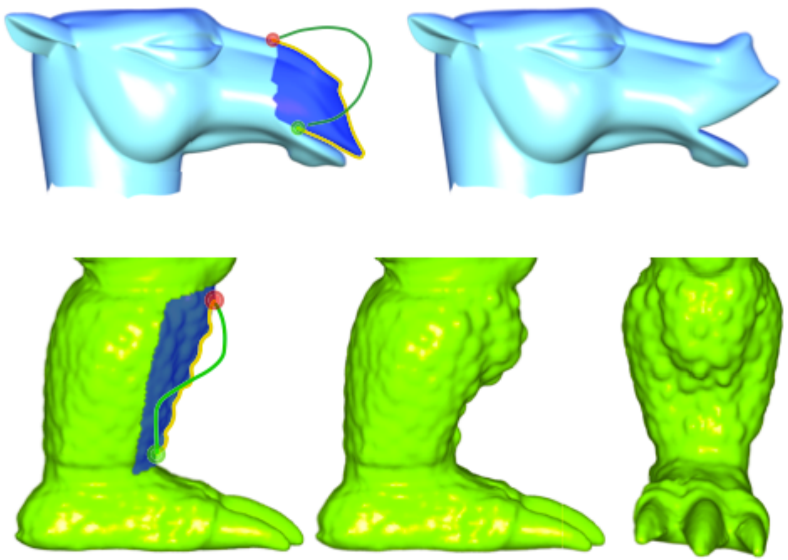


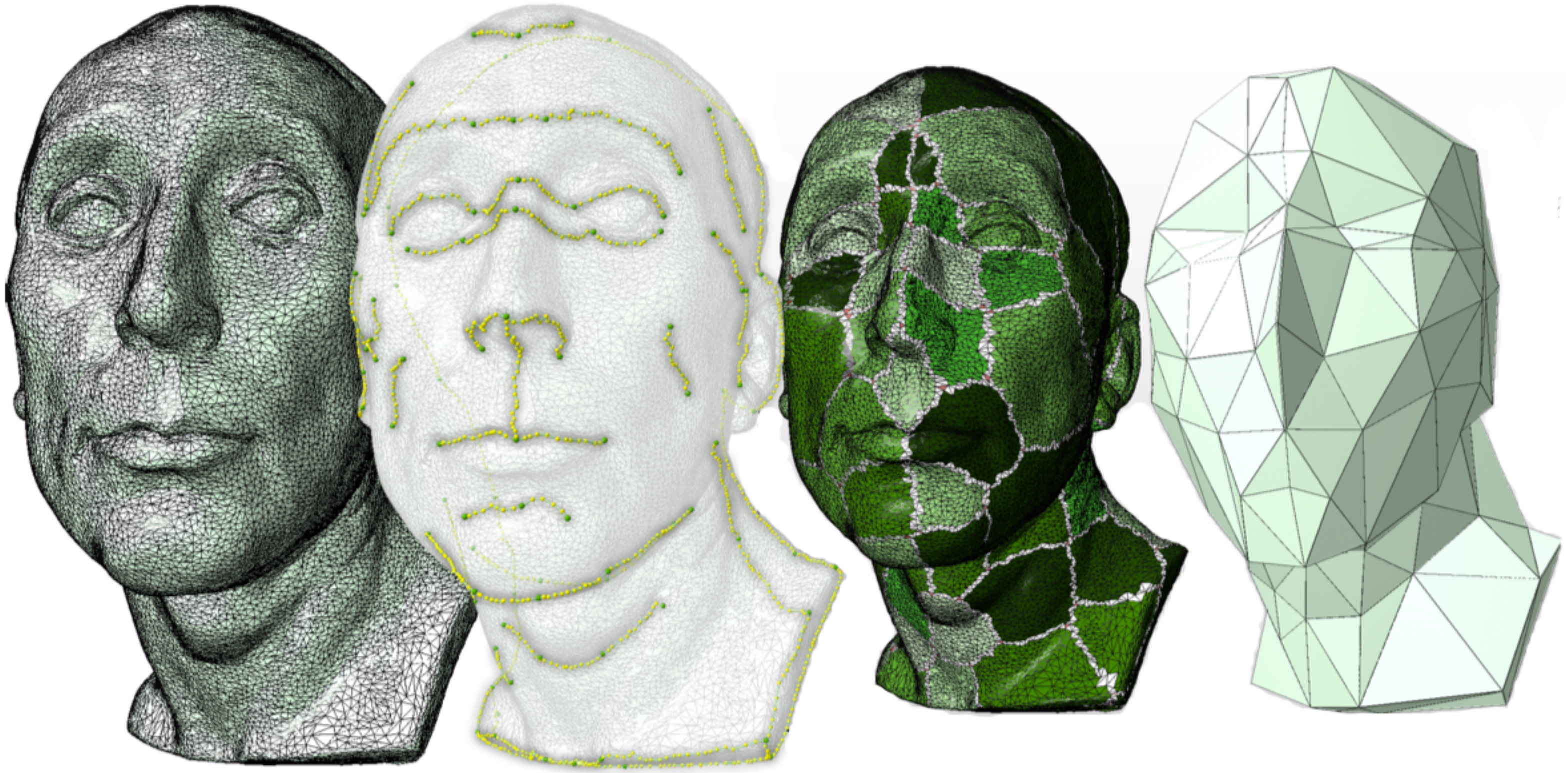
Mesh Editing and Deformation



[O. Sorkine, Eurographics 2005]

Differential Operators on Surfaces

