when $r+1 \leq i \leq n$. This shows that $(v_1, \ldots, v_n)$ are eigenvectors of h_2 for $\mu_1, \ldots, \mu_n$ (since $\mu_{r+1} = \cdots = \mu_n = 0$), and thus h_1 and h_2 have the same eigenvalues $\mu_1, \ldots, \mu_n$.

As a consequence,

$$(h_2 \circ g)(u_i) = h_2(g(u_i)) = h_2(v_i) = \mu_i v_i = f(u_i)$$

when $1 \leq i \leq n$. Since h_1, h_2, $f^* \circ f$, and $f \circ f^*$ are positive self-adjoint, $f^* \circ f = h_1^2$, $f \circ f^* = h_2^2$, and $\mu_1, \ldots, \mu_r$ are the eigenvalues of both h_1 and h_2, it follows that $\mu_1, \ldots, \mu_r$ are the singular values of f, i.e., the positive square roots of the nonnull eigenvalues of both $f^* \circ f$ and $f \circ f^*$.

Finally, since

$$f^* \circ f = h_1^2 \quad \text{and} \quad f \circ f^* = h_2^2,$$

by Lemma 12.1.2, h_1 and h_2 are unique and if f is invertible, then h_1 and h_2 are invertible and thus g is also unique, since $g = f \circ h_1^{-1}$. If h is normal, then $f^* \circ f = f \circ f^*$ and $h_1 = h_2$. $\square$

In matrix form, Theorem 12.1.3 can be stated as follows. For every real $n \times n$ matrix A, there is some orthogonal matrix R and some positive symmetric matrix S such that

$$A = RS.$$

Furthermore, R, S are unique if A is invertible. A pair (R, S) such that $A = RS$ is called a *polar decomposition of* A. For example, the matrix

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

is both orthogonal and symmetric, and $A = RS$ with $R = A$ and $S = I$, which implies that some of the eigenvalues of A are negative.

Remark: If E is a Hermitian space, Theorem 12.1.3 also holds, but the orthogonal linear map g becomes a unitary map. In terms of matrices, the polar decomposition states that for every complex $n \times n$ matrix A, there is some unitary matrix U and some positive Hermitian matrix H such that

$$A = UH.$$

12.2 Singular Value Decomposition (SVD)

The proof of Theorem 12.1.3 shows that there are two orthonormal bases $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$, where $(u_1, \ldots, u_n)$ are eigenvectors of h_1 and $(v_1, \ldots, v_n)$ are eigenvectors of h_2. Furthermore, $(u_1, \ldots, u_r)$ is an orthonormal basis of $\operatorname{Im} f^*$, $(u_{r+1}, \ldots, u_n)$ is an orthonormal basis of $\operatorname{Ker} f$,