IMPA - Instituto de Matemática Pura e Aplicada, Rio de Janeiro, RJ, Brazil

Introduction to Computational Manifolds and Applications

Part 1 - Constructions

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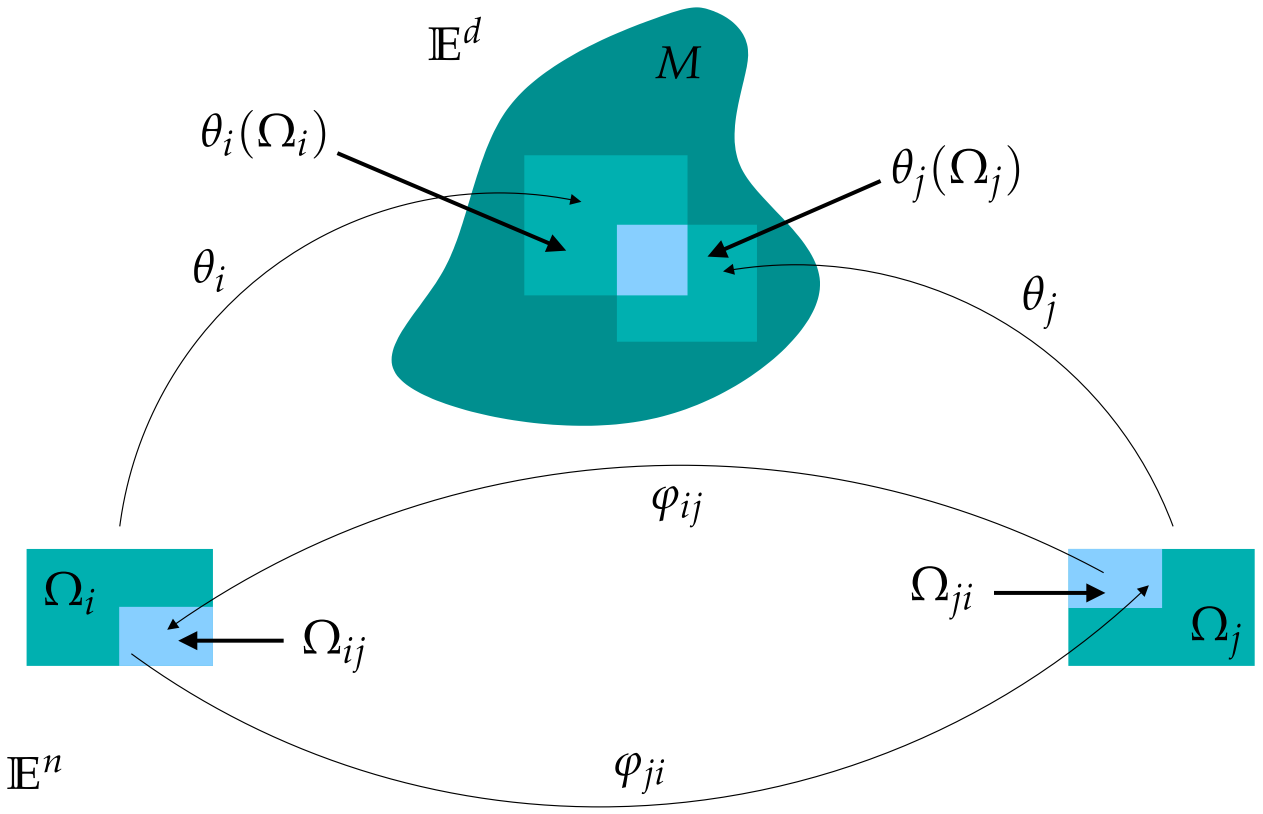
Parametric Pseudo-Manifolds

Building Parametrizations

Recall the big picture...



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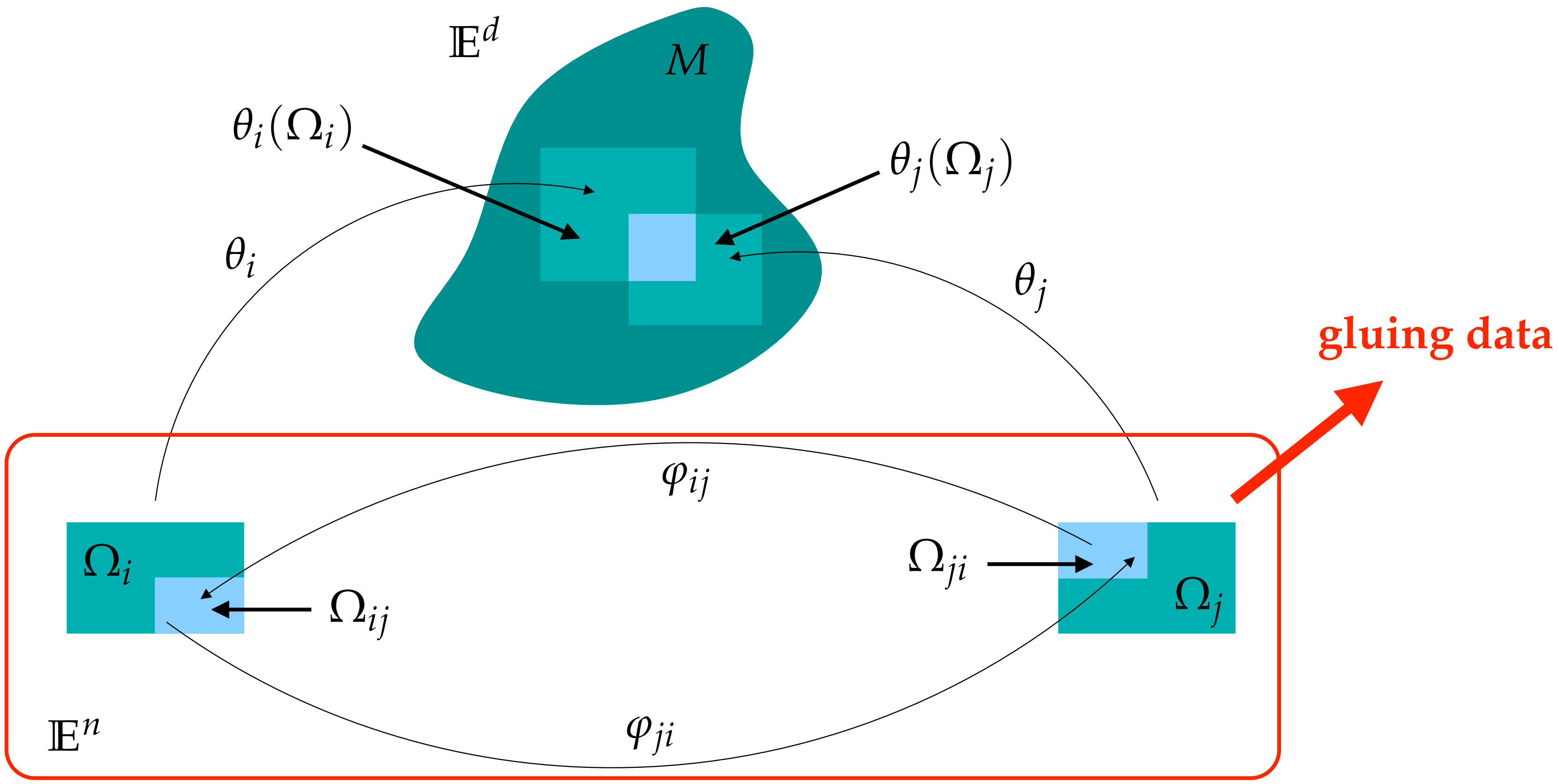
We've learned how to build a set of gluing data using distinct choices of transition maps.



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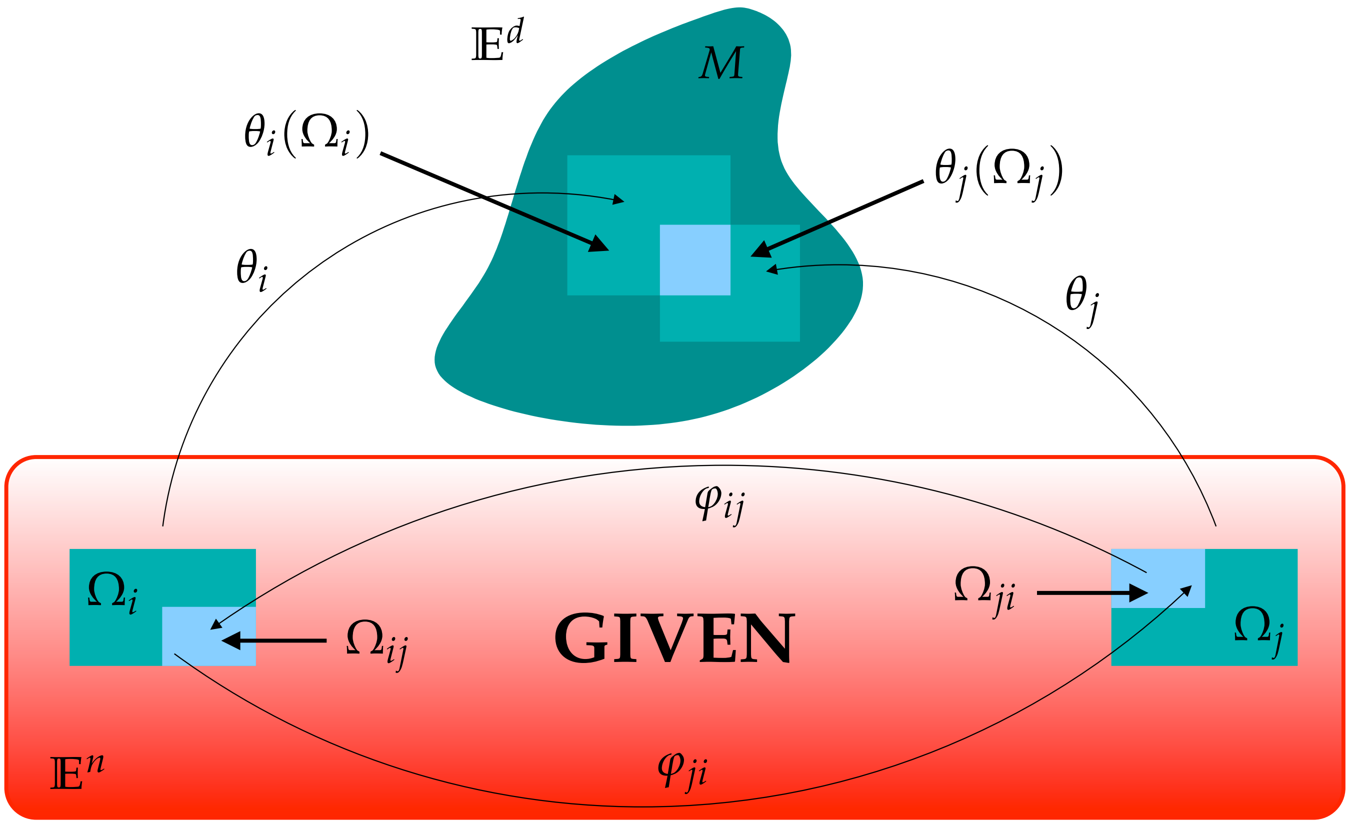
Building Parametrizations

Our goal now is to learn how to build a family, $(\theta_i)_{i \in I}$, of parametrizations.



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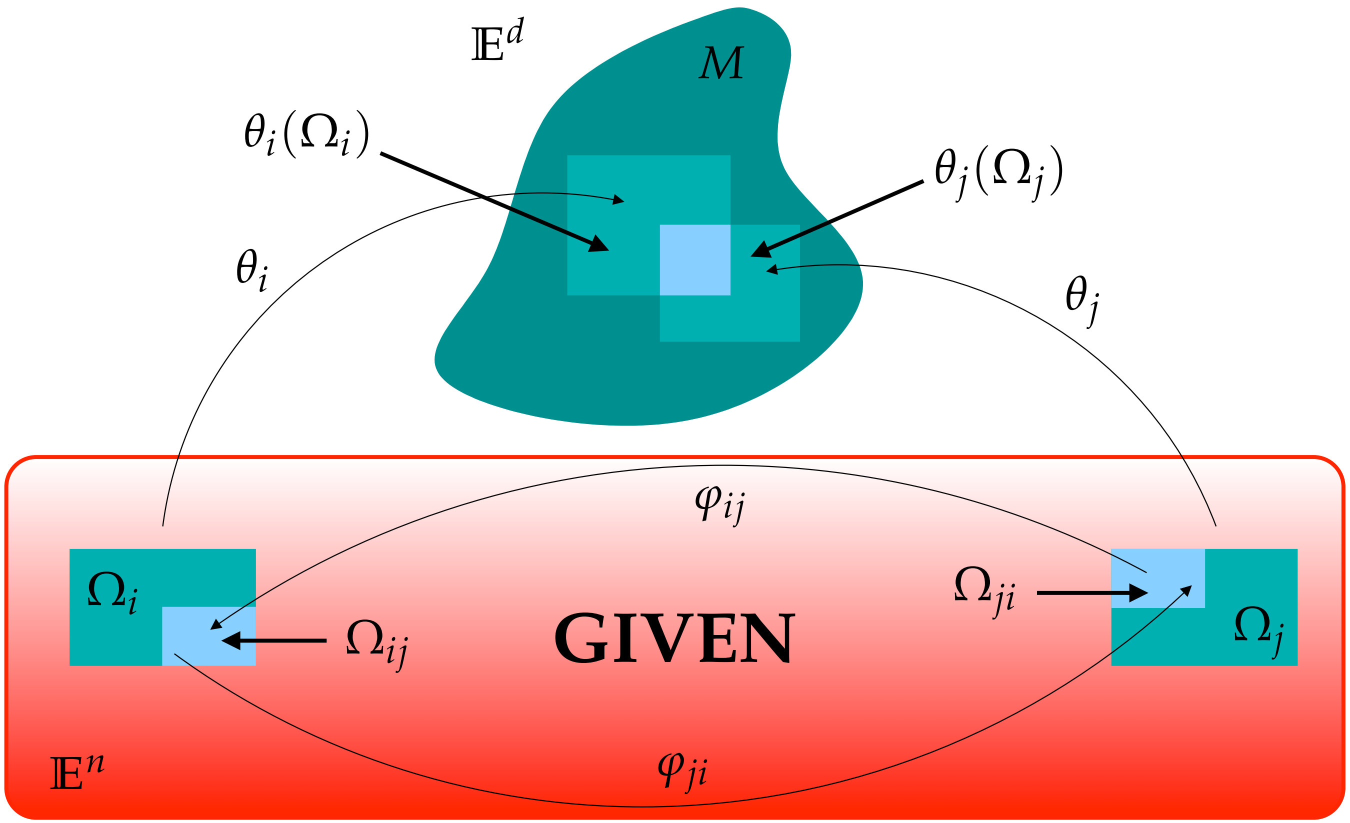
We'll define the parametrizations for one set of gluing data showed in the previous lecture.



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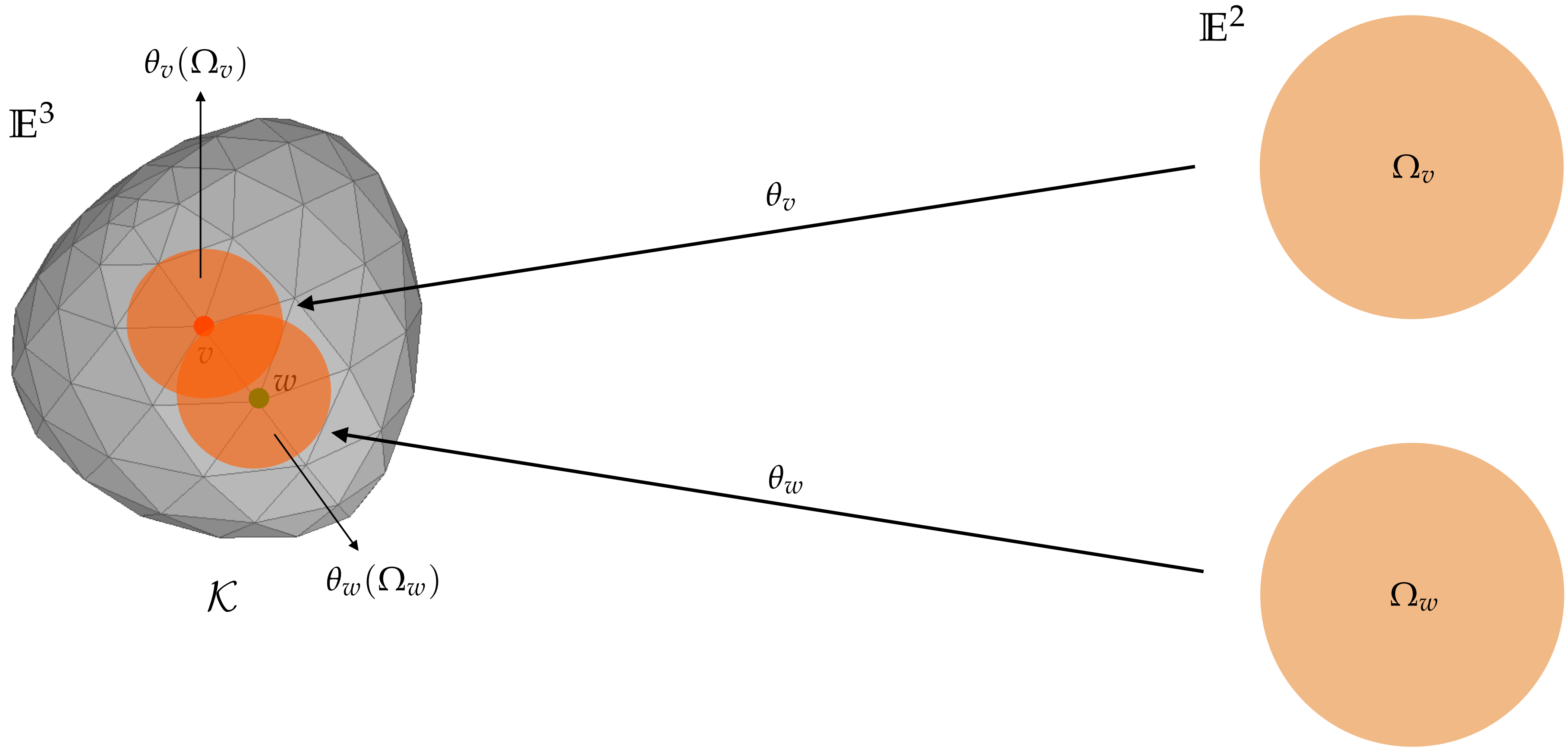

