If $\mu = 0$*, then* λ *is a real eigenvalue of* f*, and either* u *or* v *is an eigenvector of* f *for* λ*. If* W *is the subspace spanned by* u *if* $u \neq 0$*, or spanned by* $v \neq 0$ *if* $u = 0$*, then* $f(W) \subseteq W$ *and* $f^*(W) \subseteq W$*.*

Proof. Since $w = u + iv$ is an eigenvector of $f_{\mathbb{C}}$, by definition it is nonnull, and either $u \neq 0$ or $v \neq 0$. From the fact stated just before Lemma 11.2.8, $u - iv$ is an eigenvector of $f_{\mathbb{C}}$ for $\lambda - i\mu$. It is easy to check that $f_{\mathbb{C}}$ is normal. However, if $\mu \neq 0$, then $\lambda + i\mu \neq \lambda - i\mu$, and from Lemma 11.2.5, the vectors $u + iv$ and $u - iv$ are orthogonal w.r.t. $\langle -, - \rangle_{\mathbb{C}}$, that is,

$$\langle u + iv, u - iv \rangle_{\mathbb{C}} = \langle u, u \rangle - \langle v, v \rangle + 2i \langle u, v \rangle = 0.$$

Thus, we get $\langle u, v \rangle = 0$ and $\langle u, u \rangle = \langle v, v \rangle$, and since $u \neq 0$ or $v \neq 0$, u and v are linearly independent. Since

$$f(u) = \lambda u - \mu v \quad \text{and} \quad f(v) = \mu u + \lambda v$$

and since by Lemma 11.2.4 $u + iv$ is an eigenvector of f^* for $\lambda - i\mu$, we have

$$f^*(u) = \lambda u + \mu v \quad \text{and} \quad f^*(v) = -\mu u + \lambda v,$$

and thus $f(W) = W$ and $f^*(W) = W$, where W is the subspace spanned by u and v.

When $\mu = 0$, we have

$$f(u) = \lambda u \quad \text{and} \quad f(v) = \lambda v,$$

and since $u \neq 0$ or $v \neq 0$, either u or v is an eigenvector of f for λ. If W is the subspace spanned by u if $u \neq 0$, or spanned by v if $u = 0$, it is obvious that $f(W) \subseteq W$ and $f^*(W) \subseteq W$. Note that $\lambda = 0$ is possible, and this is why $\subseteq$ cannot be replaced by $=$. ☐

The beginning of the proof of Lemma 11.2.8 actually shows that for every linear map $f \colon E \to E$ there is some subspace W such that $f(W) \subseteq W$, where W has dimension 1 or 2. In general, it doesn't seem possible to prove that $W^{\perp}$ is invariant under f. However, this happens when f is normal, and in this case, other nice things also happen.

Indeed, if f is a normal linear map, recall that the proof of Lemma 11.2.8 shows that λ, μ, u, and v satisfy the equations

$$\begin{aligned} f(u) &= \lambda u - \mu v, \\ f(v) &= \mu u + \lambda v, \\ f^*(u) &= \lambda u + \mu v, \\ f^*(v) &= -\mu u + \lambda v, \end{aligned}$$

from which we get

$$\frac{1}{2}(f + f^*)(u) = \lambda u,$$