Base Mesh Construction



[Daniels et al., SMI 2011]

Differential Operators on Surfaces

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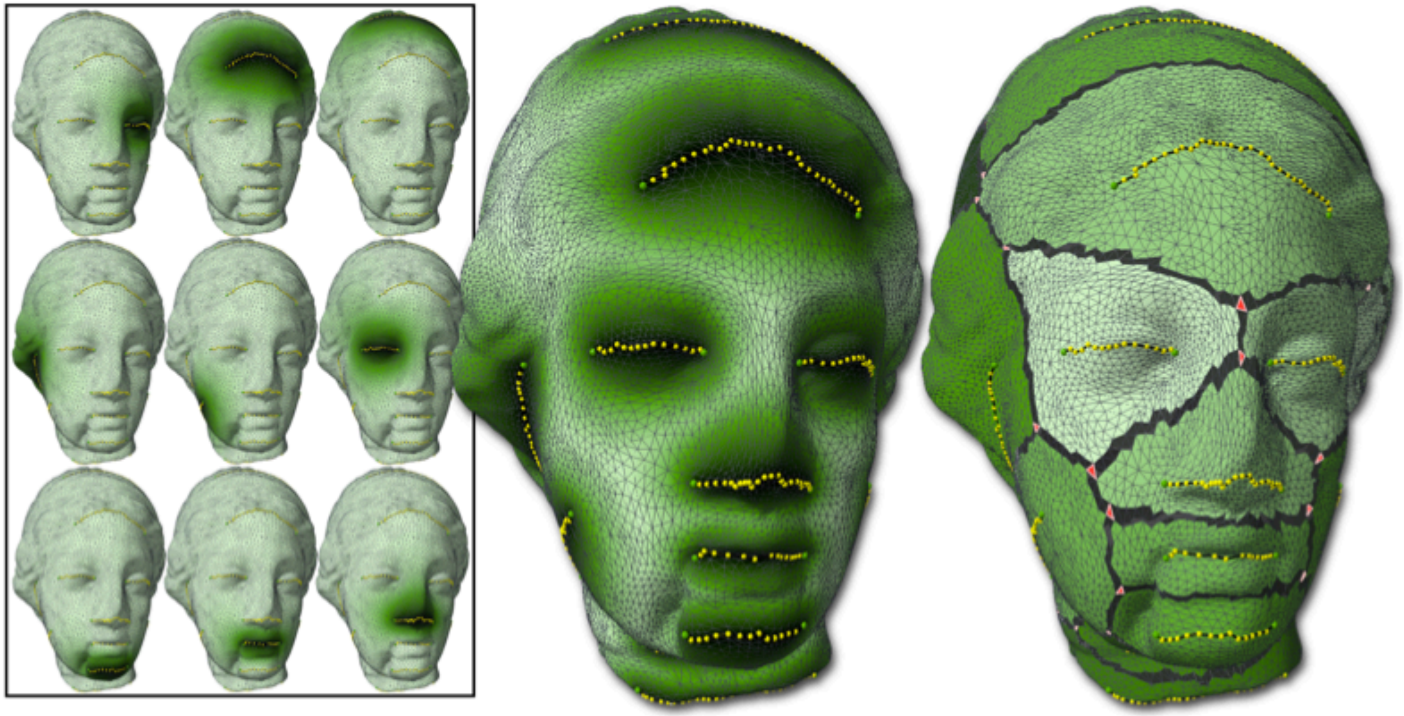