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To define $(\theta_v)_{v\in I}$, we specify a family of *shape functions* and a family of *bump functions*:

$$\left(\psi_v\right)_{v\in I} \qquad \text{and} \qquad \left(\gamma_v\right)_{v\in I} .$$

More specifically, for each $v \in I$, we define the *shape function*,

$$\psi_v : \square_v \subseteq \mathbb{E}^2 \to \mathbb{E}^3 ,$$

associated with Ω_v, as the *Bézier (surface) patch of bi-degree (m, n)* given by the expression

$$\psi_v(p) = \sum_{0\leq j\leq m} \sum_{0\leq k\leq n} B_j^m(x) \cdot B_k^n(y) \cdot b_{j,k}^v ,$$

where

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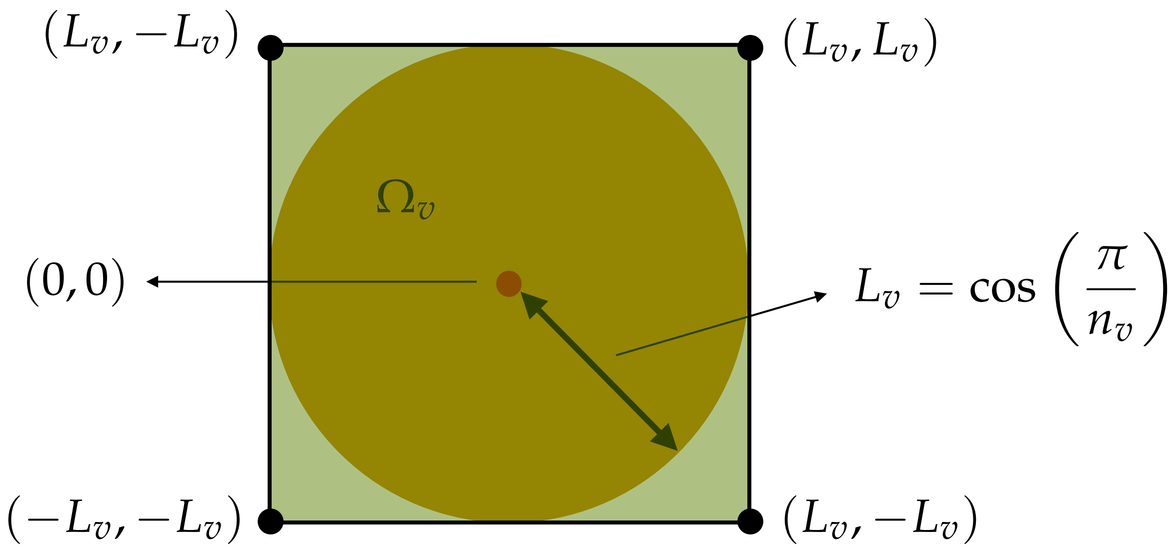
- $\square_v = [-L_v, -L_v] \times [L_v, L_v]$, with $L_v = \cos\left(\frac{\pi}{n_v}\right)$,



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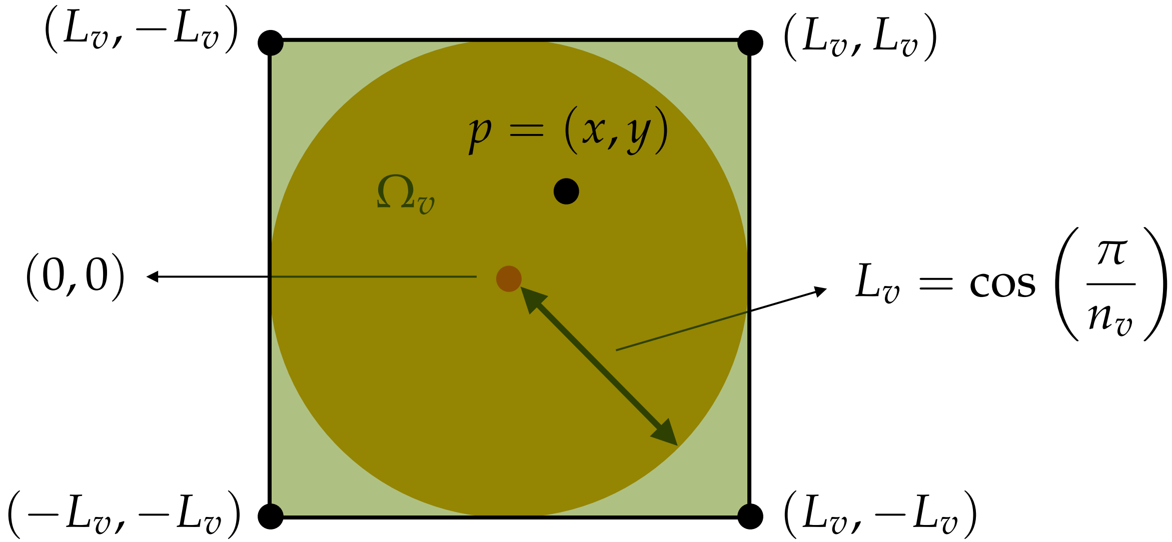
- (x, y) are the coordinates of a point p in the local coordinates system of $\square_v$,



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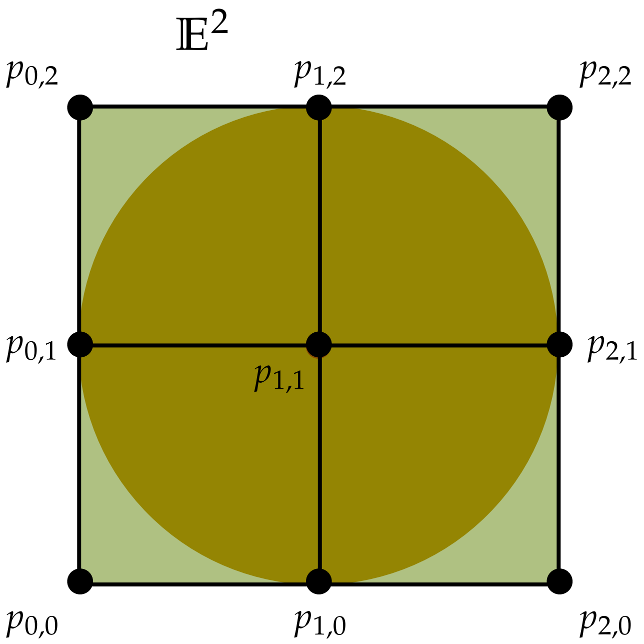
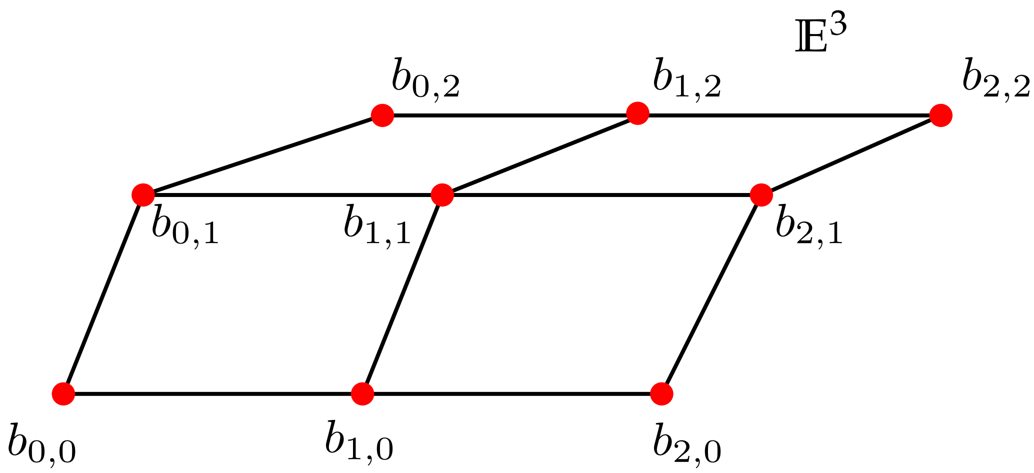
- $(b^v_{j,k}) \subset \mathbb{E}^3$ are the control points of ψ_v, with $0 \le j \le m$ and $0 \le k \le n$,

Ex: $m = 2$ and $n = 2$





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- and

$$B_i^l(t) = \binom{l}{i} \cdot \left(\frac{r-t}{r-s}\right)^{l-i} \cdot \left(\frac{t-s}{r-s}\right)^{i}$$

is the i-th Bernstein polynomial of degree l over the affine frame $[s, r]$ such that

$$s = -L_v \qquad \text{and} \qquad r = L_v\,,$$

for every $i \in \{0, 1, \ldots, l\}$, and

$$\sum_{0 \le j \le m} \sum_{0 \le k \le n} B_j^m(x) \cdot B_k^n(y) = 1\,,$$

for every $x, y \in [s, r]$.

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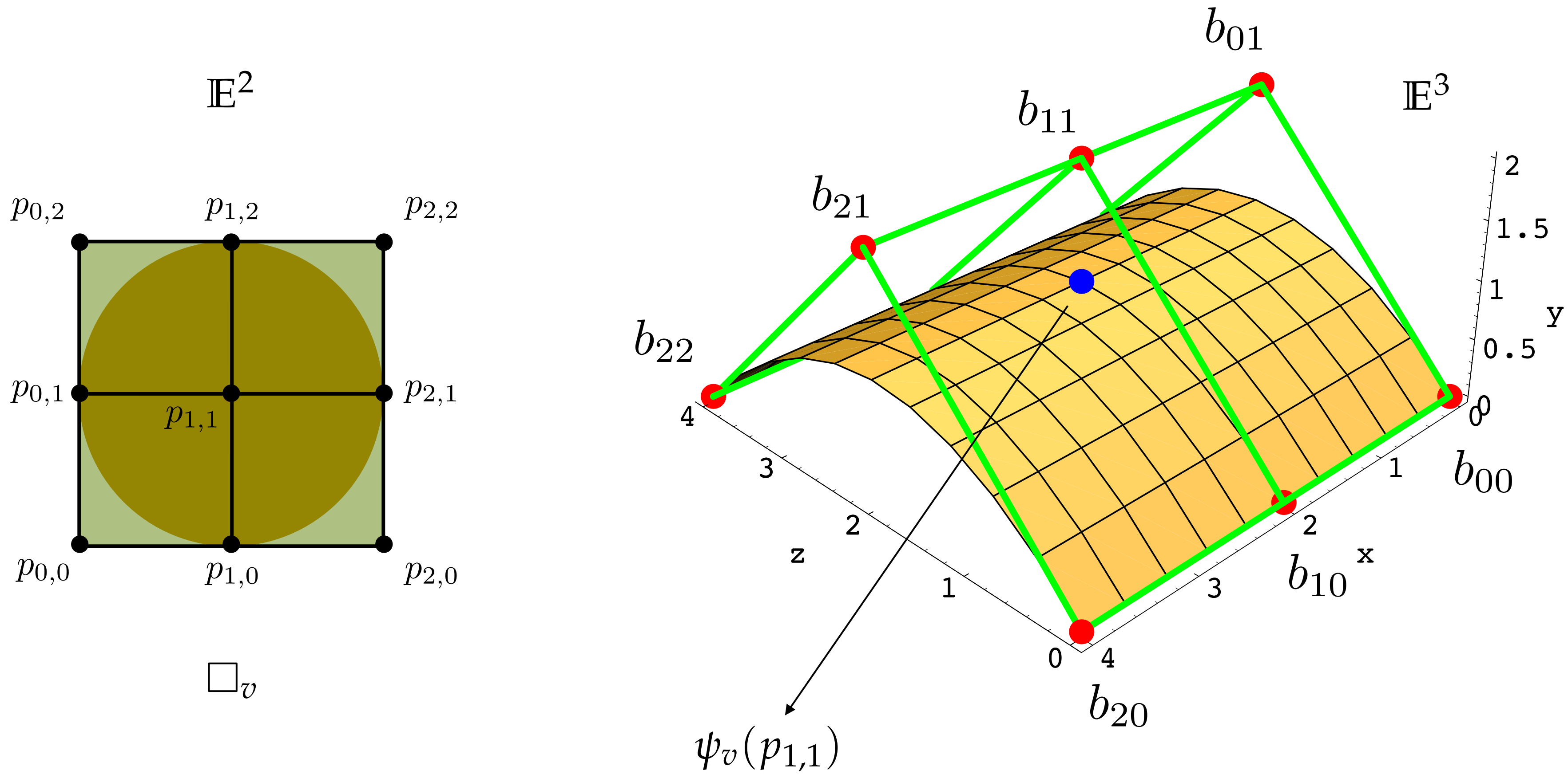
So, $\psi_v(p)$ is a convex combination of the control points, $b^v_{j,k}$.



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How can we define the control points of ψ_v?

We currently use a *least squares fitting* procedure.

The idea is to compute a *large* collection, $(p_j, p'_j)_{j\in J}$, of pairs of *parameter points* and *sample points*, respectively, where the first element, p_j, is in $\mathbb{E}^2$ and the second, p'_j, is in $\mathbb{E}^3$.

We view p'_j as the image of p_j under a *given* function, $\beta : \mathbb{E}^2 \to \mathbb{E}^3$, we wish to locally approximate using the ψ_v's. As we shall see, there are many choices for the function β.

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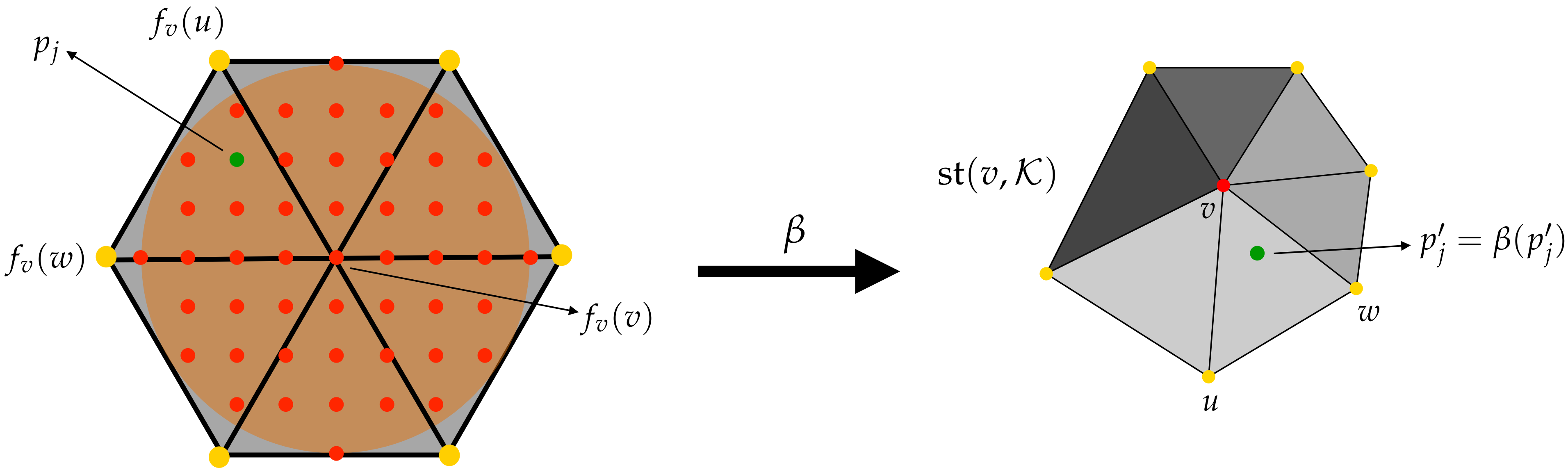
However, one of the simplest choices for the function β could be a *barycentric map* that takes each parameter point, p_j, in Ω_v to a sample point, p'_j, in the star, $\mathrm{st}(v, \mathcal{K})$, of v in $\mathcal{K}$.



