Eulerian and Hamiltonian Cycles

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Chapter 1

Directed Graphs, Paths

Recall that a *directed graph*, G, is a pair $G = (V, E)$, where $E \subseteq V \times V$.
A pair $(u, v) \in E$ is called an *edge* of G (note that $u = v$ is allowed).

Given any two nodes $u, v \in V$, a *path from u to v* is any sequence of $n + 1$ edges ($n \geq 0$)

$$(u, v_1), (v_1, v_2), \ldots, (v_n, v).$$

(If $n = 1$, a path from u to v is simply a single edge, (u, v).) A graph G is *strongly connected* if for every pair $(u, v) \in V \times V$, there is a path from u to v. A *closed path, or cycle*, is a path from some node u to itself.

We will restrict out attention to finite graphs, i.e. graphs (V, E) where V is a finite set.

Definition 1.0.1 Given a graph G, an *Eulerian cycle* is a cycle in G that passes through all the nodes (possibly more than once) and every edge of G exactly once. A *Hamiltonian cycle* is a cycle that passes through all the nodes exactly once (note, some edges may not be traversed at all).

Eulerian Cycle Problem: Given a graph G, is there an Eulerian cycle in G?

Hamiltonian Cycle Problem: Given a graph G, is there an Hamiltonian cycle in G?

Chapter 2

Eulerian Cycles

The following graph is a directed graph version of the Königsberg bridge problem, solved by Euler in 1736. The nodes A, B, C, D correspond to four areas of land in Königsberg and the edges to the seven bridges joining these areas of land. The problem is to find a closed path that crosses every bridge exactly once and returns to the starting point.

In fact, the problem is unsolvable, as shown by Euler, because some nodes do not have the same number of incoming and outgoing edges (In the undirected version of the problem, some nodes do not have an even degree.)

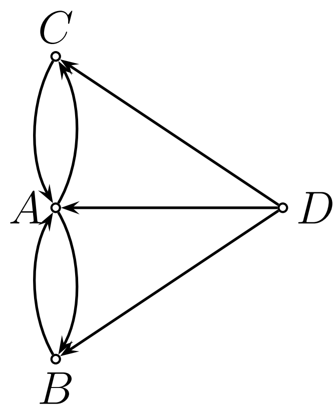

Figure 2.1: A directed graph modeling the Königsberg bridge problem

It may come as a surprise that the Eulerian Cycle Problem does have a polynomial time algorithm, but that so far, not such algorithm is known for the Hamiltonian Cycle Problem.

The reason why the Eulerian Cycle Problem is decidable in polynomial time is the following theorem due to Euler:

Theorem 2.0.2 *A graph $G = (V, E)$ has an Eulerian cycle iff the following properties hold:*

(1) The graph G is strongly connected.

(2) Every node has the same number of incoming and outgoing edges.

Proving that properties (1) and (2) hold if G has an Eulerian cycle is fairly easy. The converse is harder, but not that bad (try!).

Theorem 2.0.2 shows that it is necessary to check whether a graph is strongly connected. This can be done by computing the transitive closure of E, which can be done in polynomial time (in fact, $O(n^3)$).

Checking property (2) can clearly be done in polynomial time. Thus, the Eulerian cycle problem is in $\mathcal{P}$.

Unfortunately, no theorem analogous to Theorem 2.0.2 is know for Hamiltonian cycles.

Chapter 3

Hamiltonian Cycles

A game invented by Sir William Hamilton in 1859 uses a regular solid dodecahedron whose twenty vertices are labeled with the names of famous cities. The player is challenged to "travel around the world" by finding a closed cycle along the edges of the dodecahedron which passes through every city exactly once (this is the undirected version of the Hamiltonian cycle problem).

In graphical terms, assuming an orientation of the edges between cities, the graph D shown in Figure 3.1 is a plane projection of a regular dodecahedron and we want to know if there is a Hamiltonian cycle in this directed graph.

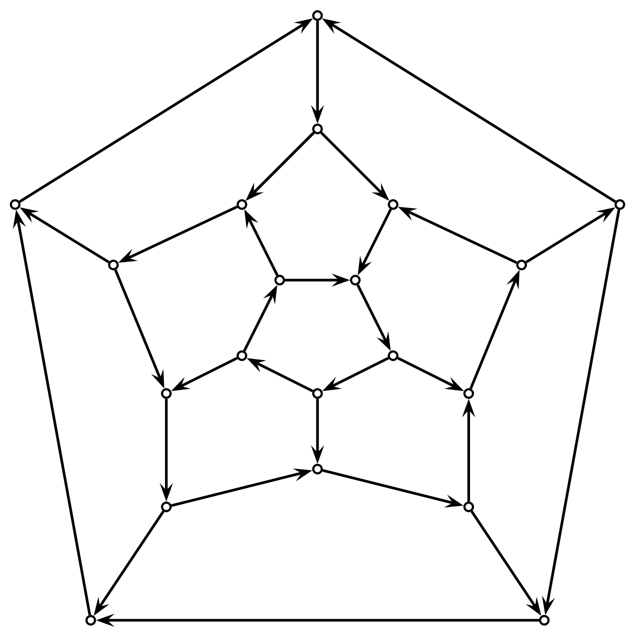

Figure 3.1: A tour "around the world."

Finding a Hamiltonian cycle in this graph does not appear to be so easy!

A solution is shown in Figure 3.2 below:

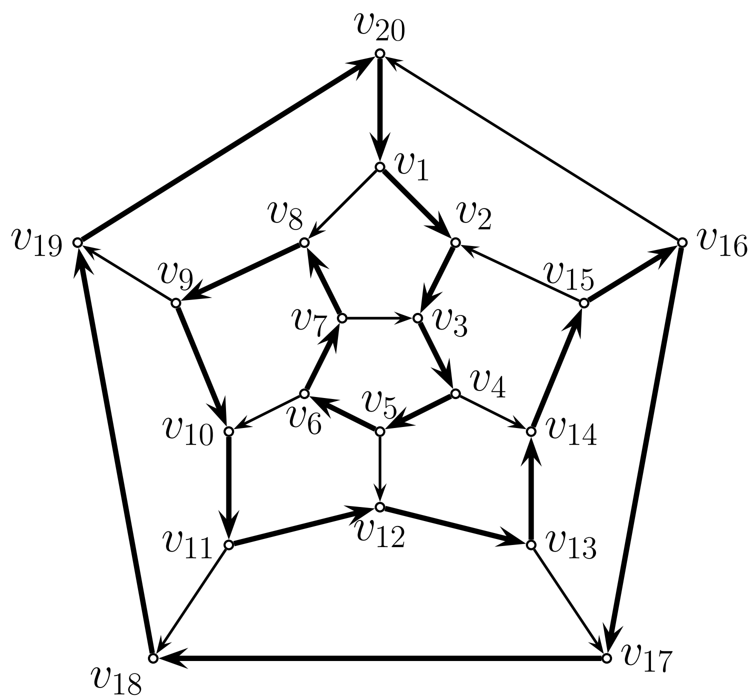

Figure 3.2: A Hamiltonian cycle in D

A solution!