(a) (b) (c)

Differential Operators on Surfaces

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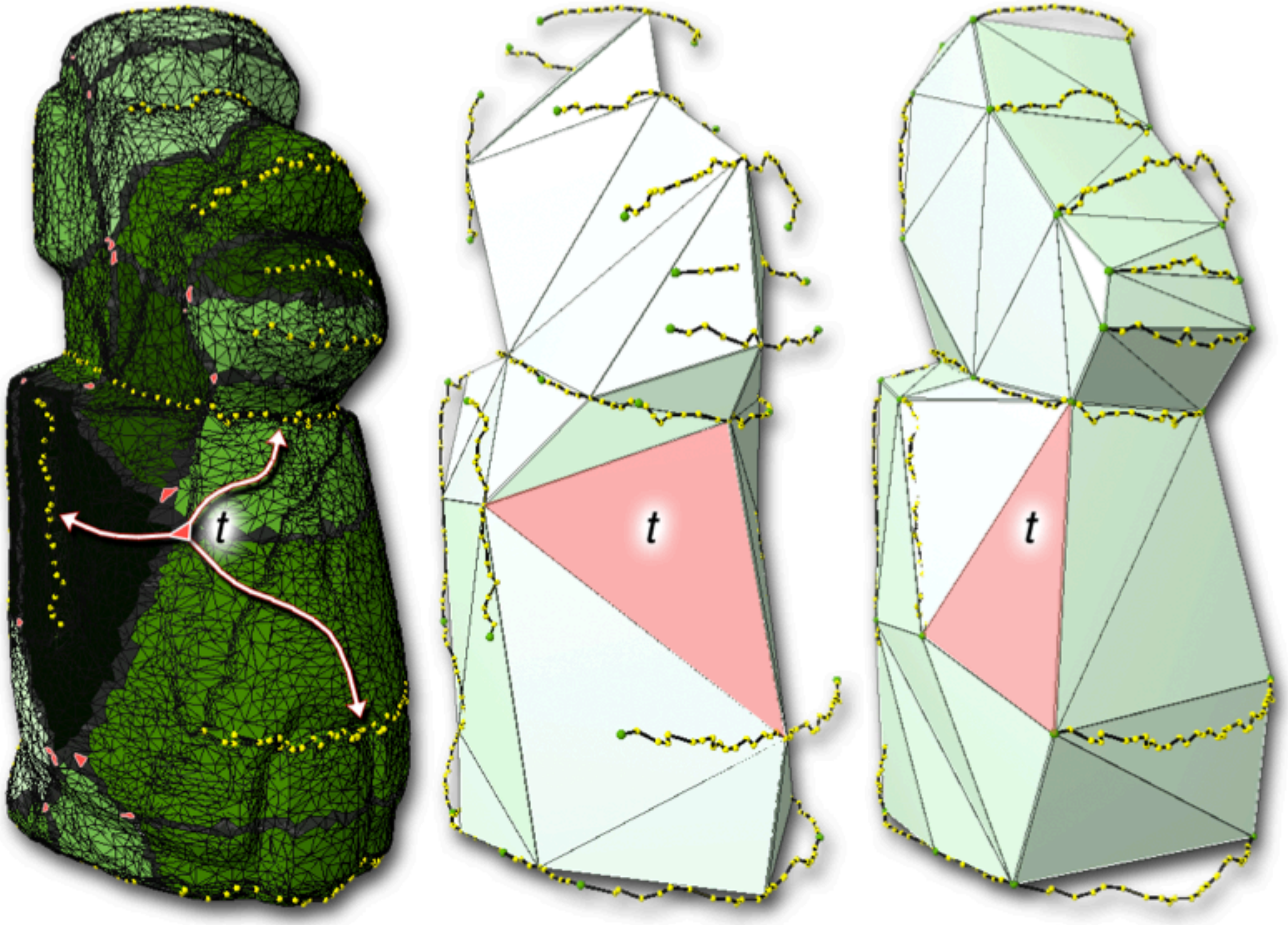

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$$L' = \Phi L$$

Φ_{ij} is the value in v_j of a kernel function (r-local) defined in v_i.