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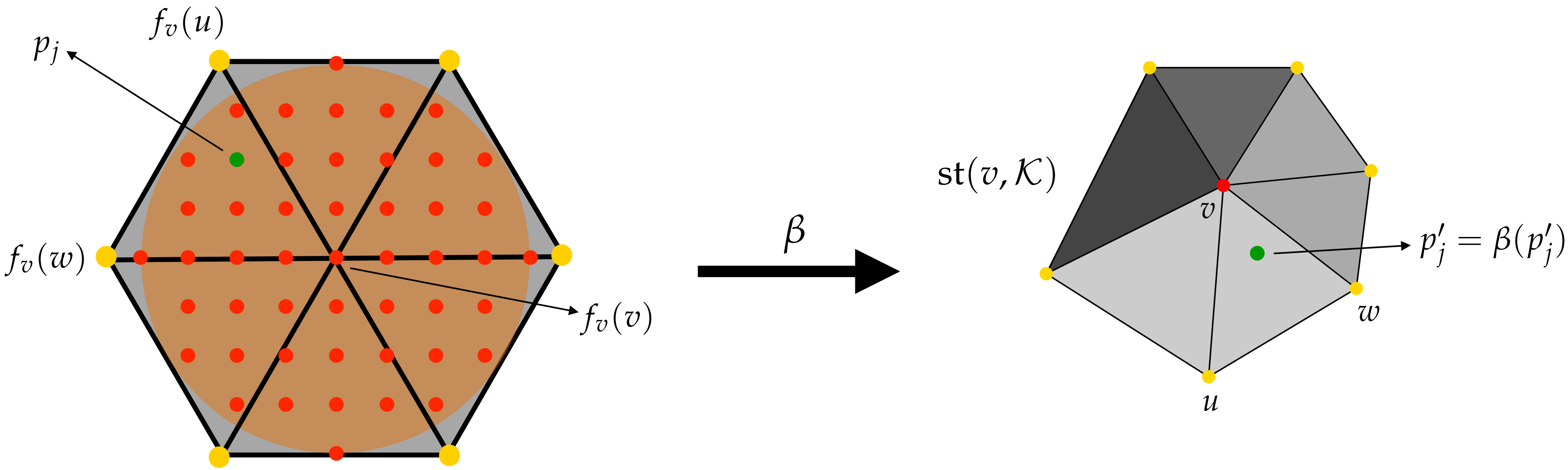
More precisely,

$$p'_j = \beta(p_j) = \beta(\lambda \cdot f_v(v) + \mu \cdot f_v(u) + \nu \cdot f_v(w)) = \lambda \cdot v + \mu \cdot u + \nu \cdot w \, ,$$

where (λ, μ, ν) are the barycentric coordinates of the point p_j w.r.t the affine frame

$$[f_v(v), f_v(u), f_v(w)] \, .$$



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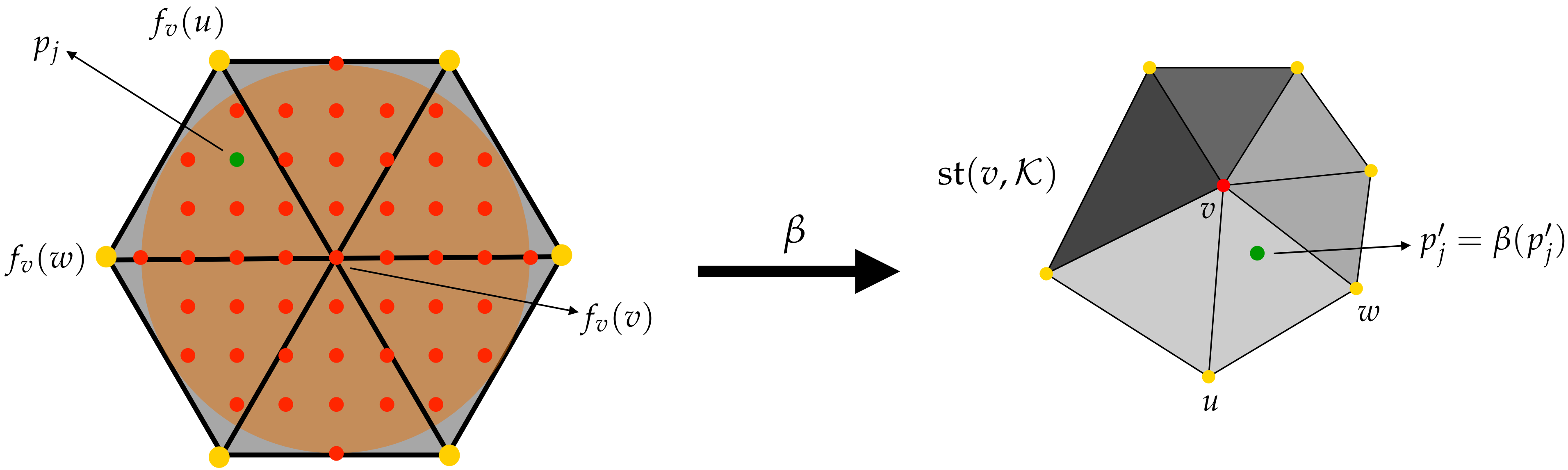
Note that β must be piecewise defined in $|\mathcal{K}_u|$ (i.e., it varies in each triangle of $\mathcal{K}_u$).



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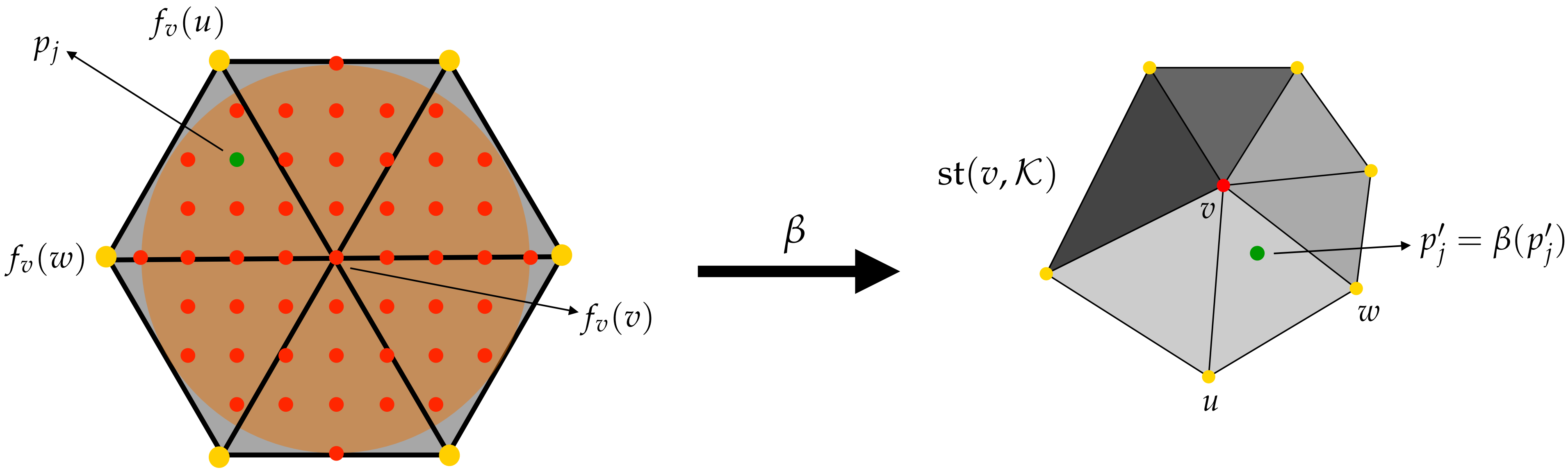
We then assemble three linear equation systems, $AX = B_l$, with $l = 1, 2, 3$, each of which has exactly E_v equations in $(m+1) \times (n+1)$ unknowns, where $m = n = n_v$.



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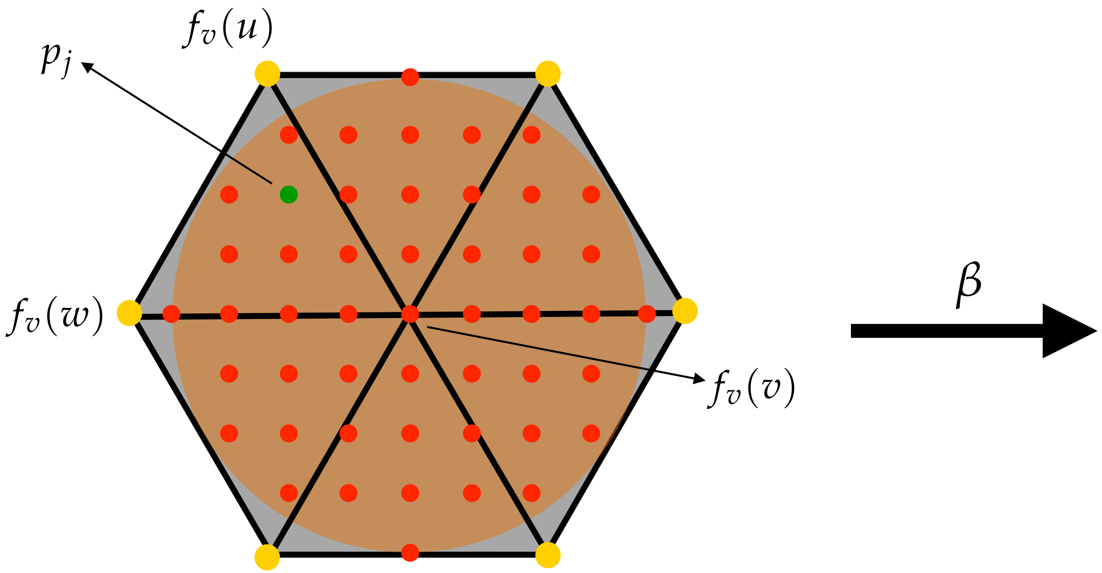
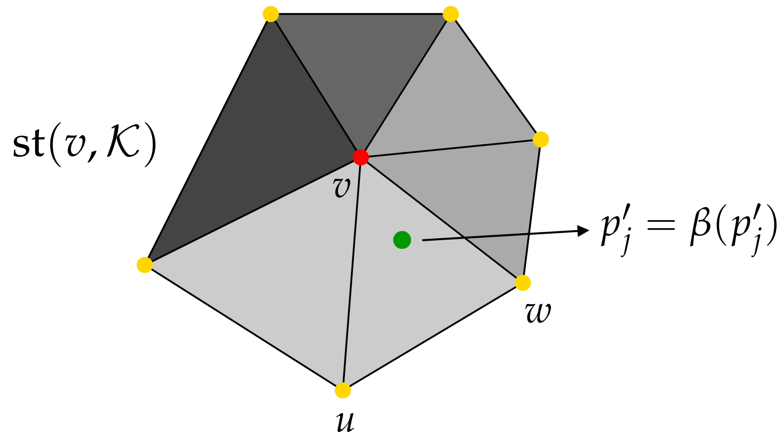
In our current implementation, we set $E_v = (2 \cdot n_v + 1)^2$. Observe that the value of E_v is, in general, not the same for any two p-domains, as it is expressed in terms of n_v.





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The linear equations of the systems $AX = B_l$, for $l = 1, 2, 3$, come from the equalities

$$p'_j = \psi_v(p_j) \Longrightarrow (x'_j, y'_j, z'_j) = \sum_{0 \le j \le m} \sum_{0 \le k \le n} B^m_j(x_j) \cdot B^n_k(y_j) \cdot (x^v_{j,k}, y^v_{j,k}, z^v_{j,k}) \,,$$

for all $j \in J$, where (x_j, y_j), (x'_j, y'_j, z'_j), and $(x^v_{j,k}, y^v_{j,k}, z^v_{j,k})$ are the coordinates of p_j, p'_j, and $b^v_{j,k}$.

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So, $AX = B_1$ consists of E_v linear equations of the form

$$x'_j = \sum_{0 \le j \le m} \sum_{0 \le k \le n} B_j^m(x_j) \cdot B_k^n(y_j) \cdot x_{j,k}^v \,,$$

$AX = B_2$ consists of E_v linear equations of the form

$$y'_j = \sum_{0 \le j \le m} \sum_{0 \le k \le n} B_j^m(x_j) \cdot B_k^n(y_j) \cdot y_{j,k}^v \,,$$

and $AX = B_3$ consists of E_v linear equations of the form

$$z'_j = \sum_{0 \le j \le m} \sum_{0 \le k \le n} B_j^m(x_j) \cdot B_k^n(y_j) \cdot z_{j,k}^v \,.$$

Each equation has $(n_v + 1)^2$ unknowns. So, A has E_v rows and $(n_v + 1)^2$ columns.

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Building Parametrizations

Note that the $(n_v + 1)^2$ unknowns of $AX = B_l$ are the l-th coordinates of the $b^v_{j,k}$'s.

Since $E_v > (n_v + 1)^2$, the system $AX = B_1$ has more equations than unknowns. So, we compute the normal equations, $A^t AX = A^t B_1$, and then solve $A^t AX = A^t B_1$ for X.

$A^t AX = A^t B_1$ admits a unique solution iff $A^t A$ has rank $(n_v + 1)^2$.

We can proceed in a similar fashion to solve $AX = B_2$ and $AX = B_3$.

Once we solve $AX = B_l$, for $l = 1, 2, 3$, we have the control points $b^v_{j,k}$, and thus ψ_v.

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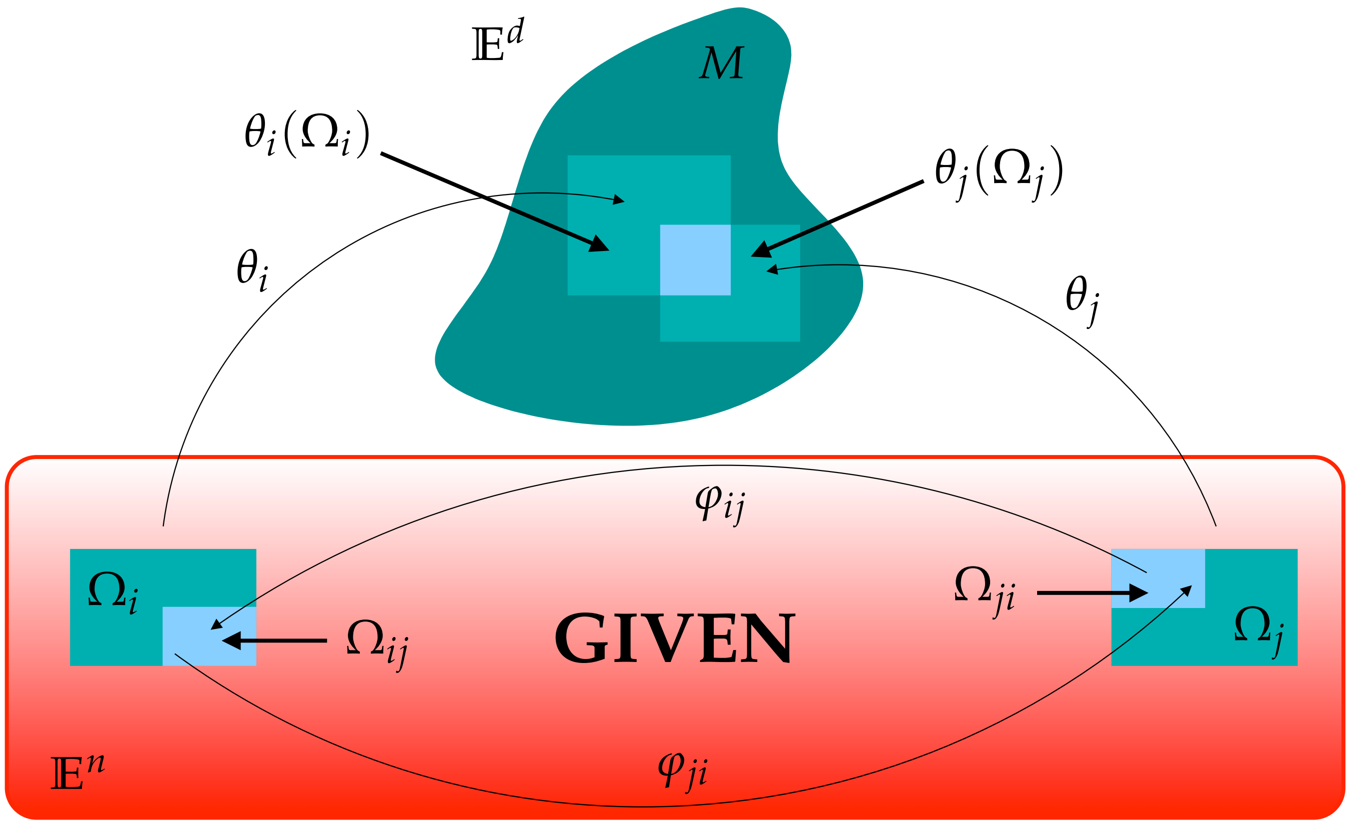
Building Parametrizations

Ultimately, we want to compute θ_v:



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Building Parametrizations

Why not let $\theta_v = \psi_v$?

The main issue here is that $\theta_v(p)$ must be the same point as $\theta_u(q)$ whenever $q = \varphi_{uv}(p)$.

However, it is *extremely* unlikely that $\psi_v(p) = \psi_u(q)$ whenever $q = \varphi_{uv}(p)$.

The reason is that the control points of ψ_v and ψ_u are computed independently.

