We now consider how the rectifying plane varies. This will uncover the torsion. According to Kreyszig [104], the term torsion was first used by de la Vallée in 1825. We leave as an easy exercise to show that the osculating plane rotates around the tangent line for points $t + \delta$ close enough to t.

15.7 Torsion (3D Curves)

Recall that the rectifying plane is the plane orthogonal to the principal normal at t passing through $f(t)$. Thus, its equation is

$$\mathbf{n} \cdot \mathbf{MP} = 0,$$

where $\mathbf{n}$ is the principal normal vector. However, things get a bit messy when we take the derivative of $\mathbf{n}$, because of the denominator, and it is easier to use the vector

$$\mathbf{N} = -(f'(t) \cdot f''(t))\, f'(t) + \|f'(t)\|^2\, f''(t),$$

which is collinear to $\mathbf{n}$, but not necessarily a unit vector. Still, we have $\mathbf{N} \cdot \mathbf{M}' = 0$, which is the important fact. Since the equation of the rectifying plane is $\mathbf{N} \cdot \mathbf{MP} = 0$ or

$$\mathbf{N} \cdot \mathbf{P} = \mathbf{N} \cdot \mathbf{M},$$

by familiar reasoning, the equation of a rectifying plane for $\delta \neq 0$ small enough is

$$\mathbf{N}(t+\delta) \cdot \mathbf{P} = \mathbf{N}(t+\delta) \cdot \mathbf{M}(t+\delta),$$

and we can easily prove that the intersection of these two planes is given by the equations

$$\begin{aligned} \mathbf{N} \cdot \mathbf{MP} &= 0, \\ \mathbf{N}' \cdot \mathbf{MP} &= \mathbf{N} \cdot \mathbf{M}' = 0, \end{aligned}$$

since $\mathbf{N} \cdot \mathbf{M}' = 0$. Thus, if $\mathbf{N}$ and $\mathbf{N}'$ are linearly independent, the intersection of these two planes is a line in the rectifying plane, passing through the point $M = f(t)$. We now have to take a closer look at $\mathbf{N}'$. It is easily seen that

$$\mathbf{N}' = -(\|\mathbf{M}''\|^2 + \mathbf{M}' \cdot \mathbf{M}''')\mathbf{M}' + (\mathbf{M}' \cdot \mathbf{M}'')\mathbf{M}'' + \|\mathbf{M}'\|^2\mathbf{M}'''.$$

Thus, $\mathbf{N}$ and $\mathbf{N}'$ are linearly independent iff $\mathbf{M}'$, $\mathbf{M}''$, and $\mathbf{M}'''$ are linearly independent. Now, since the line in question is in the rectifying plane, every point P on this line can be expressed as

$$\mathbf{MP} = \alpha\mathbf{b} + \beta\mathbf{t},$$

where α and β are related by the equation

$$(\mathbf{N}' \cdot \mathbf{b})\alpha + (\mathbf{N}' \cdot \mathbf{t})\beta = 0,$$

obtained from $\mathbf{N}' \cdot \mathbf{MP} = 0$. However, $\mathbf{t} = \mathbf{M}'/\|\mathbf{M}'\|$, and it is immediate that

$$\mathbf{b} = \frac{\mathbf{M}' \times \mathbf{M}''}{\|\mathbf{M}' \times \mathbf{M}''\|}.$$

Recalling that the radius of curvature is given by $\mathcal{R} = \|\mathbf{M}'\|^3/\|\mathbf{M}' \times \mathbf{M}''\|$, it is tempting to investigate the value of α when $\beta = \mathcal{R}$. Then the equation

$$(\mathbf{N}' \cdot \mathbf{b})\alpha + (\mathbf{N}' \cdot \mathbf{t})\beta = 0$$

becomes

$$(\mathbf{N}' \cdot (\mathbf{M}' \times \mathbf{M}''))\alpha + \|\mathbf{M}'\|^2(\mathbf{N}' \cdot \mathbf{M}') = 0.$$

Since

$$\mathbf{N}' = -(\|\mathbf{M}''\|^2 + \mathbf{M}' \cdot \mathbf{M}''')\mathbf{M}' + (\mathbf{M}' \cdot \mathbf{M}'')\mathbf{M}'' + \|\mathbf{M}'\|^2\mathbf{M}''',$$

we get

$$\mathbf{N}' \cdot (\mathbf{M}' \times \mathbf{M}'') = \|\mathbf{M}'\|^2(\mathbf{M}', \mathbf{M}'', \mathbf{M}'''),$$

where $(\mathbf{M}', \mathbf{M}'', \mathbf{M}''')$ is the mixed product of the three vectors, i.e., their determinant, and since $\mathbf{N} \cdot \mathbf{M}' = 0$, we get $\mathbf{N}' \cdot \mathbf{M}' + \mathbf{N} \cdot \mathbf{M}'' = 0$. Thus,

$$\mathbf{N}' \cdot \mathbf{M}' = -\mathbf{N} \cdot \mathbf{M}'' = (\mathbf{M}' \cdot \mathbf{M}'')^2 - \|\mathbf{M}'\|^2\|\mathbf{M}''\|^2 = -\|\mathbf{M}' \times \mathbf{M}''\|^2,$$

and finally, we get

$$\|\mathbf{M}'\|^2(\mathbf{M}', \mathbf{M}'', \mathbf{M}''')\alpha - \|\mathbf{M}'\|^2\|\mathbf{M}' \times \mathbf{M}''\|^2 = 0,$$

which yields

$$\alpha = \frac{\|\mathbf{M}' \times \mathbf{M}''\|^2}{(\mathbf{M}', \mathbf{M}'', \mathbf{M}''')}.$$

So finally, we have shown that the axis of rotation of the rectifying planes for $t + \delta$ close to t is determined by the vector

$$\mathbf{MP} = \alpha\mathbf{b} + \mathcal{R}\mathbf{t},$$

or equivalently, that

$$(\kappa\mathbf{t} + \tau\mathbf{b}) \cdot \mathbf{MP} = 0,$$

where κ is the curvature and $\tau = -1/\alpha$ is called the *torsion at* t, and is given by

$$\tau = -\frac{(\mathbf{M}', \mathbf{M}'', \mathbf{M}''')}{\|\mathbf{M}' \times \mathbf{M}''\|^2}.$$

Its inverse $\mathcal{T} = 1/\tau$ is called the *radius of torsion at* t. The vector $-\tau\mathbf{t} + \kappa\mathbf{b}$ giving the direction of the axis or rotation of the rectifying plane is called the *Darboux vector*. In summary, we have obtained the following formulae for the curvature and the torsion of a 3D-curve: