When E is a Euclidean space, we have an interesting situation regarding the value of determinants over orthornormal bases described by the following lemma. Given any basis $B = (u_1, \ldots, u_n)$ for E, for any sequence $(w_1, \ldots, w_n)$ of n vectors, we denote by $\det_B(w_1, \ldots, w_n)$ the determinant of the matrix whose columns are the coordinates of the w_j over the basis $B = (u_1, \ldots, u_n)$.

Lemma 7.8.2 *Let E be a Euclidean space of dimension n, and assume that an orientation of E has been chosen. For any sequence $(w_1, \ldots, w_n)$ of n vectors and any two orthonormal bases $B_1 = (u_1, \ldots, u_n)$ and $B_2 = (v_1, \ldots, v_n)$ of positive orientation, we have*

$$\det_{B_1}(w_1, \ldots, w_n) = \det_{B_2}(w_1, \ldots, w_n).$$

Proof. Let P be the change of basis matrix from $B_1 = (u_1, \ldots, u_n)$ to $B_2 = (v_1, \ldots, v_n)$. Since $B_1 = (u_1, \ldots, u_n)$ and $B_2 = (v_1, \ldots, v_n)$ are orthonormal, P is orthogonal, and we must have $\det(P) = +1$, since the bases have positive orientation. Let U_1 be the matrix whose columns are the coordinates of the w_j over the basis $B_1 = (u_1, \ldots, u_n)$, and let U_2 be the matrix whose columns are the coordinates of the w_j over the basis $B_2 = (v_1, \ldots, v_n)$. Then, by definition of P, we have

$$(w_1, \ldots, w_n) = (u_1, \ldots, u_n)U_2P,$$

that is,

$$U_1 = U_2P.$$

Then, we have

$$\begin{aligned}\det_{B_1}(w_1, \ldots, w_n) &= \det(U_1) = \det(U_2P) = \det(U_2)\det(P)\\ &= \det_{B_2}(w_1, \ldots, w_n)\det(P) = \det_{B_2}(w_1, \ldots, w_n),\end{aligned}$$

since $\det(P) = +1$. $\square$

By Lemma 7.8.2, the determinant $\det_B(w_1, \ldots, w_n)$ is independent of the base B, provided that B is orthonormal and of positive orientation. Thus, Lemma 7.8.2 suggests the following definition.

7.9 Volume Forms, Cross Products

In this section we generalize the familiar notion of cross product of vectors in $\mathbb{R}^3$ to Euclidean spaces of any finite dimension. First, we define the mixed product, or volume form.

Definition 7.9.1 Given any Euclidean space E of finite dimension n over $\mathbb{R}$ and any orientation of E, for any sequence $(w_1, \ldots, w_n)$ of n vectors in E, the common value $\lambda_E(w_1, \ldots, w_n)$ of the determinant $\det_B(w_1, \ldots, w_n)$