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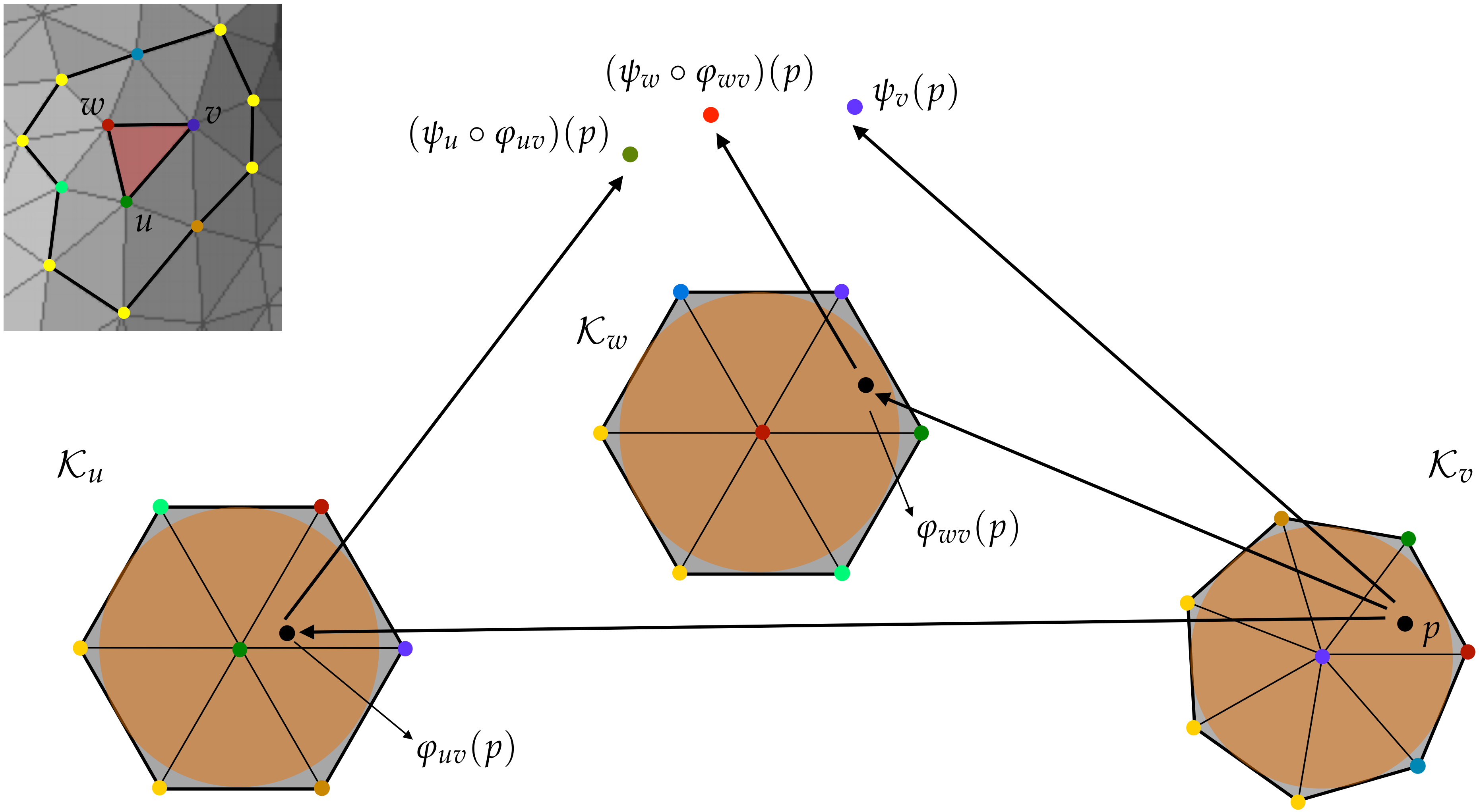

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So, what can we do?

We will use the same resource we used for the one-dimensional case: partition of unity.

For each $v \in I$, we define the *bump function,* $\gamma_v : \mathbb{E}^2 \to \mathbb{R}$*, associated with* Ω_v such that

$$\gamma_v(p) = \gamma_v(x, y) = \xi\left(\sqrt{x^2 + y^2}\right) ,$$

for every $p = (x, y) \in \mathbb{E}^2$, and $\xi : \mathbb{R} \to \mathbb{R}$ is the same map ξ of the one-dimensional case.

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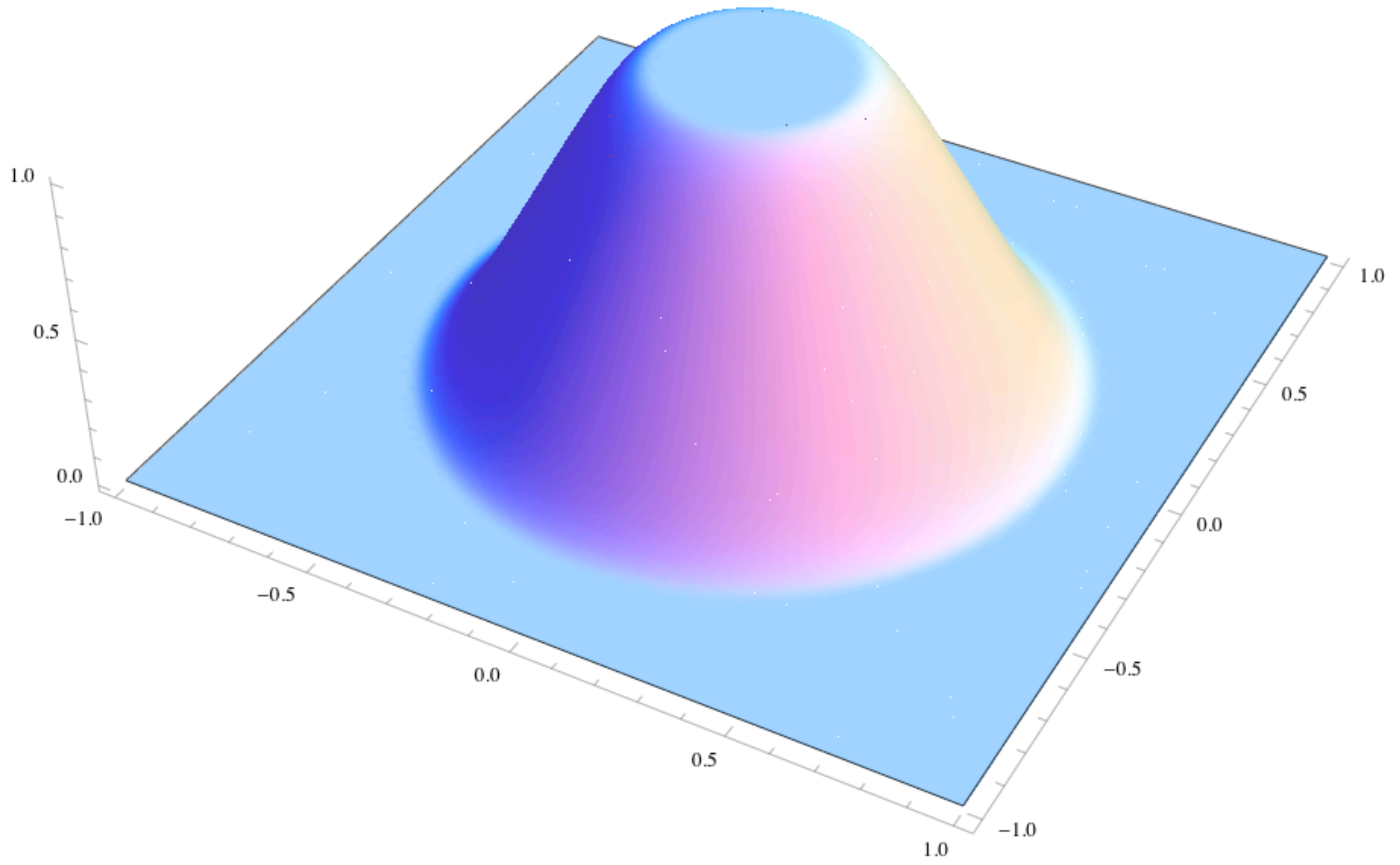

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Recall...

For every $t \in \mathbb{R}$, we define

$$\xi : \mathbb{R} \to \mathbb{R}$$

as

$$\xi(t) = \begin{cases} 1 & \text{if } t \leq H_1 \\ 0 & \text{if } t \geq H_2 \\ 1/(1 + e^{2 \cdot s}) & \text{otherwise} \end{cases}$$

where H_1, H_2 are constant, with $0 < H_1 < H_2 < 1$,

$$s = \left(\frac{1}{\sqrt{1 - H}} \right) - \left(\frac{1}{\sqrt{H}} \right) \quad \text{and} \quad H = \left(\frac{t - H_1}{H_2 - H_1} \right) .$$

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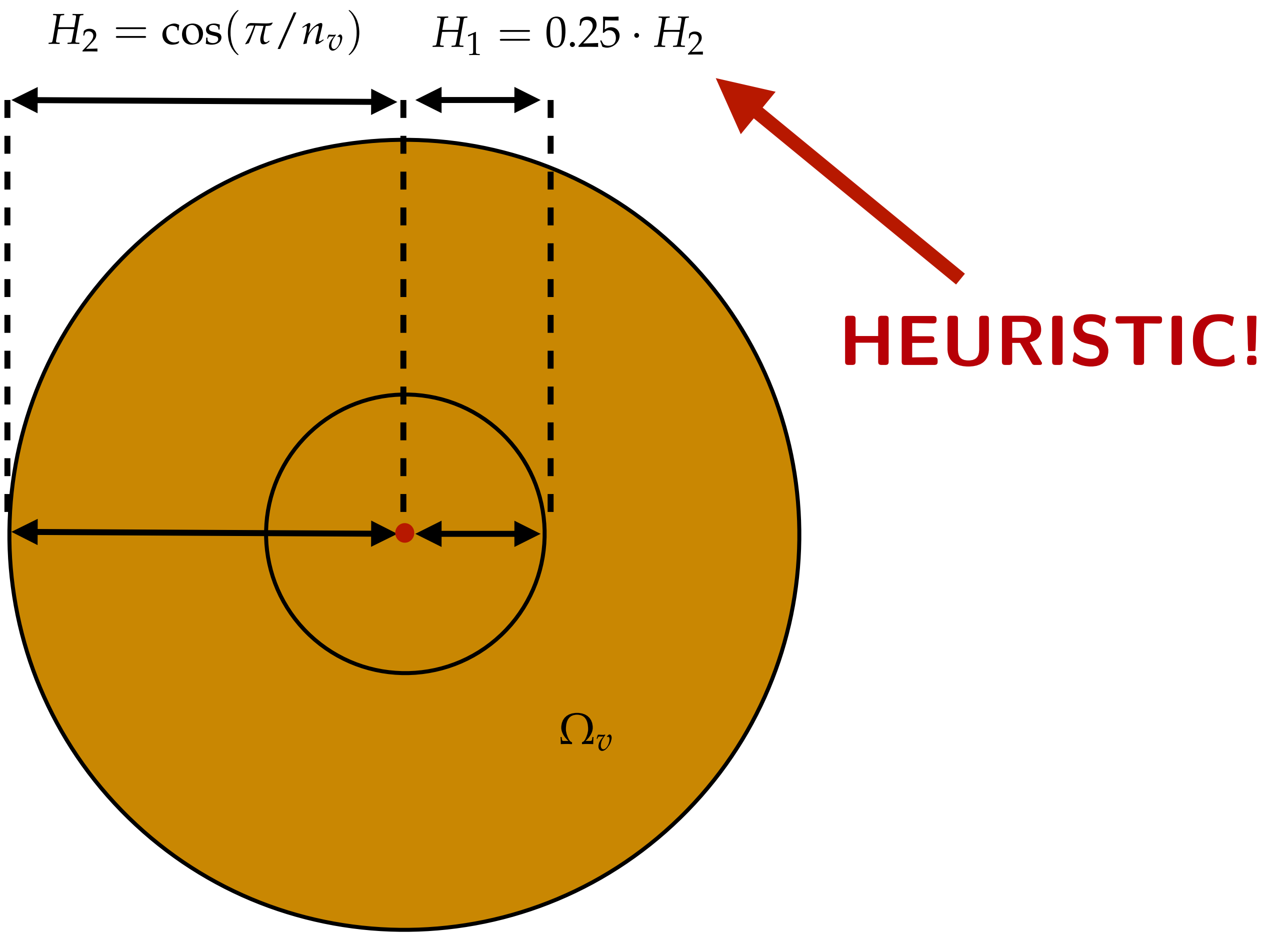

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Finally, we define

$$\theta_v : \Omega_v \to \mathbb{E}^3$$

as

$$\theta_v(p) = \frac{\sum_{z \in J_v(p)} (\psi_z \circ \varphi_{zv})(p) \cdot (\gamma_z \circ \varphi_{zv})(p)}{\sum_{z \in J_v(p)} (\gamma_z \circ \varphi_{zv})(p)} ,$$

for every $p \in \Omega_v$, where

$$J_v(p) = \{u \in I \mid p \in \Omega_{vu}\} .$$

$J_v(p)$ has *at least* one vertex (i.e., v) and *at most* 3 (i.e., v plus one or two others).

We can show that $\theta_v(p) = (\theta_u \circ \varphi_{uv})(p) = (\theta_w \circ \varphi_{wv})(p)$ whenever $p \in (\Omega_{vu} \cap \Omega_{vw})$.

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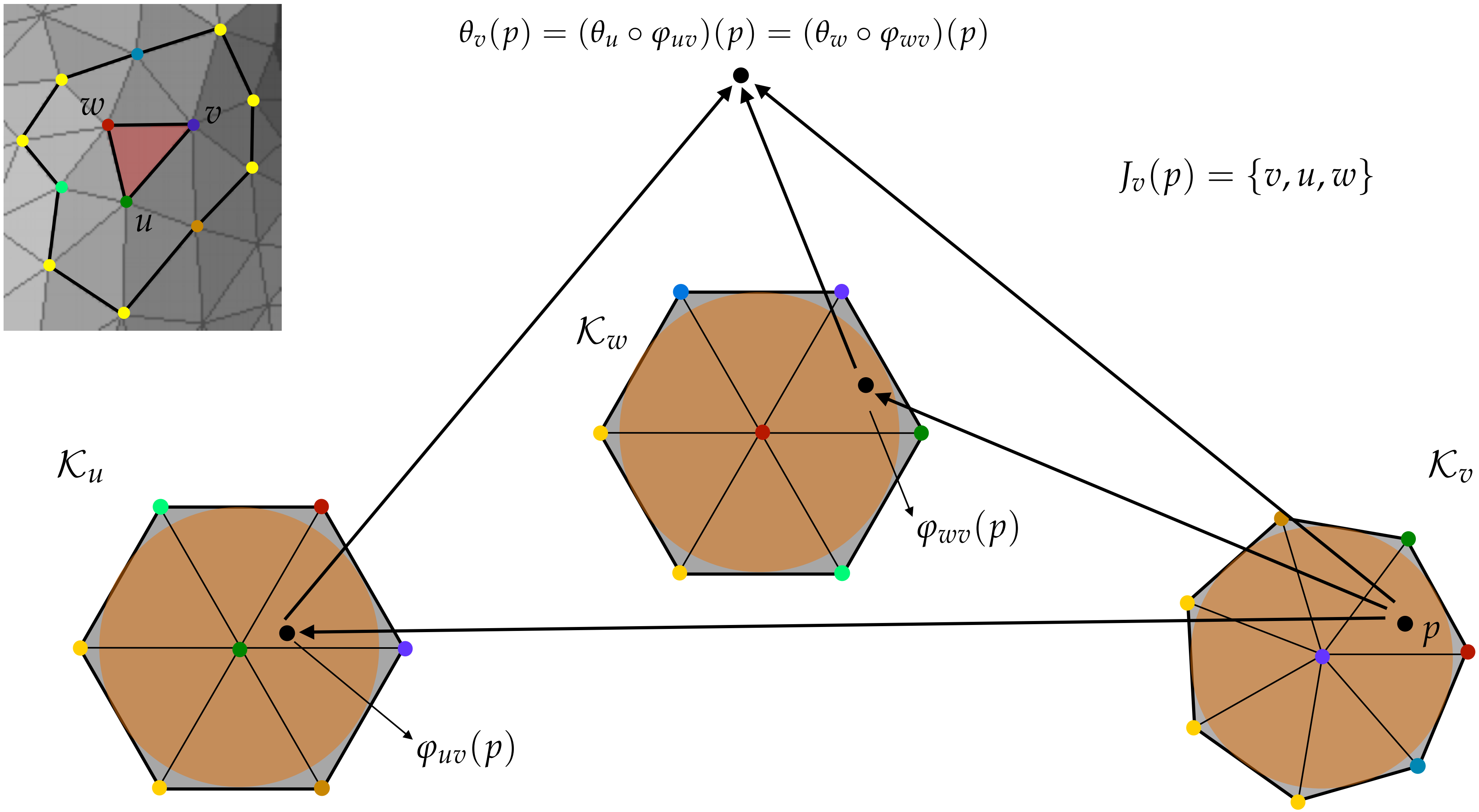

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Building Parametrizations

If $J_v(p) = \{v\}$ then

$$\begin{aligned}
\theta_v(p) &= \frac{\sum_{z \in J_v(p)} (\psi_z \circ \varphi_{zv})(p) \cdot (\gamma_z \circ \varphi_{zv})(p)}{\sum_{z \in J_v(p)} (\gamma_z \circ \varphi_{zv})(p)} \\
&= \frac{(\psi_v \circ \varphi_{vv})(p) \cdot (\gamma_v \circ \varphi_{vv})(p)}{(\gamma_v \circ \varphi_{vv})(p)} \\
&= (\psi_v \circ \varphi_{vv})(p) \\
&= (\psi_v \circ \mathrm{id}_{\Omega_v})(p) \\
&= \psi_v(p) \,.
\end{aligned}$$

Parametric Pseudo-Manifolds

Building Parametrizations

If $J_v(p) = \{v, u\}$ then

$$\begin{aligned}
\theta_v(p) &= \frac{\sum_{z \in J_v(p)} (\psi_z \circ \varphi_{zv})(p) \cdot (\gamma_z \circ \varphi_{zv})(p)}{\sum_{z \in J_v(p)} (\gamma_z \circ \varphi_{zv})(p)} \\
&= \frac{(\psi_v \circ \varphi_{vv})(p) \cdot (\gamma_v \circ \varphi_{vv})(p) + (\psi_u \circ \varphi_{uv})(p) \cdot (\gamma_u \circ \varphi_{uv})(p)}{(\gamma_v \circ \varphi_{vv})(p) + (\gamma_u \circ \varphi_{uv})(p)} \\
&= \frac{(\psi_v \circ \mathrm{id}_{\Omega_v})(p) \cdot (\gamma_v \circ \mathrm{id}_{\Omega_v})(p) + (\psi_u \circ \varphi_{uv})(p) \cdot (\gamma_u \circ \varphi_{uv})(p)}{(\gamma_v \circ \mathrm{id}_{\Omega_v})(p) + (\gamma_u \circ \varphi_{uv})(p)} \\
&= \frac{\psi_v(p) \cdot \gamma_v(p) + (\psi_u \circ \varphi_{uv})(p) \cdot (\gamma_u \circ \varphi_{uv})(p)}{\gamma_v(p) + (\gamma_u \circ \varphi_{uv})(p)} .
\end{aligned}$$

