with $\mu > 0$. Thus, we have

$$\sum_{i\in I} \frac{\mu_i}{\mu}\, \mathbf{Ox_i} = \sum_{j\in J} -\frac{\mu_j}{\mu}\, \mathbf{Ox_j},$$

with

$$\sum_{i\in I} \frac{\mu_i}{\mu} = \sum_{j\in J} -\frac{\mu_j}{\mu} = 1,$$

proving that $\sum_{i\in I}(\mu_i/\mu)x_i \in \mathcal{C}(X_1)$ and $\sum_{j\in J} -(\mu_j/\mu)x_j \in \mathcal{C}(X_2)$ are identical, and thus that $\mathcal{C}(X_1) \cap \mathcal{C}(X_2) \neq \emptyset$. $\square$

Finally, we prove a version of *Helly's theorem*.

Theorem 3.3.2 *Given any affine space E of dimension m, for every family $\{K_1, \ldots, K_n\}$ of n convex subsets of E, if $n \geq m+2$ and the intersection $\bigcap_{i\in I} K_i$ of any $m+1$ of the K_i is nonempty (where $I \subseteq \{1, \ldots, n\}$, $|I| = m+1$), then $\bigcap_{i=1}^{n} K_i$ is nonempty.*

Proof. The proof is by induction on $n \geq m+1$ and uses Radon's theorem in the induction step. For $n = m+1$, the assumption of the theorem is that the intersection of any family of $m+1$ of the K_i's is nonempty, and the theorem holds trivially. Next, let $L = \{1, 2, \ldots, n+1\}$, where $n+1 \geq m+2$. By the induction hypothesis, $C_i = \bigcap_{j\in(L-\{i\})} K_j$ is nonempty for every $i \in L$.

We claim that $C_i \cap C_j \neq \emptyset$ for some $i \neq j$. If so, as $C_i \cap C_j = \bigcap_{k=1}^{n+1} K_k$, we are done. So, let us assume that the C_i's are pairwise disjoint. Then, we can pick a set $X = \{a_1, \ldots, a_{n+1}\}$ such that $a_i \in C_i$, for every $i \in L$. By Radon's Theorem, there are two nonempty disjoint sets $X_1, X_2 \subseteq X$ such that $X = X_1 \cup X_2$ and $\mathcal{C}(X_1) \cap \mathcal{C}(X_2) \neq \emptyset$. However, $X_1 \subseteq K_j$ for every j with $a_j \notin X_1$. This is because $a_j \notin K_j$ for every j, and so, we get

$$X_1 \subseteq \bigcap_{a_j \notin X_1} K_j.$$

Symetrically, we also have

$$X_2 \subseteq \bigcap_{a_j \notin X_2} K_j.$$

Since the K_j's are convex and

$$\left(\bigcap_{a_j \notin X_1} K_j\right) \cap \left(\bigcap_{a_j \notin X_2} K_j\right) = \bigcap_{i=1}^{n+1} K_i,$$

it follows that $\mathcal{C}(X_1) \cap \mathcal{C}(X_2) \subseteq \bigcap_{i=1}^{n+1} K_i$, so that $\bigcap_{i=1}^{n+1} K_i$ is nonempty, contradicting the fact that $C_i \cap C_j = \emptyset$ for all $i \neq j$. $\square$