Differential Operators on Surfaces

A consistent discretization schemes have been proposed by Belkin:

[Belkin et al., SCG'08]

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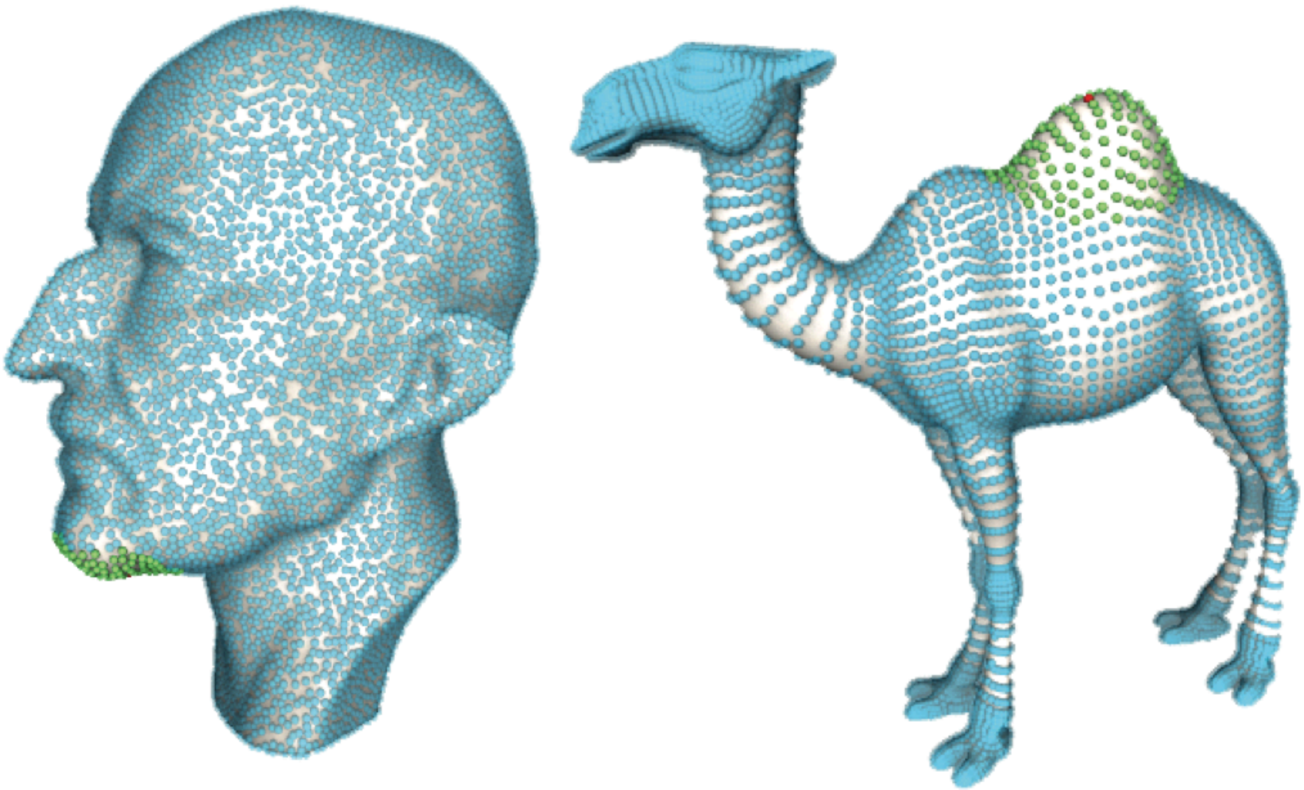
$$\mathrm{L}_K^h f(w) = \frac{1}{4\pi h^2} \sum_{t \in K} \frac{\mathrm{Area}(t)}{\#t} \sum_{p \in V(t)} e^{-\frac{\|p-w\|^2}{4h}} (f(p) - f(w))$$

Differential Operators on Surfaces

Belkin has also extended the operator for point set surfaces.



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$$\mathrm{L}_P^t f(p) = \frac{1}{(4\pi t)^{k/2} t} \sum_{\sigma \in K_{\frac{\delta}{2}}} \frac{\mathrm{A}(\sigma)}{k+1} \sum_{q \in V(\sigma)} e^{-\frac{\|p-\Phi(q)\|^2}{4t}} \left(f(p) - f(\Phi(q))\right)$$

A Delaunay triangulation is built on the tangent plane of each point of the mesh.

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Projection from
the tangent plane
back to the surface

A Delaunay triangulation is built on the tangent plane of each point of the mesh.

Differential Operators on Surfaces

Petronetto et al. have employed Smooth Particle Hydrodynamics (SPH) as discretization mechanism.

$$\langle \Delta_{\mathcal{M}} f_i \rangle = - \sum_{j \in N_i} 2 f_{ij} \frac{\hat{\mathbf{x}}_{ij}}{\left\| \hat{\mathbf{x}}_{ij} \right\|^2} \cdot \nabla W_h \left(\left\| \hat{\mathbf{x}}_{ij} \right\| \right) V_j$$

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$$\int_\Omega W_h \left(\|\mathbf{x} - \mathbf{x}'\| \right) d\mathbf{x}' = 1$$

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Normal extension

W_h is a kernel function satisfying

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Differential Operators on Surfaces

$$\int_{\Omega} W_h\left(\|\mathbf{x}-\mathbf{x}'\|\right) d\mathbf{x}' = 1$$

⇓

$$A\mathrm{v} = b$$

where $a_{ij} = W_h(\|\mathbf{x}_{ij}\|)$, $b_i = 1$, and $v_i = V_i$.

Differential Operators on Surfaces

In order to enforce a uniform distribution of area elements a regularization term is incorporated and the following minimization problem is solved:

$$\min \quad F^{\rho}(\mathrm{v}) := ||A\mathrm{v} - \mathrm{b}||^2 + \rho||\mathrm{v}||^2$$