Parametric Pseudo-Manifolds

Building Parametrizations

If $J_v(p) = \{v, u, w\}$ then

$$
\begin{aligned}
\theta_v(p) &= \frac{\sum_{z \in J_v(p)} (\psi_z \circ \varphi_{zv})(p) \cdot (\gamma_z \circ \varphi_{zv})(p)}{\sum_{z \in J_v(p)} (\gamma_z \circ \varphi_{zv})(p)} \\
&= \frac{(\psi_v \circ \varphi_{vv})(p) \cdot (\gamma_v \circ \varphi_{vv})(p) + (\psi_u \circ \varphi_{uv})(p) \cdot (\gamma_u \circ \varphi_{uv})(p) + (\psi_w \circ \varphi_{wv})(p) \cdot (\gamma_w \circ \varphi_{wv})(p)}{(\gamma_v \circ \varphi_{vv})(p) + (\gamma_u \circ \varphi_{uv})(p) + (\gamma_w \circ \varphi_{wv})(p)} \\
&= \frac{(\psi_v \circ \mathrm{id}_{\Omega_v})(p) \cdot (\gamma_v \circ \mathrm{id}_{\Omega_v})(p) + (\psi_u \circ \varphi_{uv})(p) \cdot (\gamma_u \circ \varphi_{uv})(p) + (\psi_w \circ \varphi_{wv})(p) \cdot (\gamma_w \circ \varphi_{wv})(p)}{(\gamma_v \circ \mathrm{id}_{\Omega_v})(p) + (\gamma_u \circ \varphi_{uv})(p) + (\gamma_w \circ \varphi_{wv})(p)} \\
&= \frac{\psi_v(p) \cdot \gamma_v(p) + (\psi_u \circ \varphi_{uv})(p) \cdot (\gamma_u \circ \varphi_{uv})(p) + (\psi_w \circ \varphi_{wv})(p) \cdot (\gamma_w \circ \varphi_{wv})(p)}{\gamma_v(p) + (\gamma_u \circ \varphi_{uv})(p) + (\gamma_w \circ \varphi_{wv})(p)} .
\end{aligned}
$$

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In

$$\theta_v(p) = \frac{\sum_{z \in J_v(p)} (\psi_z \circ \varphi_{zv})(p) \cdot (\gamma_z \circ \varphi_{zv})(p)}{\sum_{z \in J_v(p)} (\gamma_z \circ \varphi_{zv})(p)},$$

the term

$$(\psi_z \circ \varphi_{zv})(p)$$

can be viewed as the *contribution* of ψ_z to $\theta_v(p)$, which is weighted by $(\gamma_z \circ \varphi_{zv})(p)$.

A key observation for the proof of consistency: if $w \in J_v(p)$ then $J_w(\varphi_{wv}(p)) = J_v(p)$.

All functions involved in the definition of θ_v are C^∞.

Finally, our surface is defined as $\bigcup_{v \in I} \theta_v(\Omega_v)$.

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Building Parametrizations

There is only one issue with the construction of S: the sample points, p'_j, were located in the surface $|\mathcal{K}|$, which is piecewise-linear. As a result, S will *look* piecewise-linear too!

To improve the visual quality of S, we define the parametrization θ_v as a local approximation for a "curved" geometry. In order to do so, we assume that a parametric surface, say S', has been defined over the simplicial surface, $\mathcal{K}$. There are many choices!

Two simple choices are:

- PN triangles
- Subdivision surfaces

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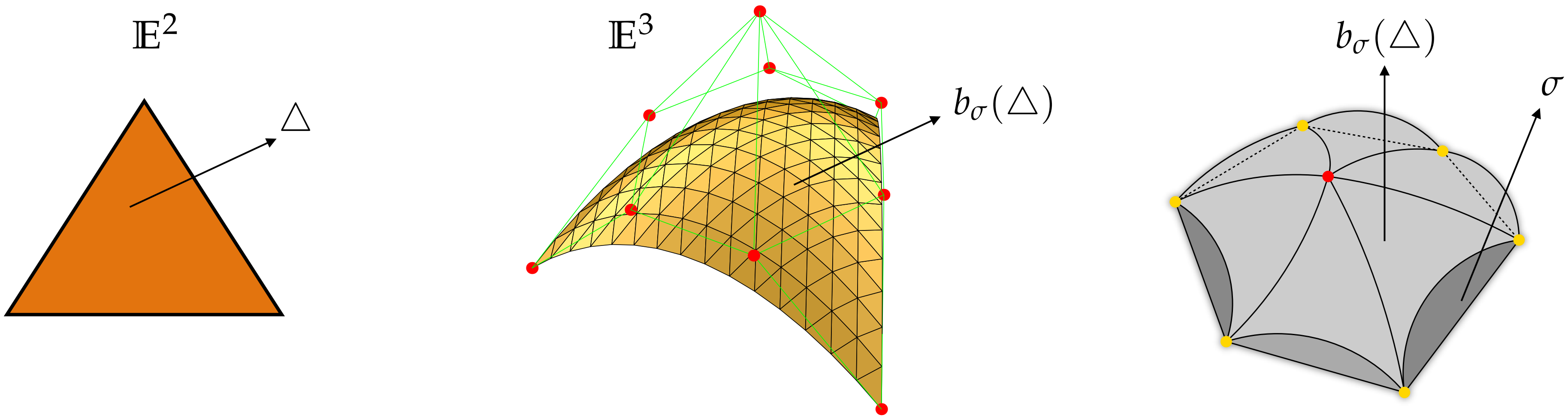
Regardless of the choice of S', we assume that S' is a union of parametric patches given by

$$b_\sigma : \triangle \subset \mathbb{E}^2 \rightarrow \mathbb{E}^3\,,$$

where each b_σ is associated with a triangle σ of $\mathcal{K}$ and is defined on a triangle, $\triangle \subset \mathbb{E}^2$; i.e,

$$S' = \bigcup_{\sigma \in \mathcal{K}^{(2)}} b_\sigma(\triangle)\,.$$



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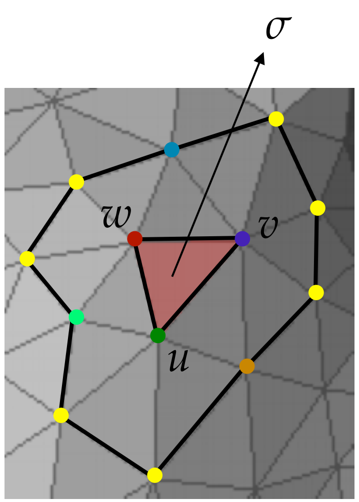
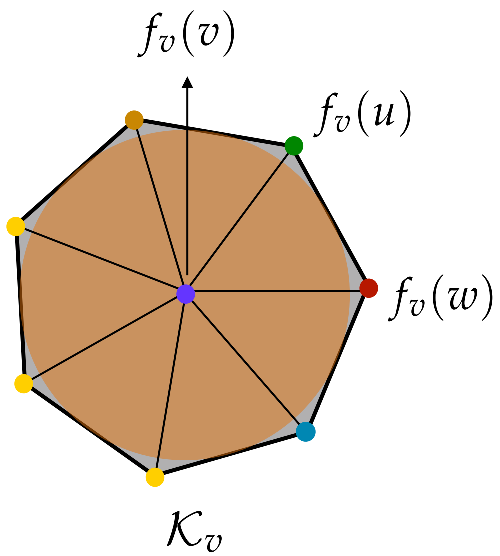
Suppose that $\sigma = [v, u, w]$.





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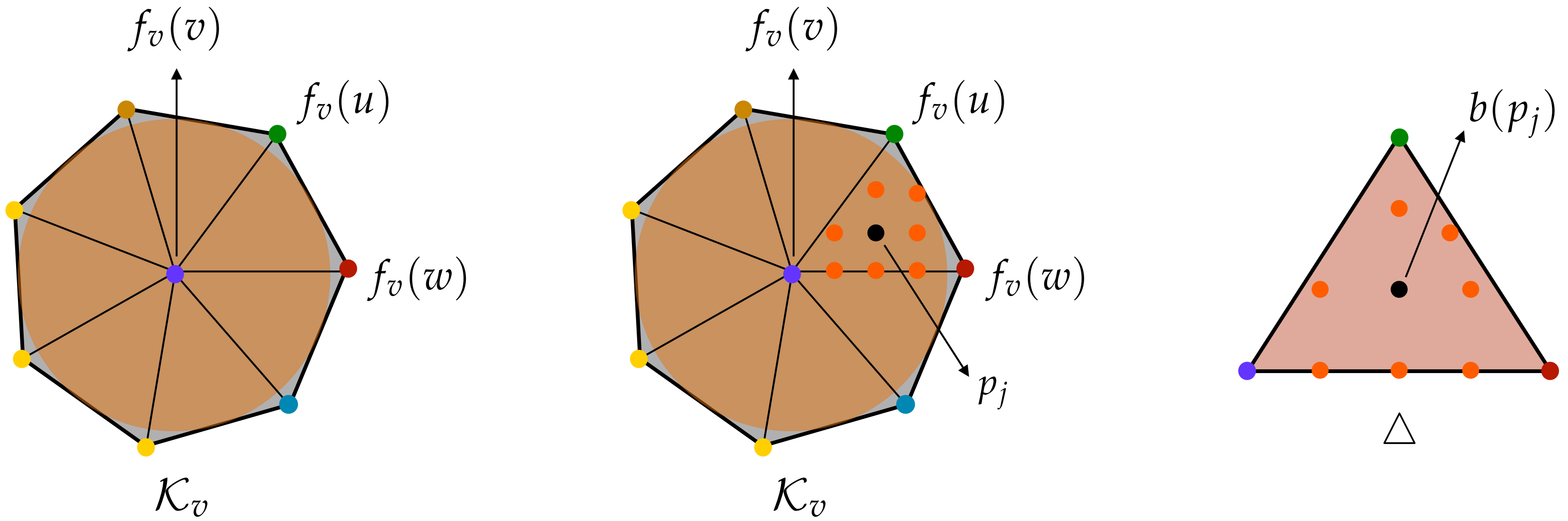
Building Parametrizations

After sampling Ω_v, we map the points p_j inside the triangle $[f_v(v), f_v(u), f_v(w)]$ to the triangle $\triangle$ using a barycentric map, say b, and then we compute the points $p'_j = (b_\sigma \circ b)(p_j)$.



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Building Parametrizations

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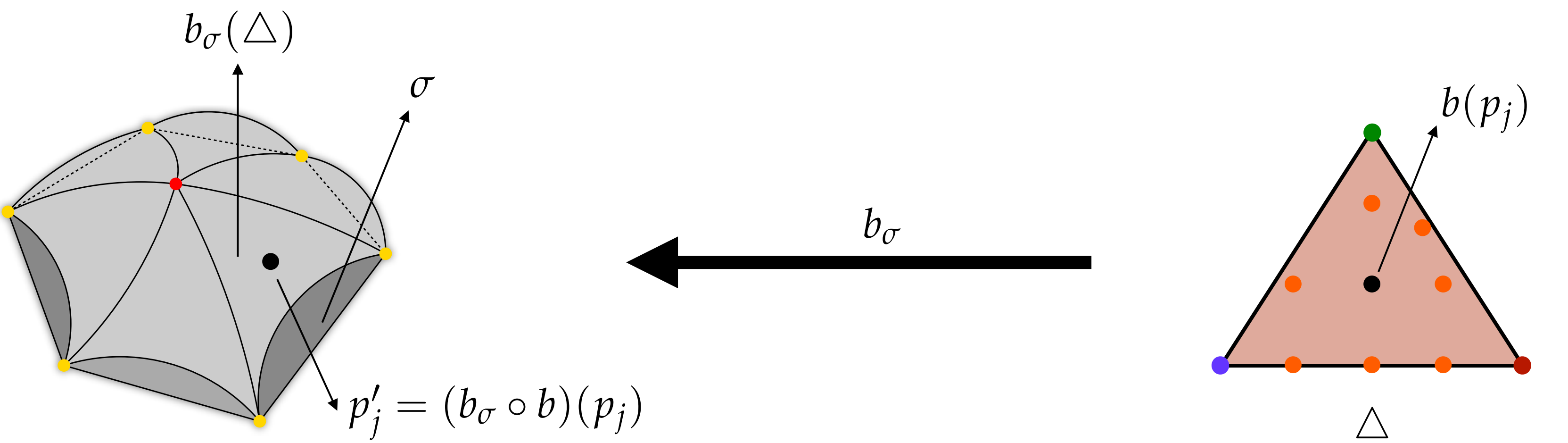

So, our *given* function β can be piecewise defined as $\beta = b_\sigma \circ b$ in each p-domain.

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Building Parametrizations

Once we have the pairs (p_j, p'_j) for each p-domain Ω_v, we can proceed as before to compute the control points of ψ_v, which is a Bézier surface patch of bi-degree (n_v, n_v).

However, since we locally approximate the shape of a "curved" geometry (i.e., the surface $S' = \bigcup_{\sigma \in \mathcal{K}^{(2)}} b_\sigma(\triangle)$), our surface, $S = \bigcup_{v \in I} \theta_v(\Omega_v)$, has a curved geometry too.

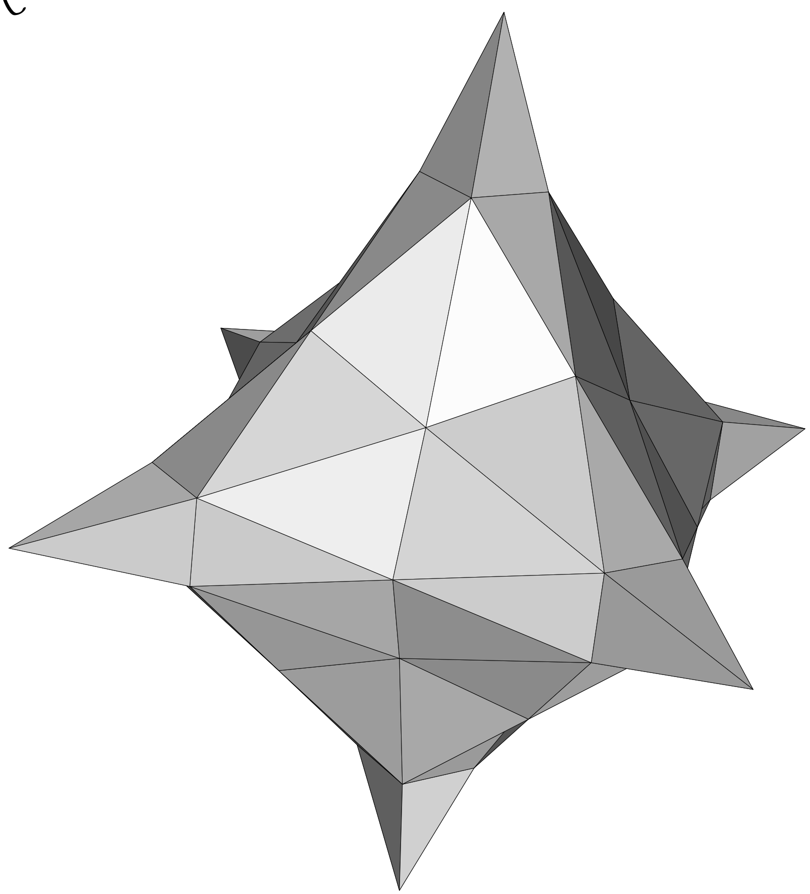
More specifically, the shape of S is very similar to the shape of S', but S is smooth (i.e., C^∞) regardless of the degree of smoothness of the surface S', which should be at least C^0.

Let us see some examples...

