with \( \mu > 0 \). Thus, we have
\[
\sum_{i \in I} \frac{\mu_i}{\mu} O x_i = \sum_{j \in J} -\frac{\mu_j}{\mu} O x_j,
\]
with
\[
\sum_{i \in I} \frac{\mu_i}{\mu} = \sum_{j \in J} -\frac{\mu_j}{\mu} = 1,
\]
proving that \( \sum_{i \in I} (\mu_i/\mu)x_i \in \mathcal{C}(X_1) \) and \( \sum_{j \in J} -(\mu_j/\mu)x_j \in \mathcal{C}(X_2) \) are identical, and thus that \( \mathcal{C}(X_1) \cap \mathcal{C}(X_2) \neq \emptyset \).

Finally, we prove a version of Helly’s theorem.

**Theorem 3.3.2** Given any affine space \( E \) of dimension \( m \), for every family \( \{K_1, \ldots, K_n\} \) of \( n \) convex subsets of \( E \), if \( n \geq m+2 \) and the intersection \( \bigcap_{i \in I} K_i \) of any \( m+1 \) of the \( K_i \)’s is nonempty (where \( I \subseteq \{1, \ldots, n\} \), \( |I| = m+1 \)), then \( \bigcap_{i=1}^{n} K_i \) is nonempty.

**Proof.** The proof is by induction on \( n \geq m+1 \) and uses Radon’s theorem in the induction step. For \( n = m+1 \), the assumption of the theorem is that the intersection of any family of \( m+1 \) of the \( K_i \)’s is nonempty, and the theorem holds trivially. Next, let \( L = \{1, 2, \ldots, n+1\} \), where \( n+1 \geq m+2 \). By the induction hypothesis, \( C_i = \bigcap_{j \in (L-i)} K_j \) is nonempty for every \( i \in L \).

We claim that \( C_i \cap C_j \neq \emptyset \) for some \( i \neq j \). If so, as \( C_i \cap C_j = \bigcap_{k=1}^{n+1} K_k \), we are done. So, let us assume that the \( C_i \)’s are pairwise disjoint. Then, we can pick a set \( X = \{a_1, \ldots, a_{n+1}\} \) such that \( a_i \in C_i \), for every \( i \in L \). By Radon’s Theorem, there are two nonempty disjoint sets \( X_1, X_2 \subseteq X \) such that \( X = X_1 \cup X_2 \) and \( \mathcal{C}(X_1) \cap \mathcal{C}(X_2) \neq \emptyset \). However, \( X_1 \subseteq K_j \) for every \( j \) with \( a_j \notin X_1 \). This is because \( a_j \notin K_j \) for every \( j \), and so, we get
\[
X_1 \subseteq \bigcap_{a_j \notin X_1} K_j.
\]

Symetrically, we also have
\[
X_2 \subseteq \bigcap_{a_j \notin X_2} K_j.
\]

Since the \( K_j \)’s are convex and
\[
\left( \bigcap_{a_j \notin X_1} K_j \right) \cap \left( \bigcap_{a_j \notin X_2} K_j \right) = \bigcap_{i=1}^{n+1} K_i,
\]
it follows that \( \mathcal{C}(X_1) \cap \mathcal{C}(X_2) \subseteq \bigcap_{i=1}^{n+1} K_i \), so that \( \bigcap_{i=1}^{n+1} K_i \) is nonempty, contradicting the fact that \( C_i \cap C_j = \emptyset \) for all \( i \neq j \).