where $\Omega^0 = I_n$. *As a consequence,*

$$e^A = \begin{pmatrix} e^{\Omega} & VU \\ 0 & 1 \end{pmatrix},$$

where

$$V = I_n + \sum_{k\geq 1} \frac{\Omega^k}{(k+1)!}.$$

Proof. A trivial induction on k shows that

$$A^k = \begin{pmatrix} \Omega^k & \Omega^{k-1}U \\ 0 & 0 \end{pmatrix}.$$

Then we have

$$\begin{aligned} e^A &= \sum_{k\geq 0} \frac{A^k}{k!}, \\ &= I_{n+1} + \sum_{k\geq 1} \frac{1}{k!} \begin{pmatrix} \Omega^k & \Omega^{k-1}U \\ 0 & 0 \end{pmatrix}, \\ &= \begin{pmatrix} I_n + \sum_{k\geq 0} \frac{\Omega^k}{k!} & \sum_{k\geq 1} \frac{\Omega^{k-1}}{k!} U \\ 0 & 1 \end{pmatrix}, \\ &= \begin{pmatrix} e^{\Omega} & VU \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

$\square$

We can now prove our main theorem. We will need to prove that V is invertible when Ω is a skew symmetric matrix. It would be tempting to write V as

$$V = \Omega^{-1}(e^{\Omega} - I).$$

Unfortunately, for odd n, a skew symmetric matrix of order n is not invertible! Thus, we have to find another way of proving that V is invertible. However, observe that we have the following useful fact:

$$V = I_n + \sum_{k\geq 1} \frac{\Omega^k}{(k+1)!} = \int_0^1 e^{\Omega t} dt.$$

This is what we will use in Theorem 14.6.4 to prove surjectivity.

Theorem 14.6.4 *The exponential map*

$$\exp\colon \mathfrak{se}(n) \to \mathbf{SE}(n)$$

is well-defined and surjective.

Proof. Since Ω is skew symmetric, e^{Ω} is a rotation matrix, and by Theorem 14.2.2, the exponential map

$$\exp\colon \mathfrak{so}(n) \to \mathbf{SO}(n)$$

is surjective. Thus, it remains to prove that for every rotation matrix R, there is some skew symmetric matrix Ω such that $R = e^{\Omega}$ and

$$V = I_n + \sum_{k\geq 1} \frac{\Omega^k}{(k+1)!}$$

is invertible. By Theorem 11.4.4, for every skew symmetric matrix Ω there is an orthogonal matrix P such that $\Omega = PD\,P^{\top}$, where D is a block diagonal matrix of the form

$$D = \begin{pmatrix} D_1 & & \dots & \\ & D_2 & \dots & \\ \vdots & \vdots & \ddots & \vdots \\ & & \dots & D_p \end{pmatrix}$$

such that each block D_i is either 0 or a two-dimensional matrix of the form

$$D_i = \begin{pmatrix} 0 & -\theta_i \\ \theta_i & 0 \end{pmatrix}$$

where $\theta_i \in \mathbb{R}$, with $\theta_i > 0$. Actually, we can assume that $\theta_i \neq k2\pi$ for all $k \in \mathbb{Z}$, since when $\theta_i = k2\pi$ we have $e^{D_i} = I_2$, and D_i can be replaced by two one-dimensional blocks each consisting of a single zero. To compute V, since $\Omega = PD\,P^{\top} = PDP^{-1}$, observe that

$$\begin{aligned} V &= I_n + \sum_{k\geq 1} \frac{\Omega^k}{(k+1)!} \\ &= I_n + \sum_{k\geq 1} \frac{PD^kP^{-1}}{(k+1)!} \\ &= P\left(I_n + \sum_{k\geq 1} \frac{D^k}{(k+1)!}\right)P^{-1} \\ &= PWP^{-1}, \end{aligned}$$

where

$$W = I_n + \sum_{k\geq 1} \frac{D^k}{(k+1)!}.$$

We can compute

$$W = I_n + \sum_{k\geq 1} \frac{D^k}{(k+1)!} = \int_0^1 e^{Dt}dt,$$

by computing

$$W = \begin{pmatrix} W_1 & & \dots & \\ & W_2 & \dots & \\ \vdots & \vdots & \ddots & \vdots \\ & & \dots & W_p \end{pmatrix}$$

by blocks. Since

$$e^{D_i} = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}$$

when D_i is a 2×2 skew symmetric matrix and $W_i = \int_0^1 e^{D_i t} dt$, we get

$$W_i = \begin{pmatrix} \int_0^1 \cos(\theta_i t)dt & \int_0^1 -\sin(\theta_i t)dt \\ \int_0^1 \sin(\theta_i t)dt & \int_0^1 \cos(\theta_i t)dt \end{pmatrix} = \frac{1}{\theta_i}\begin{pmatrix} \sin(\theta_i t)\mid_0^1 & \cos(\theta_i t)\mid_0^1 \\ -\cos(\theta_i t)\mid_0^1 & \sin(\theta_i t)\mid_0^1 \end{pmatrix},$$

that is,

$$W_i = \frac{1}{\theta_i}\begin{pmatrix} \sin\theta_i & -(1-\cos\theta_i) \\ 1-\cos\theta_i & \sin\theta_i \end{pmatrix},$$

and $W_i = 1$ when $D_i = 0$. Now, in the first case, the determinant is

$$\frac{1}{\theta_i^2}\left((\sin\theta_i)^2 + (1-\cos\theta_i)^2\right) = \frac{2}{\theta_i^2}(1-\cos\theta_i),$$

which is nonzero, since $\theta_i \neq k2\pi$ for all $k \in \mathbb{Z}$. Thus, each W_i is invertible, and so is W, and thus, $V = PWP^{-1}$ is invertible. $\square$

In the case $n = 3$, given a skew symmetric matrix

$$\Omega = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

letting $\theta = \sqrt{a^2+b^2+c^2}$, it it easy to prove that if $\theta = 0$, then

$$e^A = \begin{pmatrix} I_3 & U \\ 0 & 1 \end{pmatrix},$$

and that if $\theta \neq 0$ (using the fact that $\Omega^3 = -\theta^2\Omega$), then

$$e^\Omega = I_3 + \frac{\sin\theta}{\theta}\Omega + \frac{(1-\cos\theta)}{\theta^2}\Omega^2$$

and

$$V = I_3 + \frac{(1-\cos\theta)}{\theta^2}\Omega + \frac{(\theta-\sin\theta)}{\theta^3}\Omega^2.$$

We finally reach the best vista point of our hike, the formal definition of (linear) Lie groups and Lie algebras.