Parametric Pseudo-Manifolds

Examples

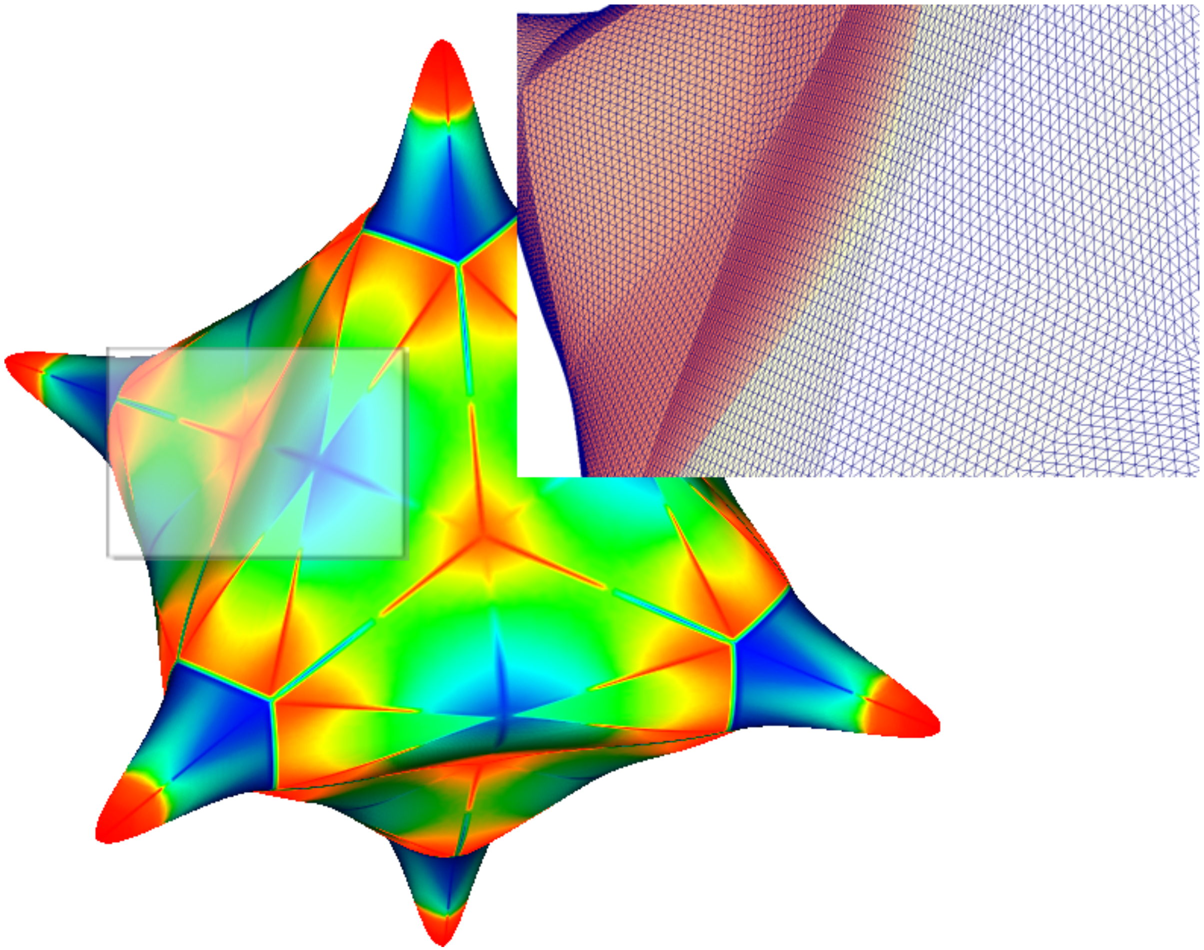
simplicial surface $\mathcal{K}$



Parametric Pseudo-Manifolds

Examples

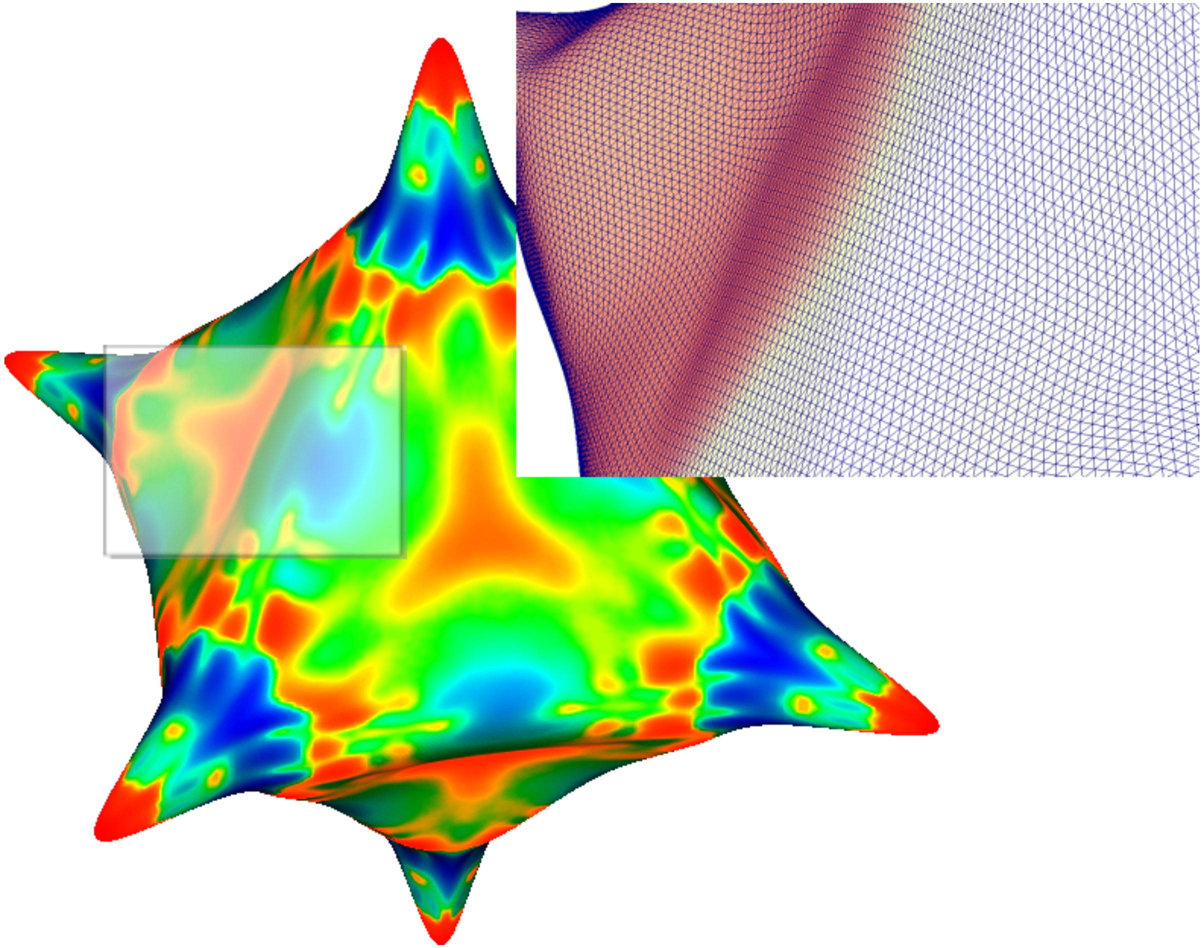
PN triangle



Parametric Pseudo-Manifolds

Examples

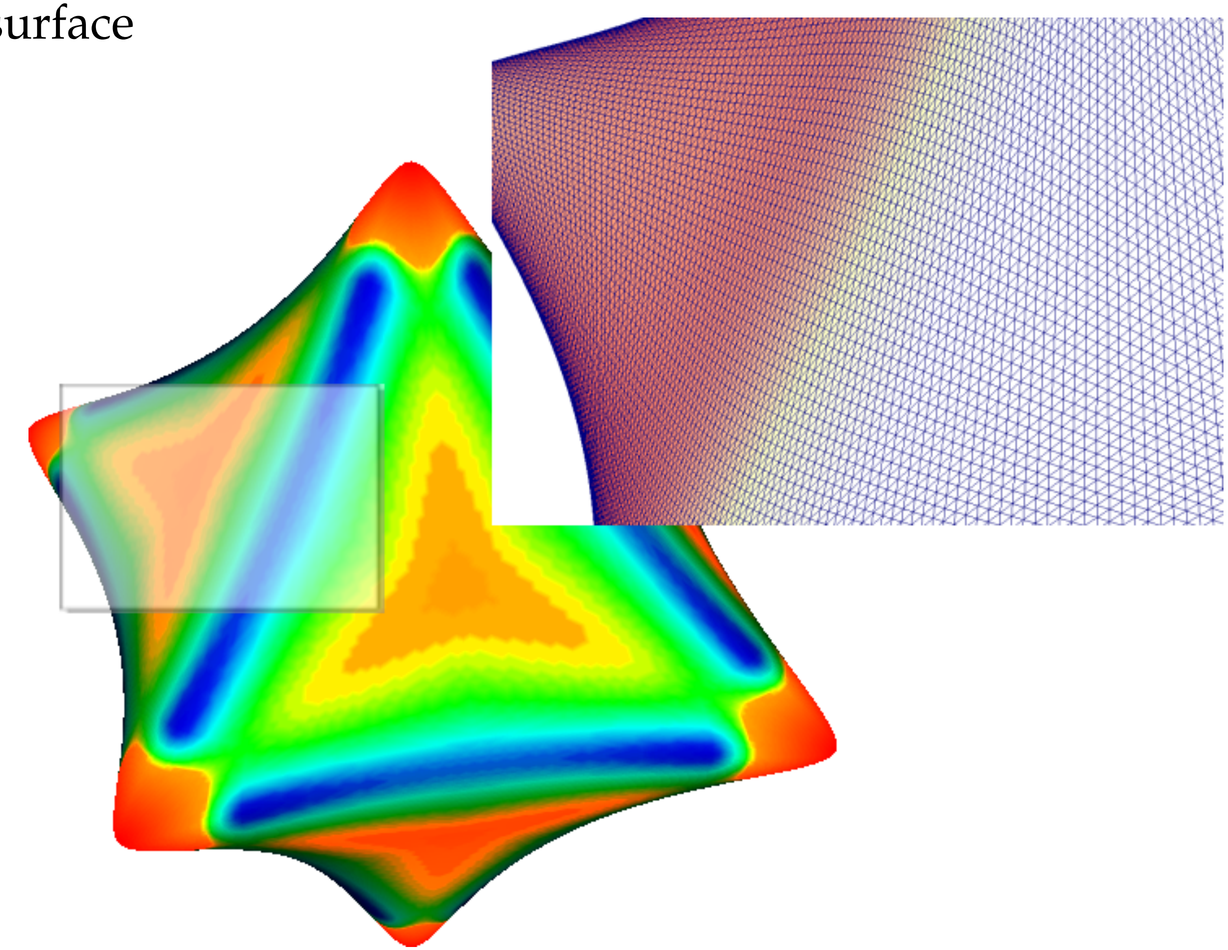
surface S



Parametric Pseudo-Manifolds

Examples

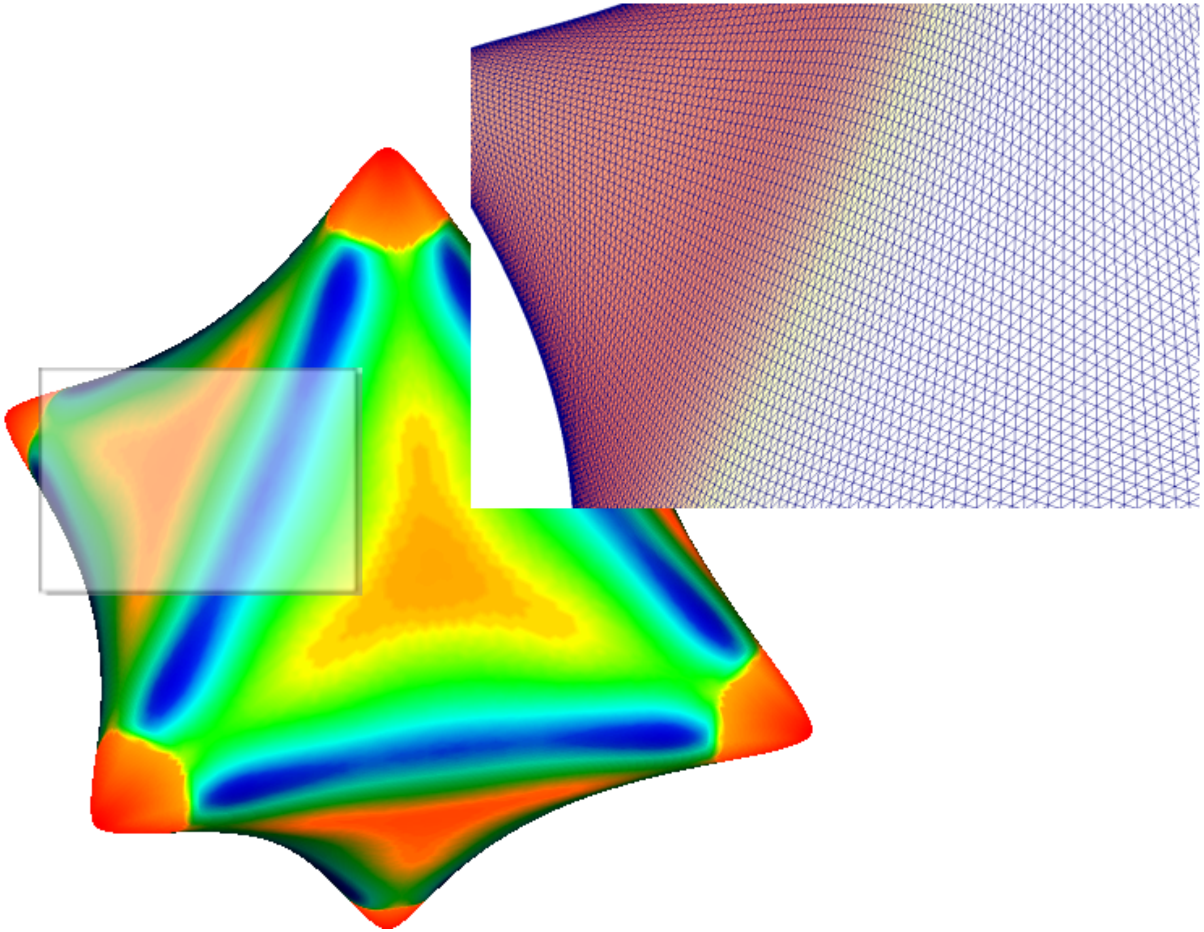
Loop subdivision surface



Parametric Pseudo-Manifolds

Examples

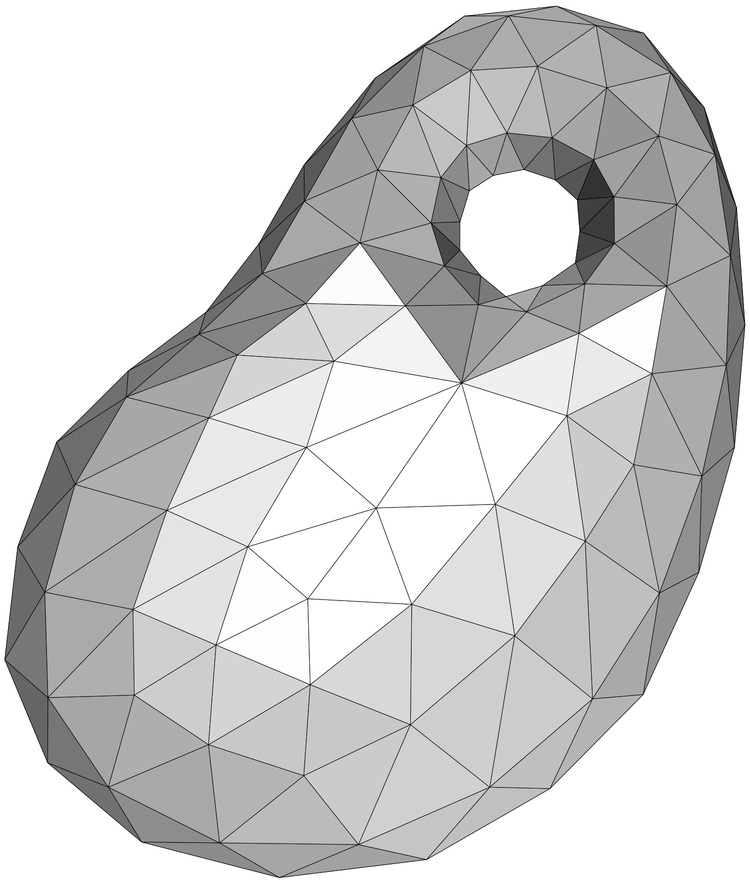
surface S



Parametric Pseudo-Manifolds

Examples

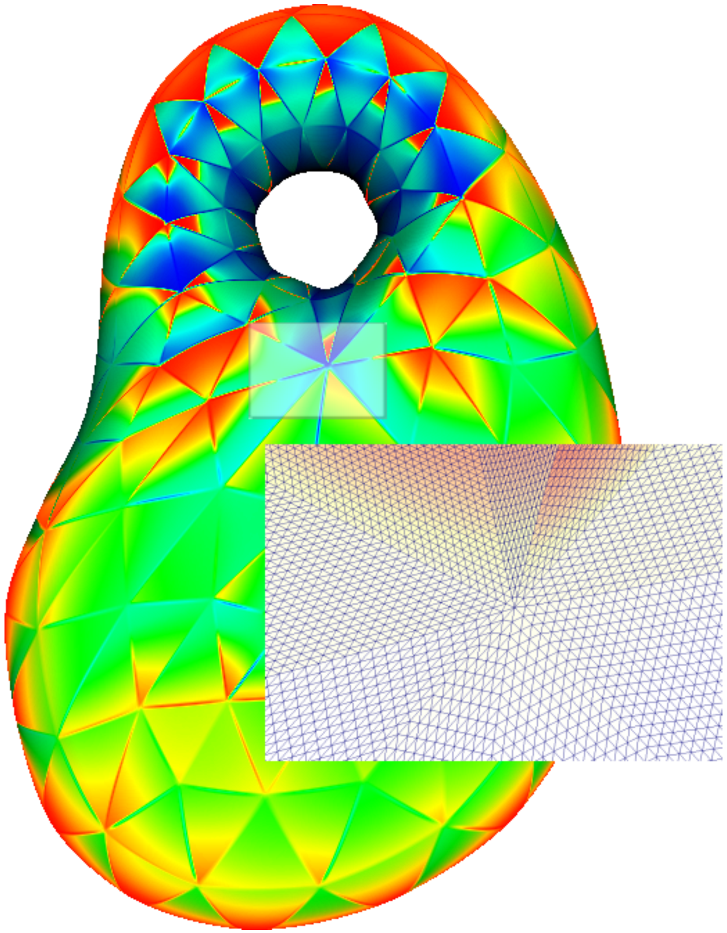
simplicial surface $\mathcal{K}$



Parametric Pseudo-Manifolds

Examples

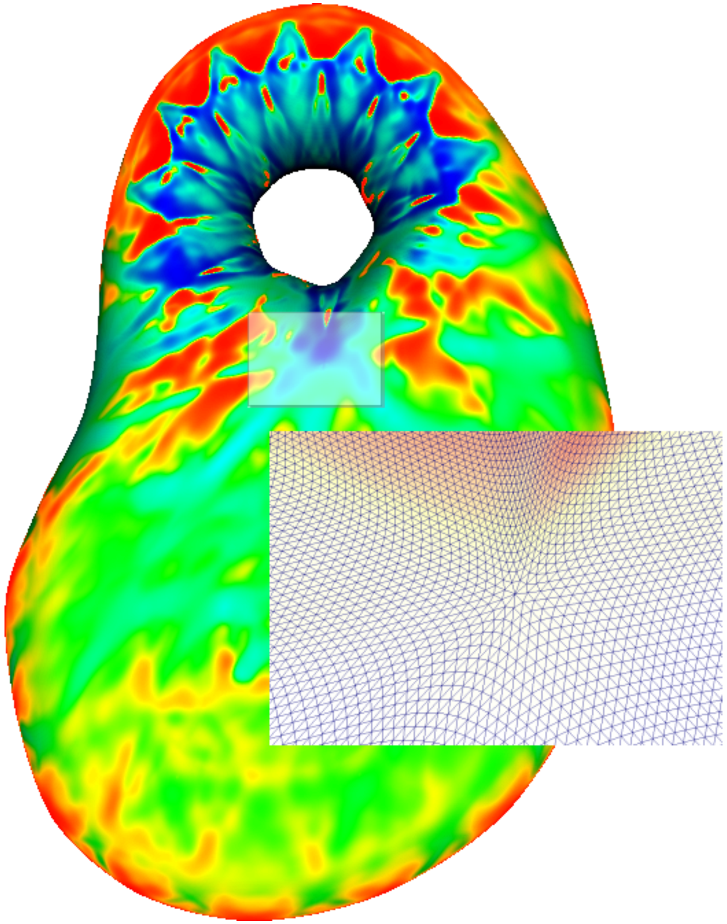
PN triangle



Parametric Pseudo-Manifolds

Examples

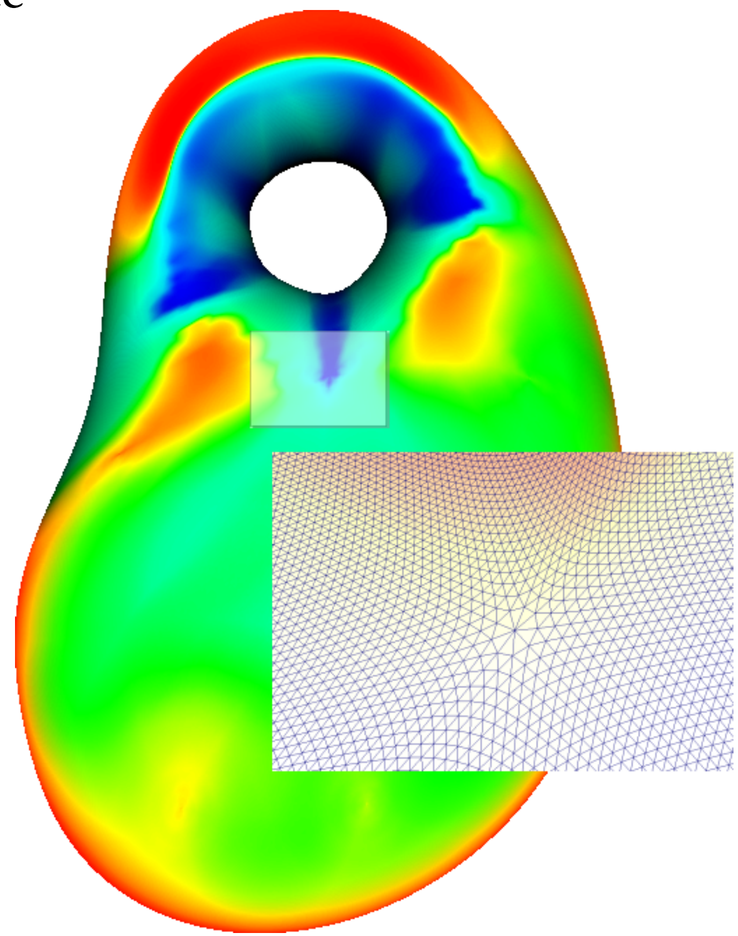
surface S



Parametric Pseudo-Manifolds

Examples

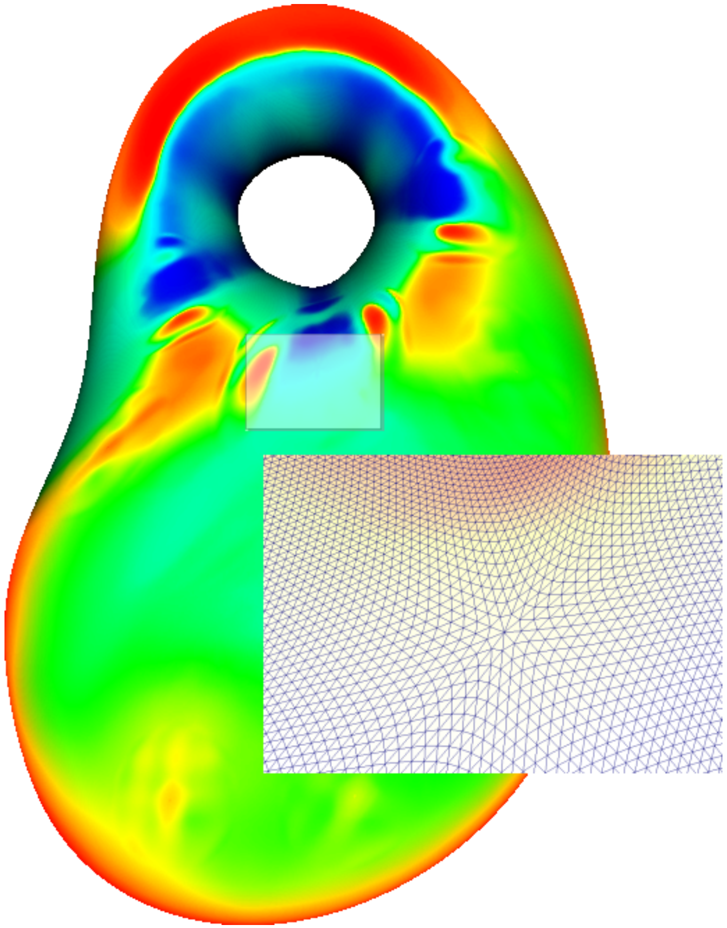
Loop subdivision surface



Parametric Pseudo-Manifolds

Examples