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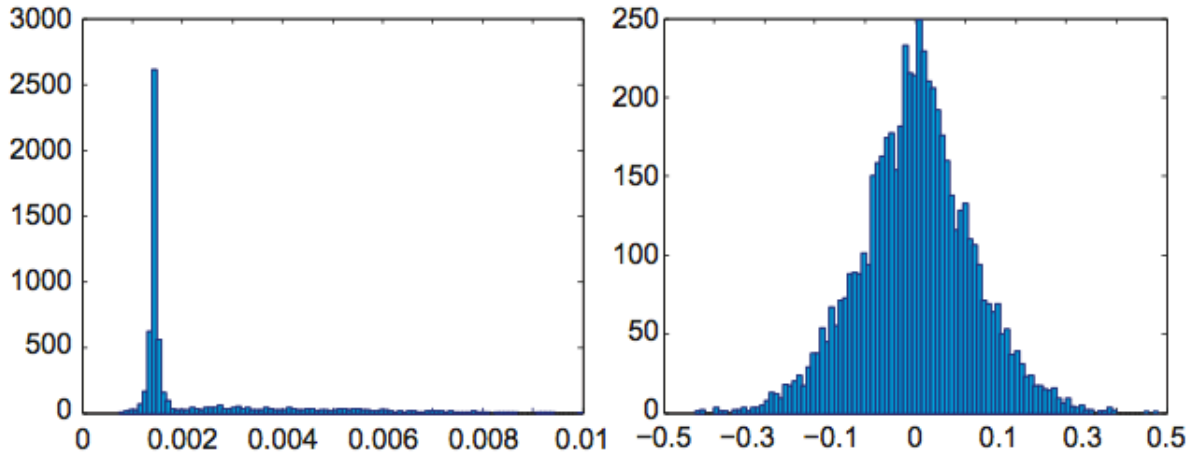
