10.3 Linear Isometries (Also Called Unitary Transformations)

In this section we consider linear maps between Hermitian spaces that preserve the Hermitian norm. All definitions given for Euclidean spaces in Section 6.3 extend to Hermitian spaces, except that orthogonal transformations are called unitary transformation, but Lemma 6.3.2 extends only with a modified condition (2). Indeed, the old proof that (2) implies (3) does not work, and the implication is in fact false! It can be repaired by strengthening condition (2). For the sake of completeness, we state the Hermitian version of Definition 6.3.1.

Definition 10.3.1 Given any two nontrivial Hermitian spaces E and F of the same finite dimension n, a function $f\colon E \to F$ is *a unitary transformation, or a linear isometry*, if it is linear and

$$\|f(u)\| = \|u\|,$$

for all $u \in E$.

Lemma 6.3.2 can be salvaged by strengthening condition (2).

Lemma 10.3.2 *Given any two nontrivial Hermitian spaces E and F of the same finite dimension n, for every function $f\colon E \to F$, the following properties are equivalent:*

(1) f is a linear map and $\|f(u)\| = \|u\|$, for all $u \in E$;

(2) $\|f(v) - f(u)\| = \|v - u\|$ and $f(iu) = if(u)$, for all $u, v \in E$.

(3) $f(u) \cdot f(v) = u \cdot v$, for all $u, v \in E$.

Furthermore, such a map is bijective.

Proof. The proof that (2) implies (3) given in Lemma 6.3.2 needs to be revised as follows. We use the polarization identity

$$2\varphi(u, v) = (1 + i)(\|u\|^2 + \|v\|^2) - \|u - v\|^2 - i\|u - iv\|^2.$$

Since $f(iv) = if(v)$, we get $f(0) = 0$ by setting $v = 0$, so the function f preserves distance and norm, and we get

$$\begin{aligned}
2\varphi(f(u), f(v))& \\
&= (1 + i)(\|f(u)\|^2 + \|f(v)\|^2) - \|f(u) - f(v)\|^2 - i\|f(u) - if(v)\|^2\\
&= (1 + i)(\|f(u)\|^2 + \|f(v)\|^2) - \|f(u) - f(v)\|^2 - i\|f(u) - f(iv)\|^2\\
&= (1 + i)(\|u\|^2 + \|v\|^2) - \|u - v\|^2 - i\|u - iv\|^2\\
&= 2\varphi(u, v),
\end{aligned}$$

which shows that f preserves the Hermitian inner product, as desired. The rest of the proof is unchanged. $\square$

Remarks:

(i) In the Euclidean case, we proved that the assumption

$$\|f(v) - f(u)\| = \|v - u\| \quad \text{for all } u, v \in E \text{ and } f(0) = 0 \tag{2'}$$

implies (3). For this we used the polarization identity

$$2u \cdot v = \|u\|^2 + \|v\|^2 - \|u - v\|^2.$$

In the Hermitian case the polarization identity involves the complex number i. In fact, the implication (2′) implies (3) is false in the Hermitian case! Conjugation $z \mapsto \overline{z}$ satisfies (2′) since

$$|\overline{z_2} - \overline{z_1}| = |\overline{z_2 - z_1}| = |z_2 - z_1|,$$

and yet, it is not linear!

(ii) If we modify (2) by changing the second condition by now requiring that there be some $\tau \in E$ such that

$$f(\tau + iu) = f(\tau) + i(f(\tau + u) - f(\tau))$$

for all $u \in E$, then the function $g\colon E \to E$ defined such that

$$g(u) = f(\tau + u) - f(\tau)$$

satisfies the old conditions of (2), and the implications (2) $\to$ (3) and (3) $\to$ (1) prove that g is linear, and thus that f is affine. In view of the first remark, some condition involving i is needed on f, in addition to the fact that f is distance-preserving.

10.4 The Unitary Group, Unitary Matrices

In this section, as a mirror image of our treatment of the isometries of a Euclidean space, we explore some of the fundamental properties of the unitary group and of unitary matrices. As an immediate corollary of the Gram–Schmidt orthonormalization procedure, we obtain the QR-decomposition for invertible matrices. In the Hermitian framework, the matrix of the adjoint of a linear map is not given by the transpose of the original matrix, but by its conjugate.

Definition 10.4.1 Given a complex $m \times n$ matrix A, the *transpose* $A^\top$ *of* A is the $n \times m$ matrix $A^\top = \left(a^\top_{i,\,j}\right)$ defined such that

$$a^\top_{i,\,j} = a_{j,\,i},$$

and the *conjugate* $\overline{A}$ *of* A is the $m \times n$ matrix $\overline{A} = (b_{i,\,j})$ defined such that

$$b_{i,\,j} = \overline{a}_{i,\,j}$$