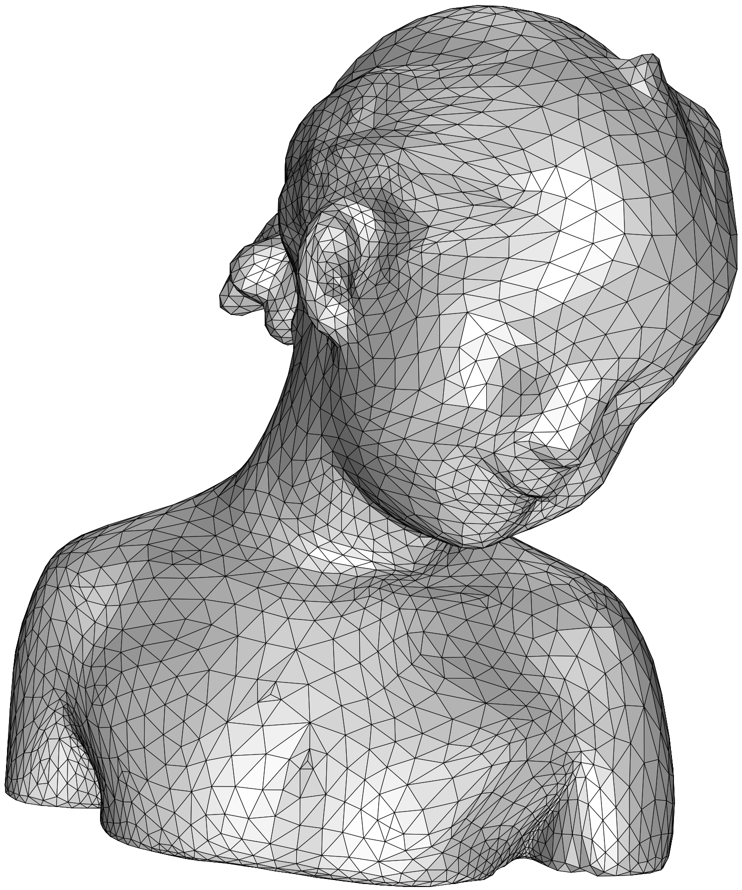
surface S



Parametric Pseudo-Manifolds

Examples

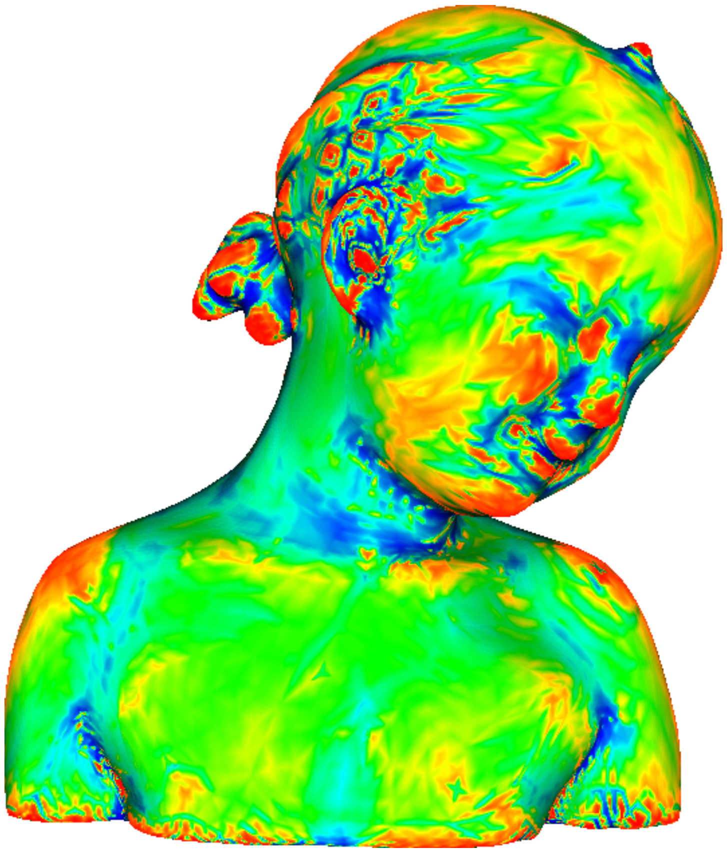
simplicial surface $\mathcal{K}$



Parametric Pseudo-Manifolds

Examples

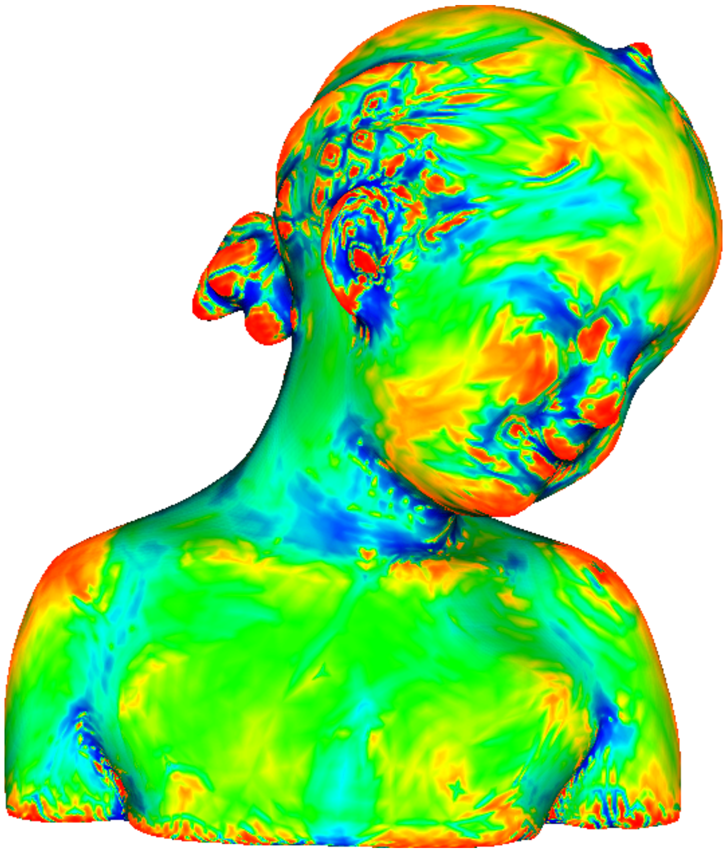
PN triangle



Parametric Pseudo-Manifolds

Examples

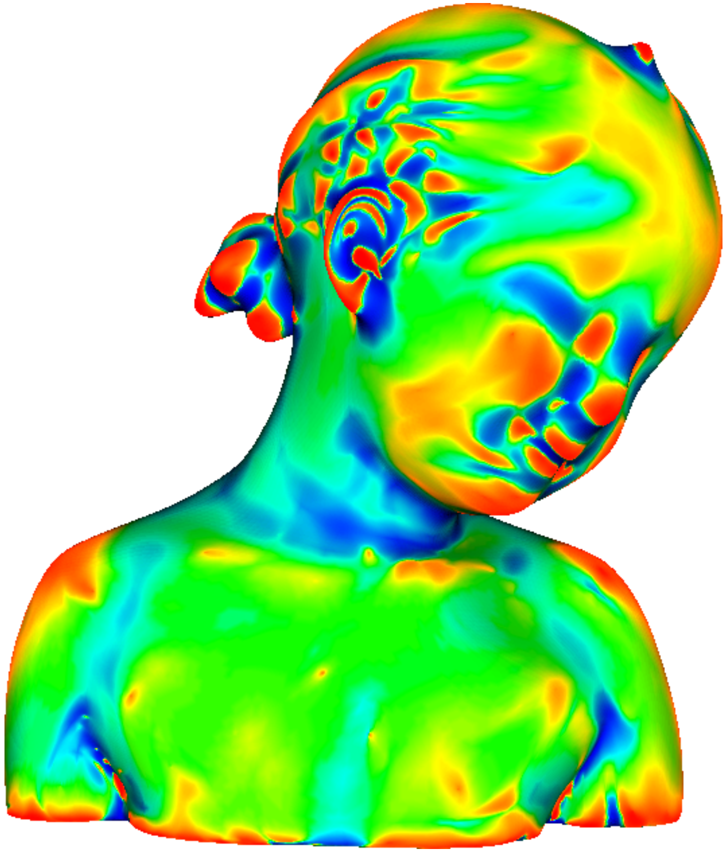
surface S



Parametric Pseudo-Manifolds

Examples

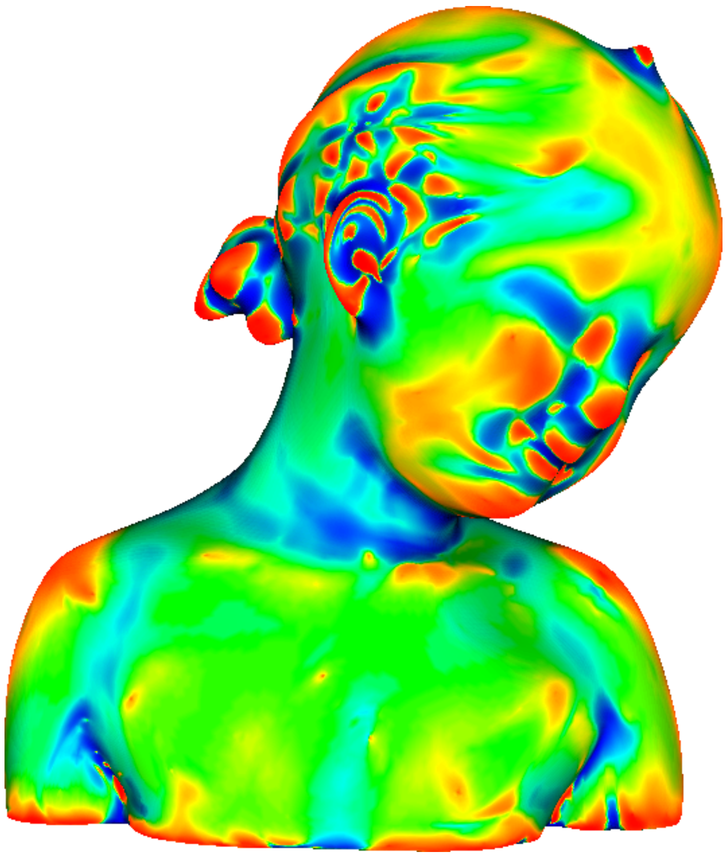
Loop subdivision surface



Parametric Pseudo-Manifolds

Examples

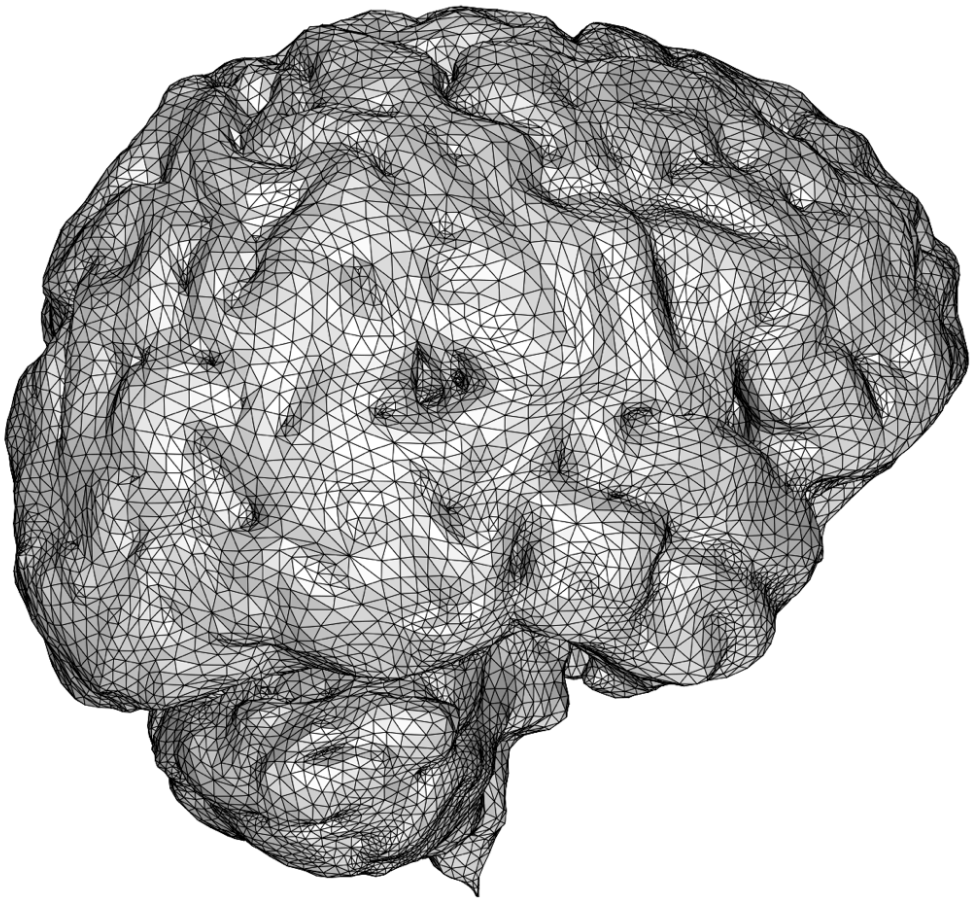
surface S



Parametric Pseudo-Manifolds

Examples

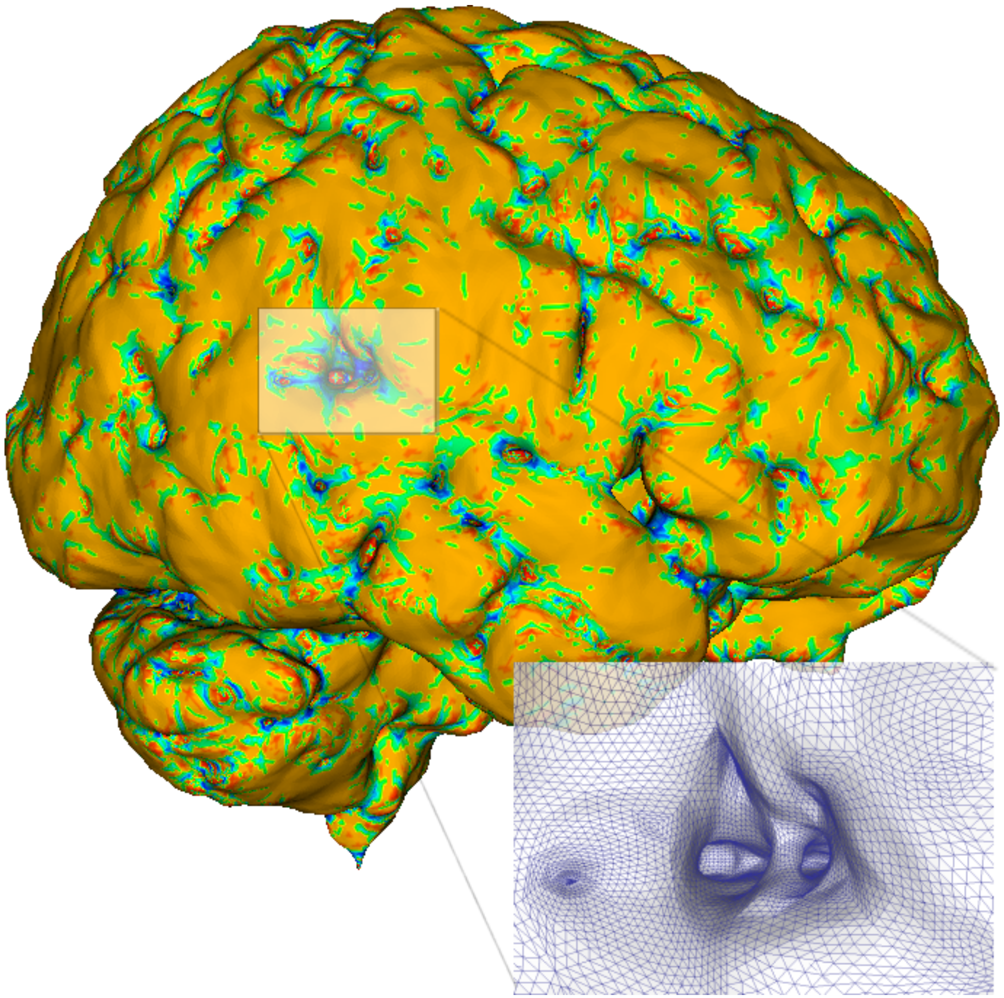
simplicial surface $\mathcal{K}$



Parametric Pseudo-Manifolds

Examples

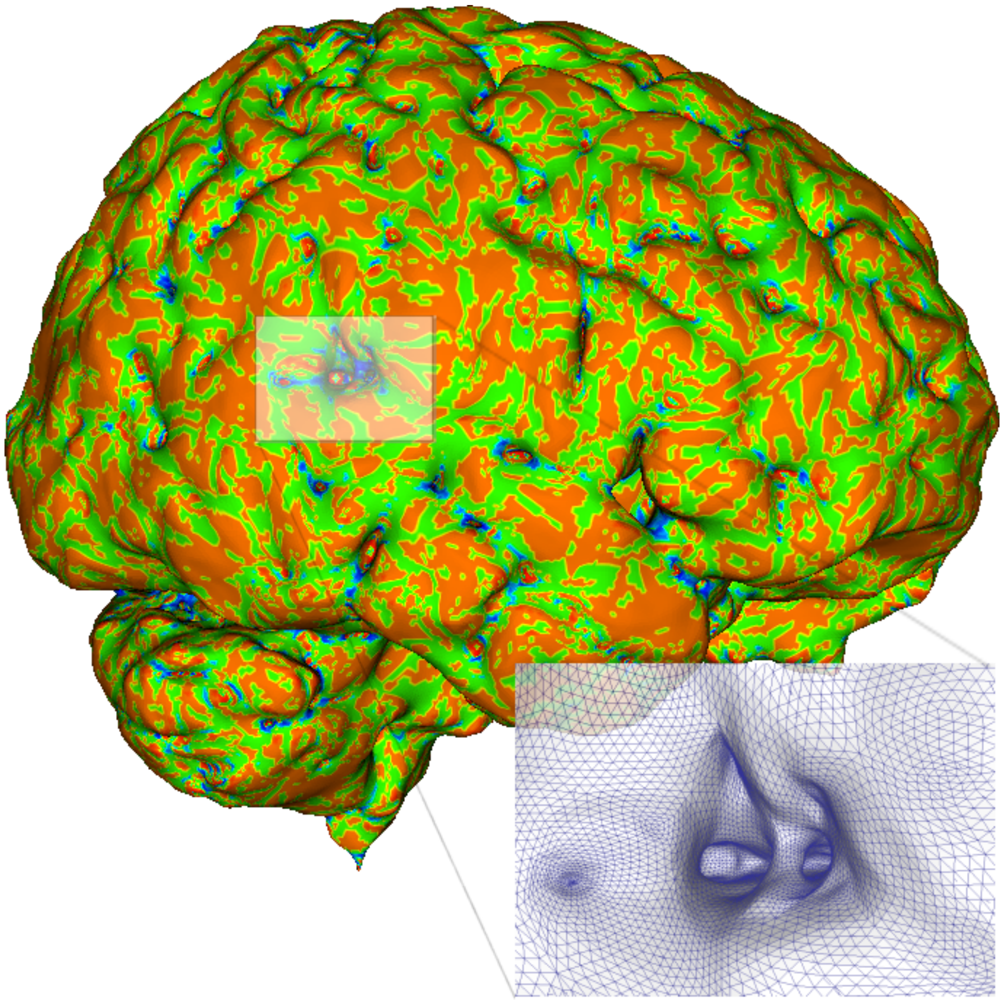
PN triangle



Parametric Pseudo-Manifolds

Examples

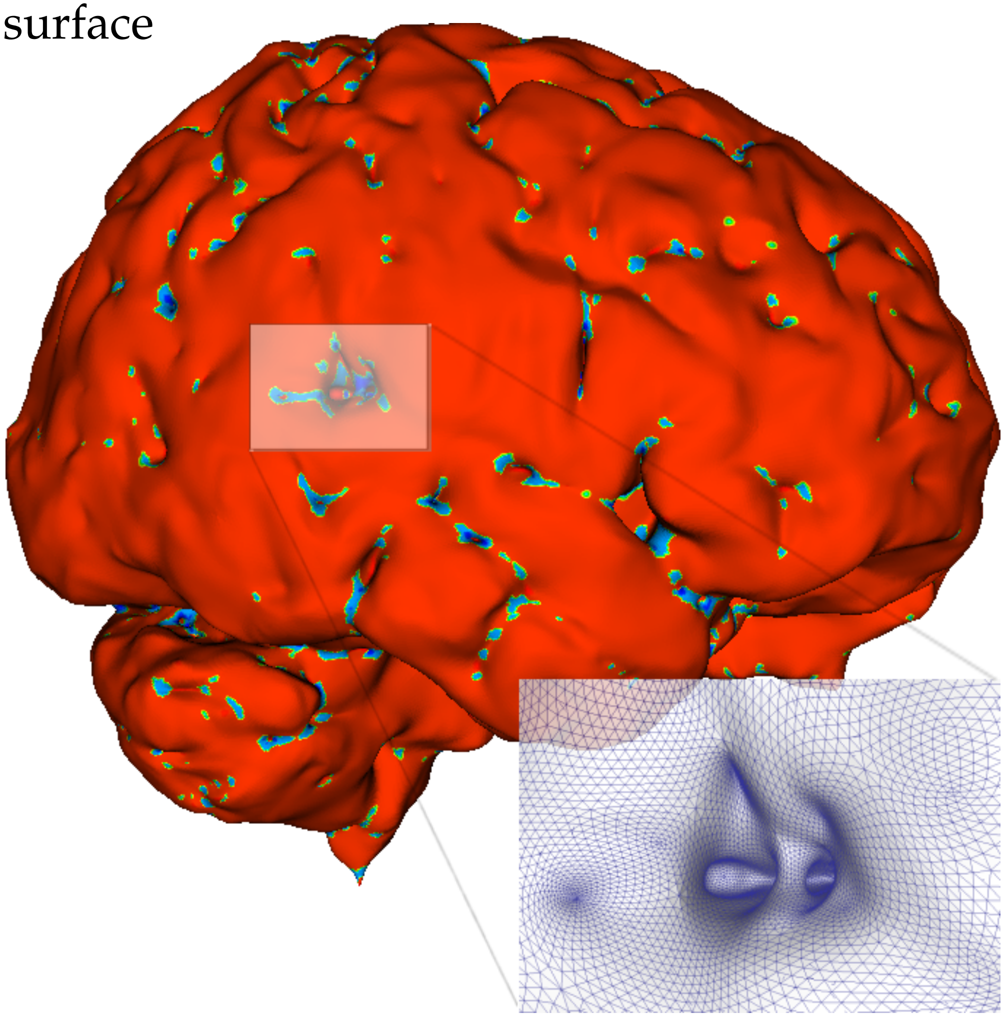
surface S



Parametric Pseudo-Manifolds

Examples

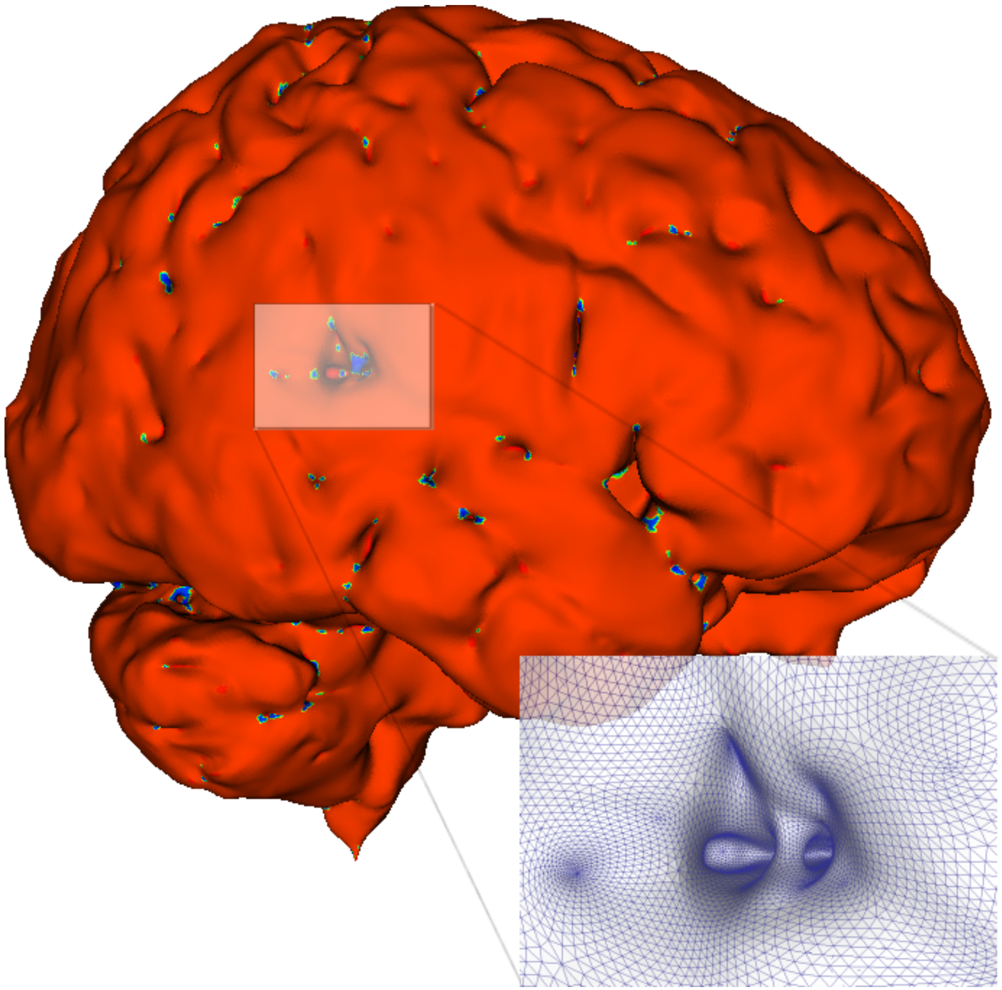
Loop subdivision surface