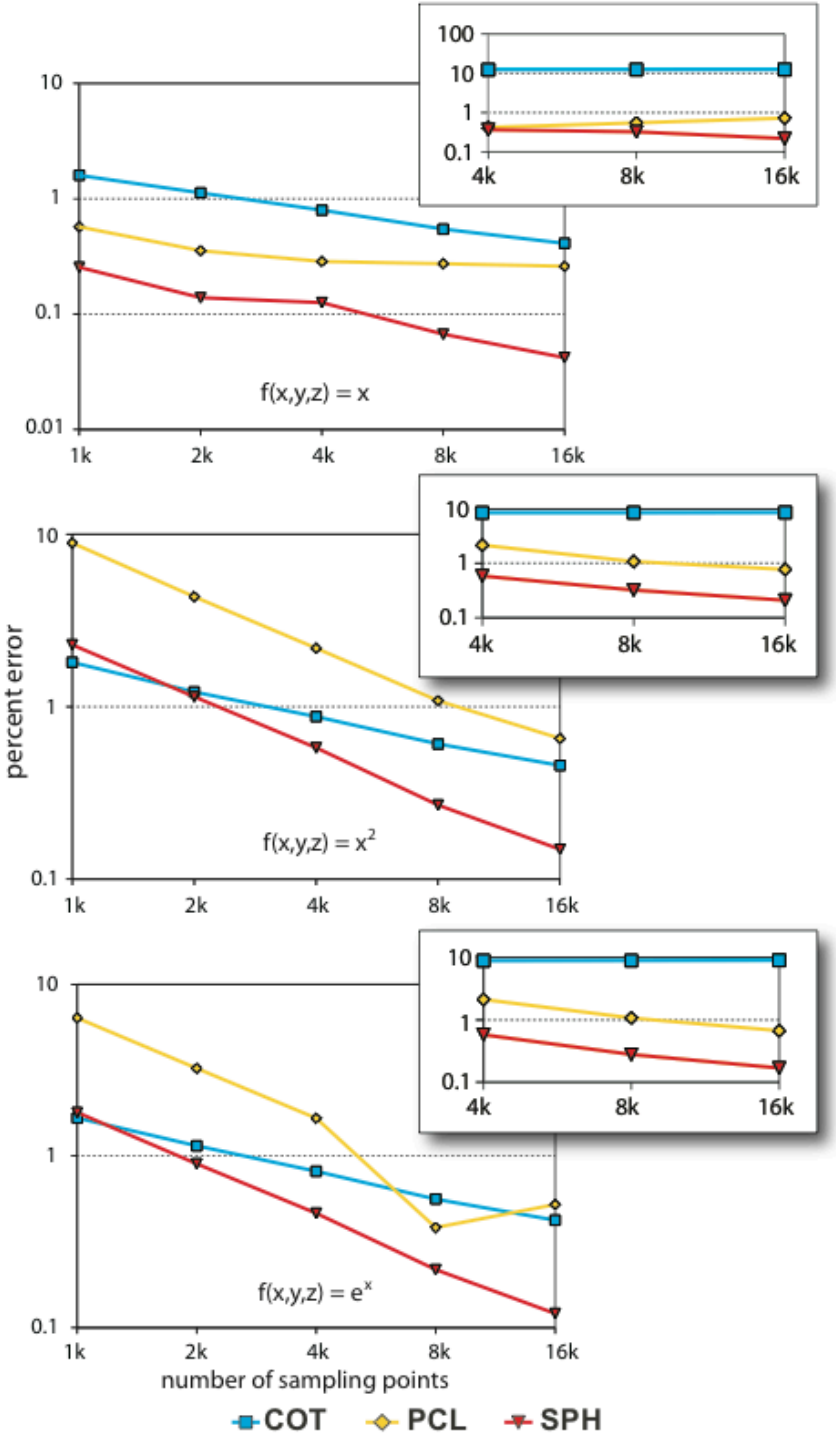
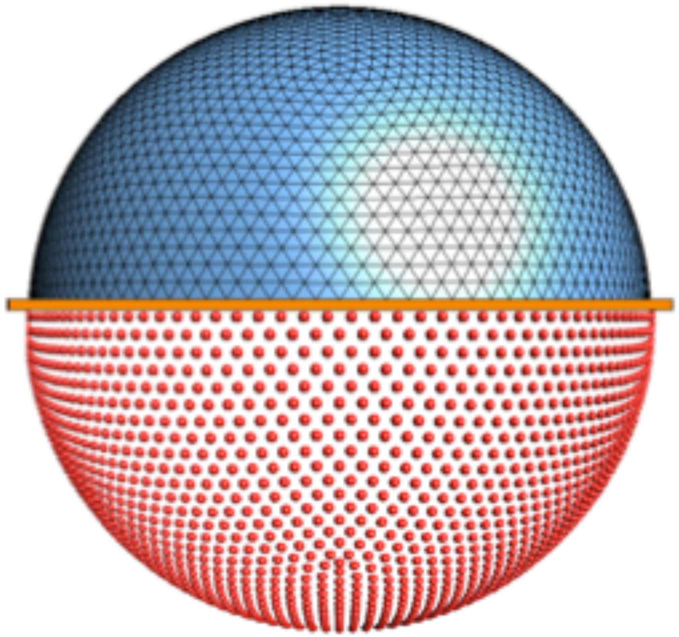
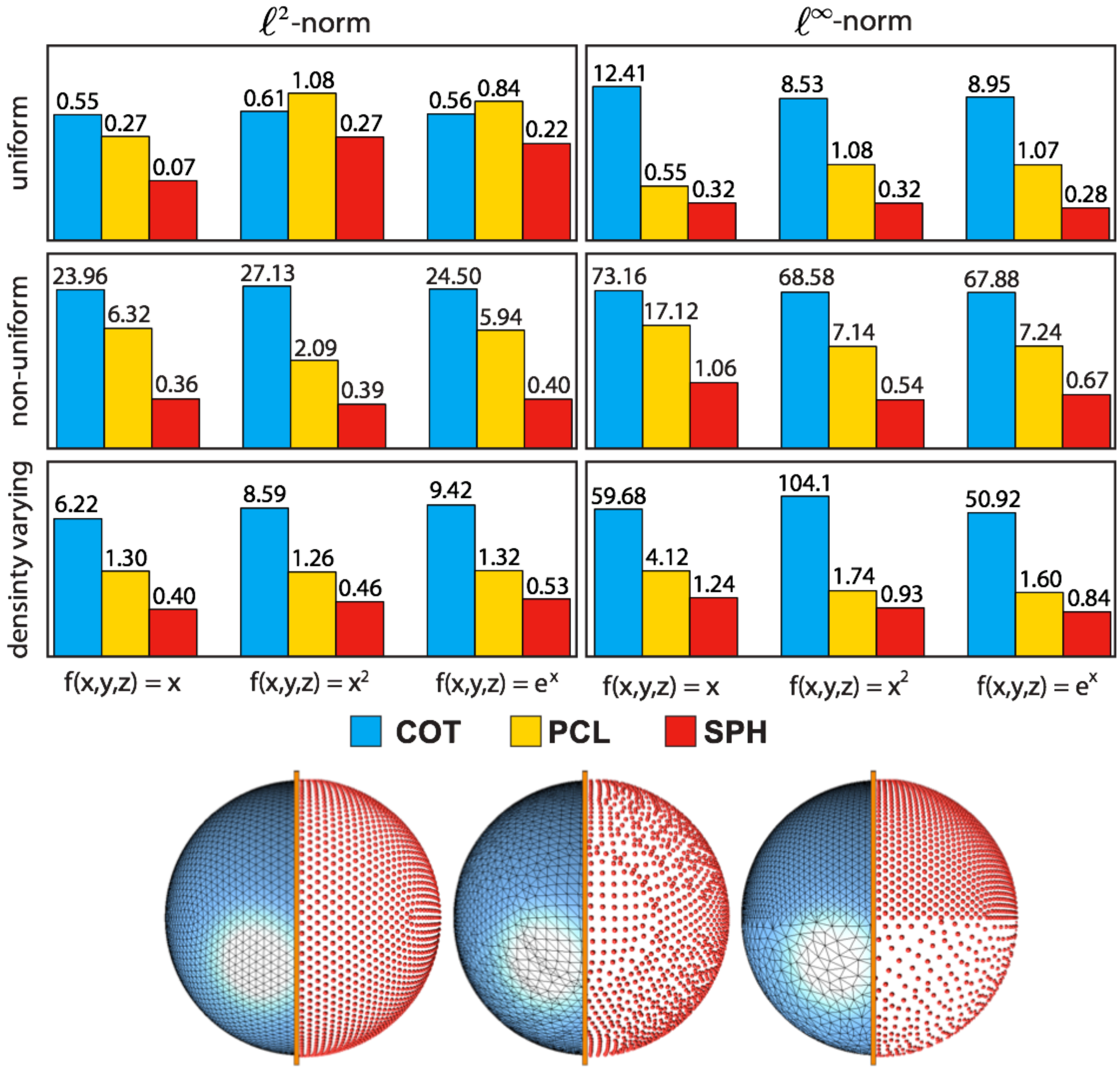


