such that each block A_i is either a one-dimensional matrix (i.e., a real scalar) or a two-dimensional matrix of the form

$$A_i = \begin{pmatrix} \lambda_i & -\mu_i \\ \mu_i & \lambda_i \end{pmatrix},$$

where $\lambda_i, \mu_i \in \mathbb{R}$, with $\mu_i > 0$.

Proof. We proceed by induction on the dimension n of E as follows. If $n = 1$, the result is trivial. Assume now that $n \geq 2$. First, since $\mathbb{C}$ is algebraically closed (i.e., every polynomial has a root in $\mathbb{C}$), the linear map $f_{\mathbb{C}}\colon E_{\mathbb{C}} \to E_{\mathbb{C}}$ has some eigenvalue $z = \lambda + i\mu$ (where $\lambda, \mu \in \mathbb{R}$). Let $w = u + iv$ be some eigenvector of $f_{\mathbb{C}}$ for $\lambda + i\mu$ (where $u, v \in E$). We can now apply Lemma 11.2.8.

If $\mu = 0$, then either u or v is an eigenvector of f for $\lambda \in \mathbb{R}$. Let W be the subspace of dimension 1 spanned by $e_1 = u/\|u\|$ if $u \neq 0$, or by $e_1 = v/\|v\|$ otherwise. It is obvious that $f(W) \subseteq W$ and $f^*(W) \subseteq W$. The orthogonal $W^\perp$ of W has dimension $n-1$, and by Lemma 11.2.7, we have $f\big(W^\perp\big) \subseteq W^\perp$. But the restriction of f to $W^\perp$ is also normal, and we conclude by applying the induction hypothesis to $W^\perp$.

If $\mu \neq 0$, then $\langle u, v\rangle = 0$ and $\langle u, u\rangle = \langle v, v\rangle$, and if W is the subspace spanned by $u/\|u\|$ and $v/\|v\|$, then $f(W) = W$ and $f^*(W) = W$. We also know that the restriction of f to W has the matrix

$$\begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix}$$

with respect to the basis $(u/\|u\|, v/\|v\|)$. If $\mu < 0$, we let $\lambda_1 = \lambda$, $\mu_1 = -\mu$, $e_1 = u/\|u\|$, and $e_2 = v/\|v\|$. If $\mu > 0$, we let $\lambda_1 = \lambda$, $\mu_1 = \mu$, $e_1 = v/\|v\|$, and $e_2 = u/\|u\|$. In all cases, it is easily verified that the matrix of the restriction of f to W w.r.t. the orthonormal basis (e_1, e_2) is

$$A_1 = \begin{pmatrix} \lambda_1 & -\mu_1 \\ \mu_1 & \lambda_1 \end{pmatrix},$$

where $\lambda_1, \mu_1 \in \mathbb{R}$, with $\mu_1 > 0$. However, $W^\perp$ has dimension $n-2$, and by Lemma 11.2.7, $f\big(W^\perp\big) \subseteq W^\perp$. Since the restriction of f to $W^\perp$ is also normal, we conclude by applying the induction hypothesis to $W^\perp$. $\square$

After this relatively hard work, we can easily obtain some nice normal forms for the matrices of self-adjoint, skew self-adjoint, and orthogonal linear maps. However, for the sake of completeness (and since we have all the tools to so do), we go back to the case of a Hermitian space and show that normal linear maps can be diagonalized with respect to an orthonormal basis.

Theorem 11.2.10 *Given a Hermitian space E of dimension n, for every normal linear map $f\colon E \to E$ there is an orthonormal basis $(e_1, \ldots, e_n)$ of*