Parametric Pseudo-Manifolds

Examples

surface S



Parametric Pseudo-Manifolds

Concluding Remarks

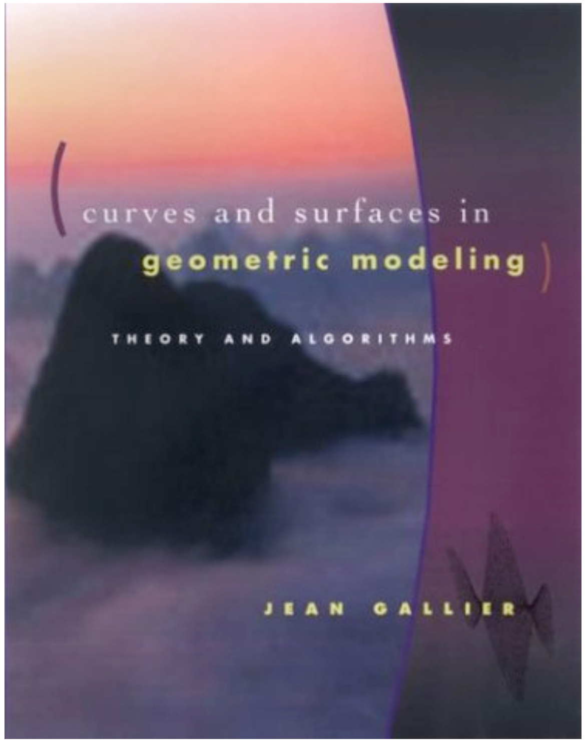
We can play with many choices for the function $\beta = b_\sigma \circ b$. But, keep in mind that we can only do so because the manifold-based approach for surface construction allows us to explicitly separate topology (i.e., gluing data) from geometry (i.e., parametrizations).

We can also use another kind of parametric surface for defining the ψ_v's. We opted for the simplest maps that could give us a C^∞-surface. Depending on the purpose, there may be better options, such as B-splines, beta-splines, box-splines, polar splines, etc.

Some of the above choices for the map ψ_v may yield C^k-surfaces only, for a small positive integer k, which may be enough for many applications you might be interested in.

Parametric Pseudo-Manifolds

Pause for a Commercial



http://www.cis.upenn.edu/~jean/geomcs-v2.pdf