Figure 1: *Histogram of area elements with (left) and without (right) the regularization term.*

Consistency analysis






ℓ²-norm
ℓ∞-norm
uniform
non-uniform
densinty varying
0.55 0.27 0.07
0.61 1.08 0.27
0.56 0.84 0.22
12.41 0.55 0.32
8.53 1.08 0.32
8.95 1.07 0.28
23.96 6.32 0.36
27.13 2.09 0.39
24.50 5.94 0.40
73.16 17.12 1.06
68.58 7.14 0.54
67.88 7.24 0.67
6.22 1.30 0.40
8.59 1.26 0.46
9.42 1.32 0.53
59.68 4.12 1.24
104.1 1.74 0.93
50.92 1.60 0.84
f(x,y,z) = x
f(x,y,z) = x²
f(x,y,z) = eˣ
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f(x,y,z) = eˣ
COT
PCL
SPH

Convergence Analysis

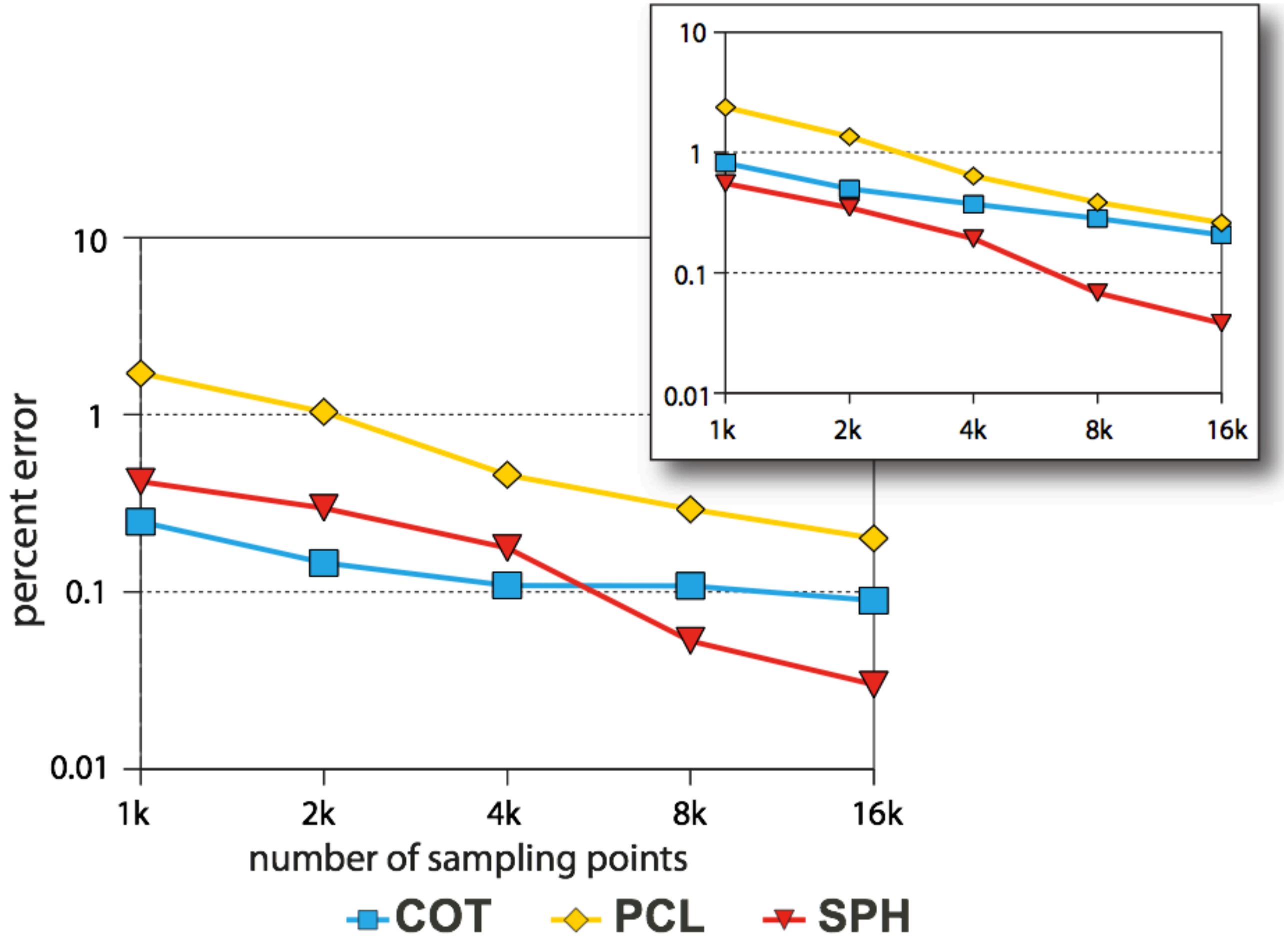
$$-\Delta_{\mathcal{M}} u = f$$

$$f(x,y,z) = 2(z-1) + 6y^2 - (1-2x-x^2)e^x$$

Convergence Analysis

$$-\Delta_{\mathcal{M}} u = f$$

$$f(x,y,z) = 2(z-1) + 6y^2 - (1-2x-x^2)e^x$$



There are at least other two approaches we have not discussed:

- Discrete exterior calculus (Desbrun)
- 3D constrained to surface approach (Kazhdan)

Those two methods have been discusses during the advanced seminars.

This Thursday:

Manifold Harmonics !!!