We claim that there is some j $(1 \leq j \leq q)$ such that

$$\lambda_j + \alpha\mu_j = 0.$$

Indeed, since

$$\alpha = \max_{1 \leq i \leq q} \{-\lambda_i/\mu_i \mid \mu_i > 0\},$$

as the set on the right hand side is finite, the maximum is achieved and there is some index j so that $\alpha = -\lambda_j/\mu_j$. If j is some index such that $\lambda_j + \alpha\mu_j = 0$, since $\sum_{i=1}^{q} \mu_i \mathbf{Oa_i} = 0$, we have

$$\begin{aligned} b = \sum_{i=1}^{q} \lambda_i a_i &= O + \sum_{i=1}^{q} \lambda_i \mathbf{Oa_i} + 0, \\ &= O + \sum_{i=1}^{q} \lambda_i \mathbf{Oa_i} + \alpha\left(\sum_{i=1}^{q} \mu_i \mathbf{Oa_i}\right), \\ &= O + \sum_{i=1}^{q} (\lambda_i + \alpha\mu_i)\mathbf{Oa_i}, \\ &= \sum_{i=1}^{q} (\lambda_i + \alpha\mu_i) a_i, \\ &= \sum_{i=1,\, i\neq j}^{q} (\lambda_i + \alpha\mu_i) a_i, \end{aligned}$$

since $\lambda_j + \alpha\mu_j = 0$. Since $\sum_{i=1}^{q} \mu_i = 0$, $\sum_{i=1}^{q} \lambda_i = 1$, and $\lambda_j + \alpha\mu_j = 0$, we have

$$\sum_{i=1,\, i\neq j}^{q} \lambda_i + \alpha\mu_i = 1,$$

and since $\lambda_i + \alpha\mu_i \geq 0$ for $i = 1, \ldots, q$, the above shows that b can be expressed as a convex combination of $q - 1$ points from S. However, this contradicts the assumption that b cannot be expressed as a convex combination of strictly fewer than q points from S, and the theorem is proved. $\square$

If S is a finite (of infinite) set of points in the affine plane $\mathbb{A}^2$, Theorem 3.2.1 confirms our intuition that $\mathcal{C}(S)$ is the union of triangles (including interior points) whose vertices belong to S. Similarly, the convex hull of a set S of points in $\mathbb{A}^3$ is the union of tetrahedra (including interior points) whose vertices belong to S. We get the feeling that triangulations play a crucial role, which is of course true!

We conclude this short chapter with two other classics of convex geometry.